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Compactifications of moduli spaces of (semi)stable bundles on singular curves: two points of view

Montserrat Teixidor i Bigas

Mathematics Department, Tufts University, Medford MA 02155, U.S.A.

E-mail: mteixido@tufts.edu

Dedicat a la memòria de Ferran Serrano, bon company i matemàtic.

Abstract

Moduli spaces of vector bundles on families of non-singular curves are usually compactified by considering (slope)semistable bundles on stable curves. Alternatively, one could consider Hilbert-stable curves in Grassmannians. We study some properties of the latter and compare them with similar properties of curves coming from the former compactification. This leads to a new interpretation of the moduli space of (semi)stable torsion-free sheaves on a fixed nodal curve. One can present it as a quotient of a moduli space of torsion-free sheaves on a variable curve in such a way that each class has a locally-free representative.

Introduction

Consider a projective non-singular curve C of a fixed genus g defined over an algebraically closed field. The set of isomorphism classes of line bundles on C is parametrised by the Jacobian which is an abelian variety. For higher rank, one can also construct moduli spaces for vector bundles so long as one restricts attention to some reasonably well-behaved bundles. The most widespread model for such a moduli space is obtained by considering slope (semi)-stable bundles. The notion of slope stability was first introduced by Mumford for vector bundles. It was later generalised by Newstead and Seshadri to torsion-free sheaves. The introduction of torsion-free sheaves is necessary if one wants to obtain a compact moduli space for a singular curve. On the other hand, working with torsion-free sheaves is often awkward and it would be more convenient for many problems to have a compact moduli space consisting entirely of vector bundles. This apparently utopic dream has in fact come true for rank two in the case of a curve with a unique node (cf. [4], [2]). The construction is as follows: A vector bundle of rank two and high degree together with a space of its sections gives a map from the curve to a Grassmannian (in the same way that a line bundle gives a map to projective space). One can then start from the other end. Consider the Hilbert scheme of curves in the Grassmannian and restrict to those curves that have a fixed stable model. The drawback then is that, while one always has a vector bundle, the curve itself varies. But the variation is bounded: In the case of a nodal curve and a rank two bundle Gieseker showed (cf. [2]) that the only new models one should consider are those obtained by adding either one or two rational components separating the node. In fact, this variation of the curve might be an advantage if one wants to deal not with a single curve but with a family of them. Caporaso (cf. [1]) used this point of view to obtain a compactification of the Jacobian over the moduli space of curves of a given genus, avoiding torsion-free sheaves altogether.

Consider now slope-stability. In the case of a reducible curve, this concept depends on the choice of a polarisation. The only polarisation that is canonically defined on a generic family of stable curves is the one associated with the dualising sheaf. Pandharipande (cf. [6]) used it to give a compactification of the moduli space of torsion-free sheaves over stable (rather than just semistable) curves. We shall refer to his construction as the slope stable compactification.

One would like to use the Hilbert scheme point of view to give a compactification over $\bar{\mathcal{M}}_g$ of the moduli space of (semi)-stable vector bundles. In view of the construction for rank one and for rank two and fixed nodal curve, one would expect this to be feasible. Let us call such a (hypothetic) construction a Hilbert-stable compactification. In order for such a compactification to be useful, its points should have a clear geometric meaning and it should be well related to the slope stable compactification. More precisely, one would expect its point to correspond to semistable curves with not too many unstable components. The immersion in the Grassmannian should be given by a vector bundle that comes close to being (semi)stable by the canonical polarisation. One would then hope to have a map from the Hilbert-stable compactification to the slope stable compactification. The image of a Hilbert-stable curve should be the semistable model of the curve. The restriction of the torsion-free sheaf to the complement of the new nodes should coincide with the restriction of the original vector bundle to the complement of the chains of rational curves and the gluing at the new nodes should provide some (but not all of the) information about the restriction of the vector bundles on these chains.

Notice that the assumptions above impose some conditions on Hilbert-semistable curves. For example, such curves should be semistable of a bounded number of types and the distribution of degrees among the different components should be one (of the finite number) possible for (semi)stability by the canonical polarisation.

In section two of this work, we show that some of these necessary conditions are in fact satisfied. We first extend some of the result of Gieseker and Morrison to rank greater than two. We obtain again that the number of new models one should consider in the case of vector bundles of given rank n is bounded. One can add chains of at most n rational curves between every pair of points corresponding to a node. Moreover, the restriction of the vector bundles to these chains must fall into a finite series of patterns. We also check that the distribution of degrees among components is compatible with the canonical polarisation.

Consider now the fibers of the map between a Hilbert-stable compactification and a slope-stable compactification. In order to study them more easily, we shall consider a larger family that has torsion-free sheaves on semistable not necessarily stable curves. The usual notion of slope stability for the canonical polarisation does not make sense in this case. We need to give an extension of this definition (cf. (1.2),(1.4)). We then study (in section 3) the properties of torsion-free sheaves that are stable by the canonical polarisation. We find some striking similarities with the analogous properties of vector bundles on curves coming from the Hilbert scheme. In particular the distribution of degrees among the components is the same and so is the structure of the vector bundle on the rational chains.

Finally in section 4 we define an equivalence relation between torsion-free sheaves on a family of curves that differ from each other in addition or deletion of rational components. This equivalence relation would identify all point in a fiber of the map from the Hilbert-stable compactification to the slope-stable compactification (although we define it in fact in the larger family that we used in section 3). Each equivalence class has a unique representative on the stable model of the curve and it has representatives that are vector bundles on some suitable models.

1. Some definitions

By a semistable curve, we mean a projective reduced and connected curve with only nodes as singularities.

We first review a few basic notions. Seshadri introduced the notion of stability for a reducible curve as follows: For a curve C with components $C(i)$, a polarisation is the choice of rational weights

$$\lambda(i), 0 < \lambda(i) < 1, \sum \lambda(i) = 1.$$

A depth one sheaf E of rank n on C is said to be (semi)stable for the given polarisation if for every subsheaf G of E with rank $n(i)$ on the component $C(i)$,

$$\chi(G)/\Big(\sum \lambda(i)n(i)\Big)(\leq) < \chi(E)/n.$$

Seshadri showed the existence of a moduli space parametrizing (equivalence classes of semi)stable torsion free sheaves on C.

For a given semistable curve C, define $d(i)$ as the degree of the canonical sheaf of C restricted to the component $C(i)$. We want to use the following weights

$$\lambda(i) = \frac{d(i)}{2g-2}\,. \tag{1.1}$$

This presents a problem when $C(i)$ is a rational curve which meets the rest of the curve at less than three points. We will not need to deal with rational curves meeting the rest at one point. When the number of points of intersection is two, then $\lambda(i) = 0$. We need a slight generalization of Seshadri's definition:

DEFINITION 1.2. Let C be a semistable curve with irreducible components $C(i)$, $i \in I$. Choose a polarization $\lambda(i), 0 \leq \lambda(i), \sum \lambda(i) = 1$ and if $\lambda(i) = 0$ then $C(i)$ is a smooth rational component meeting the rest of the curve at exactly two points. A sheaf E of depth one and constant rank n on C is said to be Seshadri (semi)stable for the polarization λ if and only if for every subsheaf G such that rank $G_{|C(i)} = n(i)$,

$$\chi(G)/\sum \lambda(i)n(i)(\leq) < \chi(E)/n \text{ when} \sum \lambda(i)n(i) \neq 0, n,$$

$$\chi(G) \leq 0 \text{ when } \sum \lambda(i)n(i) = 0$$

and

$$\chi(G) < \chi(E) \text{ when} \sum \lambda(i)n(i) = n, G \neq E.$$

Let us check first that we are not so far off from Seshadri's original definition.

Lemma 1.3

Let C be a curve which has a rational component R intersecting the rest in exactly two points. Let $\lambda = (\lambda(i))$ be a polarization of C of the type introduced in (1.2) (i.e. with weight zero on the component R). For a given $\varepsilon, 0 < \varepsilon < 1$ consider a new polarization $\lambda' = (\lambda'(i))$ such that $\lambda'(i) = \lambda(i)(1-\varepsilon)$ for $C(i) \neq R$ and the λ' weight on R is ε. If a sheaf E is stable for the polarization λ, then it is stable for the polarization λ' for sufficiently small ε. If a sheaf E is stable for the polarization λ' for sufficiently small ε, then it is semistable for the polarization λ. In particular, if the rank and Euler-Poincare characteristic are relatively prime, λ' and λ stability are equivalent.

Proof. The condition for λ stability for a sheaf E means that for every subsheaf G of E,

$$\chi(G) < \left(\sum \lambda(i)n(i)\right)\chi(E)/n \tag{1.3.1}$$

where the $n(i)$ are the ranks of G on the components $C(i)$. Note that $\chi(G)$ is an integer and the $n(i)$ can take only a finite number of values. Therefore, if ε is small enough, the inequality

$$\chi(G) < \left(\sum \lambda(i)(1-\varepsilon)n(i) + \varepsilon n'\right)\chi(E)/n \tag{1.3.2}$$

will be satisfied too with n' the rank of G on R. Therefore, E is also λ'-stable.

Conversely, if (1.3.2) holds for sufficiently small ε, then

$$\chi(G) \leq \left(\sum \lambda(i)n(i)\right)\chi(E)/n$$

and hence E is semistable. □

Definition 1.4. Let C be a semistable curve without rational components attached at a single point. The canonical polarisation on C is a polarisation as above with weight $\lambda(i) = \frac{deg\omega_{|C(i)}}{deg\omega}$. Here ω denotes the dualising sheaf

Recall 1.5. We review the condition of Hilbert stability: Choose a $d >> 0$ and $m >> 0$. Let C be a curve of genus g. Let V be a vector space of dimension $\alpha = d + n(1-g)$. Denote by G the Grassmannian of subspaces of codimension n of V and $\mathcal{E}$ the universal rank n subbundle of $V \otimes \mathcal{O}_G$. Assume that C is a subscheme of G and let E be the restriction of $\mathcal{E}$ to C. Then, V gives rise to a subspace of $H^0(E)$. Consider the map from $S^m(\bigwedge^n V) \to H^0((\bigwedge^n E)^{\otimes m})$. The triple (C, E, V) is m-Hilbert-(semi)-stable if this map is (semi)-stable by the action of the linear group.

The condition for m-Hilbert (semi)-stability has a numerical interpretation as follows: consider any one parameter subgroup λ of the general linear group of the vector space V. Take a basis B of V diagonalizing the action of λ. Denote by r_i the weights of the elements of such a basis. Let $w_B(m)$ be the minimum among the sums of the weights of the elements of basis of $H^0((\bigwedge^n E)^{\otimes m})$ obtained as images of elements of $S^m(\bigwedge^n H^0(E))$. Write $P(m) = \chi((detE)^{\otimes m}) = md + 1 - g$. Then,

$$w_B(m) < (\leq) nmP(m)\sum r_i/\alpha\,. \tag{1.5.1}$$

2. Some properties of Hilbert stable curves

In this section we consider a reduced semistable curve in the Grassmannian embedded by a complete linear system. We study some properties of either the curve or the vector bundle that are a consequence of m-Hilbert stability (for large m) of the corresponding point in the Hilbert scheme.

We first prove that, if the curve is non-singular, (semi)stability of the Hilbert point implies slope (semi)stability. The converse of this was proved for rank two in [4] but is still open for larger rank. In [11], we provided an alternative proof in the case of rank two.

We show next that if the curve is reducible, the distribution of the degrees of the vector bundles among the different components is in fact one of the possible distributions of degrees allowed by slope stability with the canonical polarisation (cf. (1.4)).

Finally we study the structure of rational components. Our results generalise those of Gieseker (compare our (2.4),(2.5) and (2.6) with [2]). On the other hand, this structure is analogous to what is obtained in the case of slope stability (compare with section 3).

For ranks one and two, it has been proved (cf. [3] Theorem 1.0.1 and [4] Theorem 1.2) that any point of the Hilbert scheme that is m-Hilbert stable for arbitrarily large m corresponds to a semistable curve embedded by a complete linear system. It seems likely that this is also the case for higher rank and then some of our assumptions are redundant. We plan to consider this question in the future.

Notations 2.1. In this paragraph, we use the notations $w_B(m), P(m), \alpha$ introduced above. We often write $w(m) = w_B(m)$ when the basis is clear.We shall be defining several one parameter subgroups with specified weights r_i. Denote by b_i the dimension of the subspace of $H^0((\bigwedge^n E)^{\otimes m})$ whose elements have weight at most i. Then,

$$w_B(m) = \sum i(b_i - b_{i-1}) = w_{\max} b_{w_{\max}} - \sum_{i=0}^{w_{\max}-1} b_i.$$

Here, $w_{\max}$ is the maximum weight of a section. Notice then that $b_{w_{\max}}$ is the dimension of $H^0((\bigwedge^n E)^{\otimes m})$ that we denoted by $P(m) = md + 1 - g$. Hence,

$$w_B(m) = w_{\max}(md + 1 - g) - \sum_{i=0}^{w_{\max}-1} b_i. \tag{2.1.1}$$

We shall denote by $O(m)$ a polynomial of degree at most one in m.

Proposition 2.2

Let C be an irreducible non-singular curve embedded in the Grassmannian by a complete linear system. Assume that C corresponds to an m-Hilbert (semi)stable point in the Hilbert scheme of the Grassmannian for m large enough. Then, the corresponding vector bundle E is (semi)stable.

Proof. Let E' be a subsheaf of E of rank n'. Define a one parameter subgroup by giving weight zero to the elements in a basis of $H^0(E')$ and weight one to a complement. Consider the morphism $S^m(\wedge^n V) \to H^0((\wedge^n E)^{\otimes m})$. An element in the image has weight i if it is the product of $nm-i$ sections of E' and i sections of E not in $H^0(E')$. As the rank of E' is n', the minimum weight of a section is $m(n-n')$. So,

$$w(m) \geq m(n-n')\big(md + n(1-g)\big).$$

By Hilbert (semi)stability (cf. (1.5.1))

(2.2.1)

$$w(m) \leq (nm/\alpha)\big(md + n(1-g)\big)\Big(\sum r_i\Big) = (nm/\alpha)\big(md + n(1-g)\big)\big(\alpha - h^0(E')\big).$$

Then, we get

$$m(n-n') \leq nm - (nm/\alpha)h^0(E').$$

Hence,

$$h^0(E')/n' \leq \alpha/n.$$

As $\chi(E') \leq h^0(E')$, this shows that E' does not contradict semistability of E.

Assume now that the point is strictly m-Hilbert stable. If $h^0(E')/n' < \alpha/n$, then E' does not contradict stability of E. If $h^0(E')/n' = \alpha/n$, Hilbert stability would imply that there is a strict inequality for the linear term in m in (2.2.1). Using the condition $h^0(E')/n' = \alpha/n$, we see that this is impossible. □

Proposition 2.3

Let C be a semistable curve embedded in the Grassmannian by a complete linear system (i.e $V = H^0(E)$). Assume that (C, E, V) is m-Hilbert semistable for large enough m and sufficiently big d. Then the degrees of E on each component of C are one of the sets of degrees allowed by the canonical polarisation.

Proof. Let E' be the sheaf of sections of E which vanish outside of a given union of components C' of C. We want to show that the stability condition is satisfied for this special E'.

Consider a basis $X_1, \dots, X_j$ of $H^0(E') \cap V$. Complete it to a basis $X_1, \dots, X_\alpha$ of V. Define a one parameter subgroup of V by $\lambda(t)(X_i) = X_i$ for $i \leq j$ and $\lambda(t)(X_i) = tX_i$ for $i \geq j+1$. For this special one parameter subgroup, the r_i are zero and one. The weights of a basis in $H^0((\wedge^n E)^{\otimes m})$ obtained as images of elements in $S^m(\wedge^n V)$ vary between 0 and mn. Write C'' for the union of all components of C not contained in C'. Denote by D the divisor of intersection of C' and C'' and let k be its degree. Any section of $\wedge^n(E)$ with weights less than nm vanishes on C''. Then, $b_i \leq h^0((\wedge^n E)^{\otimes m}_{|C'}(-(nm-i)D))$. Let g' be the genus of C' and d' the degree of E'. Then, either $h^0((\wedge^n E)^{\otimes m}_{|C'}(-(nm-i)D)) = 0$ or

$$\begin{aligned} h^0\big((\wedge^n E)^{\otimes m}_{|C'}(-(nm-i)D)\big) &= md' - (nm-i)k + 1 - g' \\ &\quad + h^1\big((\wedge^n E)^{\otimes m}_{|C'}(-(nm-i)D)\big) \\ &\leq md' - (nm-i)k + 1. \end{aligned}$$

If $md' - (nm-1)k + 1 \geq 0$, the inequality above is valid for all values of i. Hence, from (2.1.1)

$$\begin{aligned} w_B &\geq nm(dm+1-g) - \sum_{i=0}^{nm-1} \big(md' - (nm-i)k + 1\big) \\ &= m^2(nd - nd') + (mn-1)mnk/2 + O(m) \\ &= m^2(nd - nd' + n^2k/2) + O(m). \end{aligned}$$

On the other hand,

$$\begin{aligned} nmh^0\big((\wedge^n E)^{\otimes m})\big) \frac{\sum r_i}{h^0(E)} &\leq nm(md+1-g)\ \frac{\alpha - h^0(E')}{\alpha} \\ &= nm(md+1-g)\Big(1 - \frac{h^0(E')}{\alpha}\Big). \end{aligned}$$

Therefore the Hilbert-stability condition gives rise to

$$nd - nd' + n^2k/2 \leq nd\Big(1 - \frac{h^0(E')}{\alpha}\Big).$$

Hence,

$$\frac{h^0(E')}{\alpha} \leq \big(d' - n(k/2)\big)/d$$

or equivalently

$$h^0(E') \leq \frac{\big(d' - n(k/2)\big)\alpha}{d}.$$

Note that

$$h^0(E') \geq \chi(E') = d' - nk + n(1-g') = d' - nk/2 - n(g' - 1 + k/2).$$

Also, by definition

$$\alpha = \chi(E) = d + n(1-g) = d - n(g-1).$$

Using these equations,we get

$$\chi(E')/\chi(E) = \chi(E')/\alpha \leq \big(d' - n(k/2)\big)/d = [\chi(E') + n(g'-1+k/2)]/[\chi(E) + n(g-1)].$$

This condition can be rewritten as

$$\chi(E') \leq \frac{(g' - 1 + k/2)}{g-1}\chi(E). \tag{2.3.1}$$

This is exactly the condition that E' does not contradict Seshadri stability for the weights defined above (cf. (1.1)). As these conditions completely determine the distribution of degrees (cf. [10] Proposition (1.2)), the degree of such an E is one of the possible choices for bundles stable by the canonical polarization.

Note that if $md' - (nm-1)k + 1 < 0$ for large m, then $d' - nk < 0$. So $\chi(E') < n(1-g')$. If C' is not rational or if $k \geq 3$, then $g' - 1 + k/2 > 0$ and (2.3.1) is also satisfied. The case C rational and $k \leq 2$ is dealt with in (2.4)- (2.6) below. □

Proposition 2.4

Let C be a semistable curve embedded in the Grassmannian by a complete linear system. Assume that C corresponds to an m-Hilbert stable point for large enough m. Then C has no rational components attached at one point only.

Proof. Assume the opposite. Then, there is a rational component $\mathbf{P}$ of C attached at one point Q only to the rest of the curve. Write the restriction of the vector bundle to $\mathbf{P}$ as $\mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_n)$, $a_1 \geq \ldots \geq a_n$. As the vector bundle gives an immersion of the curve in the Grassmannian, $a_n \geq 0$ and $a_1 \geq 1$. Write $d' = a_1 + \ldots + a_n$.

Consider the subsheaf F of E whose restriction to $\mathbf{P}$ is $\mathcal{O}(a_1(-Q))$ and that is zero outside $\mathbf{P}$. Define a one parameter subgroup by giving weight zero to the sections of F and weight one to the elements of a complement in V. Consider the weights induced on $H^0((\bigwedge^n E)^{\otimes m})$. These weights vary between $m(n-1)$ and mn. The space of sections with weight at most $k, m(n-1) < k < nm$ has dimension at most $h^0(\mathbf{P}, (\wedge^n E)^{\otimes m}(-(nm-k)Q)) = md' + 1 + (k - mn)$ if this number is positive. As $k \geq m(n-1)$ and $d' \geq 1$, $md' + 1 + k - mn \geq 1$. Then from (2.1.1)

$$\begin{aligned} w &\geq mnP(m) - \sum_{k=m(n-1)}^{mn-1} h^0\Big(\mathbf{P}, (\wedge^n E)^{\otimes m}\big(-(nm-k)Q\big)\Big) \\ &= m^2(nd - d' + 1/2) + O(m). \end{aligned}$$

The Hilbert-stability condition gives

$$\begin{aligned} w(m) &\leq [nmP(m)/\alpha] \sum r_i = [nmP(m)/\alpha][\alpha - h^0(\mathbf{P}, \mathcal{O}(a_1(-Q))] \\ &= nmP(m)(1 - a_1/\alpha) = m^2(nd - nda_1/\alpha) + O(m). \end{aligned}$$

Hence, we get,

$$nd - d' + 1/2 \leq nd - nda_1/\alpha.$$

Equivalently,

$$a_1 \leq (\alpha/dn)(d' - 1/2) < d'/n.$$

This is not compatible with the definition of the a_i. □

Proposition 2.5

Let C be a semistable curve embedded in the Grassmannian by a complete linear system. Assume that it corresponds to an m- Hilbert stable point of the Hilbert scheme for m sufficiently large. If $\mathbf{P}$ is a rational curve attached to the rest at two points Q and R, then, $E_{|\mathbf{P}} = \mathcal{O}^a \oplus \mathcal{O}(1)^{(n-a)}, n - a \geq 1$.

Proof. Write $E_{|\mathbf{P}} = \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_n), a_1 \geq \ldots \geq a_n$. Write $d' = a_1 + ... + a_n$. As the bundle gives an immersion of the curve in the Grassmannian, $a_n \geq 0, a_1 \geq 1$. It is then enough to show $a_1 \leq 1$. Let j be the last index such that $a_j \geq 2$. Assume $j \geq 1$. Define a subsheaf F of E by the condition $F_{|\mathbf{P}} = E_{|\mathbf{P}}(-R-Q), F_{|C-\mathbf{P}} = 0$. Define a one parameter subgroup by giving weight 0 to the sections of $H^0(F)$, weight one to the sections of a complement. Notice that

$$H^0(F) \subset H^0\big((\mathcal{O}(a_1) \oplus ... \oplus \mathcal{O}(a_j))\big(-P-Q\big)\big).$$

The minimum weight of a section of $H^0((\wedge^n E)^{\otimes m})$ is $m(n-j)$. The dimension of the locus of sections with weight k, is bounded by

$$b_k \leq h^0\Big((\wedge^n E)^{\otimes m}_{|\mathbf{P}}\big(-(nm-k)(R+Q)\big)\Big), \ m(n-j) \leq k \leq mn - 1.$$

Now $h^0((\wedge^n E)^{\otimes m}_{|\mathbf{P}}(-(nm-k)(R+Q)) = md' + 1 - 2(nm-k)$ as this number is positive. Hence from (2.1.1),

$$w \geq nm(md+1-g) - \sum_{k=m(n-j)}^{mn-1} [md' + 1 - 2(nm-k)] = m^2(nd - jd' + j^2) + O(m).$$

On the other hand, Hilbert stability implies that

$$\begin{aligned} w(m) &\leq \big(nmP(m)/\alpha\big)\Big(\sum r_i\Big) \\ &= nm(md+1-g)\left[1 - \frac{h^0\big(\mathbf{P}, \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_j)(-(P+Q))\big)}{\alpha}\right] \\ &= m^2\Big[nd - \frac{nd(a_1 + \ldots + a_j - j)}{\alpha}\Big] + O(m). \end{aligned}$$

Hence,

$$\frac{nd(a_1 + \ldots + a_j - j)}{\alpha} \leq jd' - j^2.$$

Equivalently,

$$(a_1 + \ldots + a_j - j) \leq \frac{\alpha}{nd}\,(jd' - j^2)\,.$$

From the definition of α, $\alpha/d < 1$. From the definition of j, $d' \leq a_1 + ... + a_j + n - j$. Then

$$a_1 + ... + a_j - j < \frac{j}{n}(d' - j) < \frac{j}{n}(a_1 + ... + a_j + n - 2j).$$

Hence

$$(n-j)(a_1 + ... + a_j) < 2j(n-j).$$

Therefore

$$a_1 + ... + a_j < 2j$$

contradicting the choice of j. $\square$

Proposition 2.6

Let C be a semistable curve embedded in the Grassmannian by a complete linear system. Assume that the point corresponding to C is m-Hilbert semistable for m large enough. Assume that C contains a chain $\mathbf{P}$ of rational curves (i.e. $\mathbf{P} = \mathbf{P_1} \cup \ldots \cup \mathbf{P_k}$ each $\mathbf{P_i}$ intersects $\mathbf{P_{i-1}}$ and $\mathbf{P_{i+1}}$ at points Q_i and Q_{i+1}; $\mathbf{P_1}$ intersects the rest of C at Q_1 and $\mathbf{P_k}$ intersects the rest of C at Q_{k+1}). Then $k \leq n$. If C minus the chain is not connected and the bundle is strictly stable, then $k \leq n-1$. Moreover, the restriction of E to any subchain is a direct sum of line bundles of degrees zero and one.

Proof. From [8] Proposition (3.1), the restriction of E to the chain is a direct sum of line bundles. The proof of (2.5) can be generalized to show that these line bundles must have degree zero or one. As C is a curve in the Grassmannian, E cannot be trivial on any rational curve. Therefore, there are no more than n rational curves.

Assume now that the chain disconnects the curve into two components C_1 and C_2. Denote by d_1, d_2 the degree of the restriction of E to C_1, C_2. We want to show that there are then at most $n-1$ rational curves in the chain. Assume that there were n. Write the restriction of E to $\mathbf{P}$ as a direct sum of n line bundles $L_1 \oplus \ldots \oplus L_n$. We showed already that the restriction of each L_i to each component $\mathbf{P_j}$ is of the form $\mathcal{O}$ or $\mathcal{O}(1)$ with at least one $\mathcal{O}(1)$ and two $\mathcal{O}(1)$ cannot appear on the same L_j. Hence, we can assume that L_i is of the form $\mathcal{O}(1)$ on the component $\mathbf{P_i}$ and is $\mathcal{O}$ on all other components. Consider the subsheaf E' of E whose restriction to $\mathbf{P}$ consists of the direct sum of the sheaf of sections of the L_i vanishing at Q_{i+1}, the restriction to C_1 coincides with the restriction of E to C_1 and the restriction to C_2 is zero. Define a one parameter subgroup of E by giving weight zero to the sections of $H^0(E') \cap V$ and weight one to the sections of a complement. An element of $H^0((\bigwedge^n E)^m)$ has weight at most i if it includes a product of at least $nm-i$ sections of E'. Hence, for $i \geq 1$ the dimension of the space of sections with weigh at most i is at most

$$\begin{aligned} b_i &\leq h^0\big((\wedge^n E)^m_{|\mathbf{P}\cup C_1}(-(nm-i)Q_n)\big) \\ &= md_1 + i + 1 - g_1 + h^1\big((\wedge^n E)^m_{|\mathbf{P}\cup C_1}(-(nm-i)Q_n)\big) \\ &\leq md_1 + i + 1. \end{aligned}$$

Therefore,

$$w(m) \geq nm(md+1-g) - \sum_{i=1}^{nm-1}(md_1+i+1-g_1) = m^2(nd - nd_1 - n^2/2) + O(m).$$

From the condition for Hilbert stability,

$$nd - nd_1 - n^2/2 \leq (nd/\alpha)\sum r_i = nd(1 - h^0(E')/\alpha).$$

Therefore, $dh^0(E')/\alpha \leq d_1 + n/2$ or equivalently

$$h^0(E') \leq \alpha(d_1 + n/2)/d.$$

Interchanging the roles of C_1 and C_2 and reversing the order of the L_i, one gets

$$h^0(E'') \leq \alpha(d_2 + n/2)/d.$$

We now use that $d = d_1 + d_2 + n, g = g_1 + g_2$. Then

$$\alpha = d_1 + n(1-g_1) + d_2 + n(1-g_2) \leq h^0(E') + h^0(E'') \leq (d_1 + d_2 + n)\alpha/d = \alpha.$$

Therefore, these inequalities must in fact be equalities and the vector bundle is not strictly stable. □

3. Some consequences of Seshadri's stability for the canonical polarization

In this paragraph, we study some properties of Seshadri semistable sheaves for the canonical polarization when restricted to a chain of rational components. These are analogous to the corresponding properties for m-Hilbert stability (cf. (2.4)-(2.6)).

Lemma 3.1

Let E be a Seshaderi semistable sheaf on a curve C and let C_i be a rational component of C with corresponding weight $\lambda_i = 0$. Then $E_{|C_i} = \mathcal{O}(1)^a \oplus \mathcal{O}^{n-a}$.

Proof. Write $E_{|C_i} = \mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_n)$, $a_1 \geq \ldots \geq a_n$. Denote by P_1 and P_2 the two nodes of C_i. Consider the subsheaf G_1 of E which vanishes on every component different from C_i and whose restriction to C_i is $\mathcal{O}(a_1)(-P_1 - P_2)$. By the stability condition for $E, \chi(G_1) = a_1 - 2 + 1 \leq 0$. So $a_1 \leq 1$.

Consider then the subsheaf G_2 obtained by taking $\mathcal{O}(a_1) \oplus \ldots \oplus \mathcal{O}(a_{n-1})$ on C_i and the sections that glue with this subsheaf in the remaining components. Then $\chi(G_2) = \chi(E) - \chi(\mathcal{O}(a_n)) = \chi(E) - a_n - 1 < \chi(E)$. Hence $a_n > -1$, so $a_n \geq 0$. □

Lemma 3.2

Let E be a Seshadri-semistable sheaf for the canonical polarisation on a curve C. Let $\mathbf{P}$ be a chain of rational components as in (2.6). Assume that the restriction of E to the chain is locally free. Then, it is a direct sum of line bundles of degrees 0 *and* 1.

Proof. The fact that E is a direct sum of line bundles was proved in [8] Proposition 3.1. The fact that the degrees of these line bundles are 0 and 1 is similar to the proof of (3.1). □

Lemma 3.3

Consider a chain of rational curves as in (2.6). *Let E be a torsion-free sheaf on the curve. Let V_1 be a subspace in the fiber of Q_1. Let $F_1 \subset \ldots \subset F_n$ be a complete flag at Q_{k+1}. Then, there exists a decomposition of the sheaf as direct sum of n rank one torsion free sheaves on the chain compatible with the subspace and the flag. By this we mean that V_1 is the direct sum of the fibers at Q_1 of a few of these torsion free sheaves and each F_i is the direct sum of a few fibers at Q_{k+1}.*

Remark. This is a refinement of Lemma 3.2 in [8].

Proof. We use induction on k. Assume $k=1$. Let $E_{|\mathbf{P_1}} = \mathcal{O}(a_1)^{e'_1} \oplus \ldots \oplus \mathcal{O}(a_t)^{e'_t}$ with $a_1 > \ldots > a_t$. Define E_1 as the unique subsheaf of $\mathcal{O}(a_1)^{e'_1}$ whose fiber at Q_1 coincides with $V_1 \cap (\mathcal{O}(a_1)^{e'_1})_{Q_1}$ and has rank equal to the dimension of this space. Define $E_2 = \mathcal{O}(a_1)^{e'_1}$. Define E_3 as the sum of $\mathcal{O}(a_1)^{e'_1}$ and the subsheaf of $\mathcal{O}(a_1)^{e'_1} \oplus \mathcal{O}(a_2)^{e'_2}$ whose fiber at Q_1 coincides with $V_1 \cap (\mathcal{O}(a_1)^{e'_1} \oplus \mathcal{O}(a_2)^{e'_2})_{Q_1}$ and has rank equal to the dimension of $V_1 \cap (\mathcal{O}(a_1)^{e'_1} \oplus \mathcal{O}(a_2)^{e'_2})_{Q_1} + \mathcal{O}(a_1)^{e'_1}$... Define E_{2t} as $E\mid_{Q_1}$. Denote by e_i the rank of E_i. Define a new filtration of E_{Q_2}. One can find e_1 subspaces $F_l, l = i(1) < \ldots < i(e_1)$ such that $F_l \cap (E_1)_{Q_2} \neq F_{l-1} \cap (E_1)_{Q_2}$. This comes from the fact that $(E_1)_{Q_2} = (E_1)_{Q_2} \cap F_n$ and each F_i has codimension one in F_{i-1}. Choose $v_l \epsilon F_l \cap (E_1)_{Q_2} - F_{l-1} \cap (E_1)_{Q_2}$.

We proceed now by induction on j. One can find e_j subspaces $F_l, l = i(e_1+\ldots+e_{j-1}+1) < \ldots < i(e_1+\ldots+e_j)$ such that $F_l \cap (E_j)_{Q_2} \neq F_{l-1} \cap (E_j)_{Q_2} + F_l \cap (E_{j-1})_{Q_2}$, choose v_l in $F_l \cap (E_j)_{Q_2} - F_{l_{-1}} \cap (E_j)_{Q_2} - F_l \cap (E_{j-1})_{Q_2}$. For every l, there is a minimum j for which $F_l \cap (E_j)_{Q_2} \neq F_{l-1} \cap (E_j)_{Q_2}$. The minimality implies automatically $F_l \cap (E_j)_{Q_2} \neq F_{l-1} \cap (E_j)_{Q_2} + F_l \cap (E_{j-1})_{Q_2}$ and so i is a permutation of the set $\{1, \ldots, n\}$. It is clear then that there is a decomposition of $E_{|C_1}$ with directions at Q_2 in the directions of the basis chosen in that way.

Assume now the result for $k = M-1$ and prove it for M. Denote by Q_i^1 and Q_i^2 the inverse images of the node Q_i on the components C_{i-1} and C_i obtained when we normalize at Q_i. There are subspaces V_i^1 and V_i^2 of $(E_{|C_i})_{Q_i^1}$ and $(E_{|C_i})_{Q_i^2}$ and an isomorphism f_i between $(E_{|C_i})_{Q_i^1}/V_i^1$ and $(E_{|C_i})_{Q_i^2}/V_i^2$ that defines E at Q_i.

Define subsheaves E_i of $E_{|C_M}$ exactly as in the case $k=1$ above by replacing V_1 by V_M^2. Define a permutation of the F_l exactly as in the case $k=1$. Consider the flag of $E_{Q_{M+1}}$ given by

$$F_{i_{(1)}} \cap (E_1)_{Q_{M+1}} \subset \ldots \subset F_{i(e_1)} \cap (E_1)_{Q_{M+1}} \subset (E_1)_{Q_{M+1}} + F_{i(e_1+1)} \cap (E_2)_{Q_{M+1}} \subset$$

$$\subset \ldots \subset (E_{e_{2t-1}})_{Q_{M+1}} + F_{i_{(n)}}.$$

There is a decomposition of $E_{|\mathbf{P}_M}$ into a direct sum of line bundles whose successive sums give rise to this flag at Q_{M+1} and such that V_M^2 is the direct sum of the fibers of a few of these line bundles. Denote by $v_1, \ldots, v_n$ the basis that this decomposition induces at the fiber at Q_{M+1}. Denote by $w_1, \ldots, w_n$ the basis that it induces at Q_M^2. Consider now the chain $P_1 \cup \ldots \cup P_{M-1}$. Consider the flag induced at Q_M^1 by the inverse image by f_M of the flag above. This flag has as first space V_M^1. We can obtain a complete flag with n subspaces at Q_M^1 if we add a complete flag of the subspace V_M^1. Applying the induction hypothesis, there is a decomposition of the restriction of E to $P_1 \cup \ldots \cup P_{M-1}$ adjusted to this flag. Denote by $w'_1, \ldots, w'_n$ the basis that this decomposition induces at Q_M. Then, $w'_i = w_i + \sum l_{ij} w_j$. Consider the image of the w'_i by f_{M-1}. Modify the decomposition of $E_{|P_M}$ to fit this basis. In so doing, we can disregard the l_{ij} for either i or j between $e(2k)+1$ and $e(2(k+1))$. Therefore, this modification is possible. □

Corollary 3.4 (Compare with (2.6))

With the notations of (2.6), (3.2), let a be the number of components on which E is not trivial (i.e. of the form $\mathcal{O}^n$). Let b be the number of nodes on which E is not locally free. Then $a + b \leq n$.

Proof. From (3.3), we know that the restriction of E to the chain of rational components is a direct sum of rank one torsion free sheaves. The proof of (3.1) can be generalized to show that each of the rank one sheaves can fail to be either locally free or trivial but not both on at most one component. From this, the fact that $a + b \leq n$ follows.

Assume now that the chain disconnects the curve into two components C_1 and C_2. Write the restriction of E to the chain as $L_1 \bigoplus \ldots \bigoplus L_n$ where each L_i is a torsion-free sheaf of rank one on the chain. None of the L_i can fail to be locally free or trivial more than once. Assume there are n rational curves on the chain and the sheaf is not locally free on any of them. Then, each of the L_i is not trivial on one component. Define the analogue of E' in (2.6) Assume first that L_i is of the form $\mathcal{O}(1)$ on the component $\mathbf{P}_j$.

Denote by L_i^1 the sheaf consisting of the sections of $L_{i|\mathbf{P}_1 \cup \ldots \cup \mathbf{P}_j}$ that vanish at Q_{j+1}. Denote by L_i^2 the sheaf consisting of the sections of $L_{i|C_j \cup \ldots \cup C_n}$ that vanish at Q_j.

Assume that L_i fails to be locally free at the node Q_j. Denote by L_i^1 the sheaf consisting of the sections of $L_{i|\mathbf{P}_1 \cup \ldots \cup \mathbf{P}_j}$. Denote by L_i^2 the sheaf consisting of the sections of $L_{i|\mathbf{P}_{j+1} \cup \ldots \cup \mathbf{P}_n}$.

Define E' from the restriction of E to C_1 and the L_i^1 extended by zero. Define E'' from the restriction of E to C_2 and the L_i^2 extended by zero. Then, either E' or E'' contradicts stability of E. □

4. The moduli space as a set of equivalence classes of vector bundles

We studied in section 2 Hilbert points in the Grassmannian. We saw that Hilbert stability imposes restrictions on the possible semistable curves that can appear. We can consider the stable model of such a curve. Then, a torsion-free sheaf needs to take the place of the vector bundle. On the other hand, given such a stable curve and a torsion-free sheaf on it, one can consider in general many semistable curves whose stable model is the curve given. Consider for example an irreducible curve with just one node. Take a torsion-free sheaf of rank two. Assume that the fiber of the sheaf at the node is a direct sum of two copies of the maximal ideal at the point. This curve can have two admissible semistable models with a chain of either one or two rational curves. In the case of one rational component, the vector bundle on it is of the form $\mathcal{O}(1)^2$. In the case of two rational components, the vector bundle on each of them is of the form $\mathcal{O}(1) \oplus \mathcal{O}$ and the two $\mathcal{O}(1)$ don't glue together.

The purpose of this section is to clarify the relationship between the different semistable models that can give rise to the same stable curve with a fixed torsion-free sheaf. In order to do so, we set up an equivalence relation between them. We include, along with Hilbert-admissible vector bundles, some torsion-free sheaves on partial normalisations of a nodal curve.

The main idea of the definition that follows is that one replaces some of the non local-freeness at a node P in the original sheaf E by a summand of the form $\mathcal{O}(1)^a$ in an additional $\mathbf{P}^1$. The rest of the non local-freeness is divided between the two nodes that appear on the new $\mathbf{P}^1$. The gluing on the locally free part must respect the original gluing of E. All curves C are assumed to be semistable and have a common stable model C_0. Moreover they don't have rational components attached at one point only. Hence they differ from each other by addition or deletion of chains of rational curves. The equivalence will be generated by the relation defined in (4.1).

(4.1). Assume that one of the curves C' is obtained from C by adding one single rational component $\mathbf{P^1}$ at a node P of C. Denote by P_1 P_2 the two preimages of P in the partial normalization of C at P. Denote by P_1^1 and P_2^1 the points of $\mathbf{P}^1$ that glue with P_1 and P_2 respectively. Consider the two pairs (C, E) and (C', E') of the curve and a torsion-free sheaf. For a vector bundle F and a point P, we shall denote by F_P the vector space fiber of F at P (rather than the stalk of the sheaf F at the point P). We recall that E is obtained from the following data

a) a torsion free sheaf F on the partial normalisation of C at P
b) Subspaces V_1 and V_2 of F_{P_1} and F_{P_2} (of the same dimension)
c) an isomorphism $F_{P_1}/V_1 \xrightarrow{\varphi} F_{P_2}/V_2$.

Similarly E' is obtained from.

a') a torsion free sheaf F' on the partial normalisation of C at P and a vector bundle T on $\mathbf{P}^1$
b') subspaces V_1', V_1^1 of F'_{P_1} and $T_{P_1^1}$ and V_2', V_2^1 of F'_{P_2} and $T_{P_2^1}$
c') isomophisms $\varphi_1 : {F'}_{P_1}/V_1' \to T_{P_1^1}/V_1^1$ and $\varphi_2 : F'_{P_2}/V_2' \to T_{P_2^1}/V_2^1$.

If the following conditions are satisfied, we'll say that (C,E) and (C',E') are equivalent.

i) $F' = F, T = \mathcal{O}(1)^a \oplus \mathcal{O}^{n-a}$. This latter condition implies that there is a canonical isomorphism
$$\psi : (T/\mathcal{O}(1)^a)_{P_1^1} \to (T/\mathcal{O}(1)^a)_{P_2^1}$$

ii)
$$V_1' \subset V_1, V_2' \subset V_2$$
$$V_1^1 \cap (\mathcal{O}(1)^a)_{P_1^1} = 0, V_2^1 \cap (\mathcal{O}(1)^a)_{P_2^1} = 0$$
$$\psi(V_1^1) \cap V_2^1 = 0$$
where the last equality is understood in the quotient.

iii)
$$\varphi_1(V_1/V_1') = [\mathcal{O}(1)^a_{P_1^1} + V_1^1 + \psi^{-1}(V_2^1)]/V_1^1$$
$$\varphi_2(V_2/V_2') = [\mathcal{O}(1)^a_{P_2^1} + \psi(V_1^1) + V_2^1]/V_2^1$$

iv) Note that from the inclusions $V_i' \subset V_i$ in ii), one obtains isomorphisms
$$F_{P_1}/V_1 \cong [F_{P_1}/V_1']/[V_1/V_1']$$
and
$$F_{P_2}/V_2 \cong [F_{P_2}/V_2']/[V_2/V_2'].$$
From condition iii), φ_1, φ_2 induce isomorphisms of the quotients
$$[F_{P_1}/V_1']/[V_1/V_1'] \to [T_{P_1^1}/V_1^1]/[(\mathcal{O}(1)^a_{P_1^1} + V_1^1 + \psi^{-1}(V_2^1))/V_1^1]$$
and
$$[F_{P_2}/V_2']/[V_2/V_2'] \to [T_{P_2^1}/V_2^1]/[(\mathcal{O}(1)^a_{P_2^1} + \psi(V_1^1) + V_2^1)/V_2^1].$$
The morphism ψ induces a map between the two target spaces above. We require then that the following diagram commutes
$$\begin{array}{ccc} [F'_{P_1}/V_1']/[V_1/V_1'] & \stackrel{\bar\varphi_1}{\to} & [T_{P_1^1}/V_1^1]/[(\mathcal{O}(1)^a_{P_1^1} + V_1^1 + \psi^{-1}(V_2^1))/V_1^1] \\ \downarrow \varphi & & \downarrow \bar\psi \\ [F'_{P_2}/V_2']/[V_2/V_2'] & \stackrel{\bar\varphi_2}{\to} & [(T_{P_2^1}/V_2^1]/[\mathcal{O}(1)^a_{P_2^1} + \psi(V_1^1) + V_2^1)/V_2^1]\,. \end{array}$$

Theorem 4.2

The moduli space of torsion-free (a_i) *semistable sheaves on* C_0 *parametrizes equivalence classes of torsion-free sheaves on the family of curves defined in* (4.1) *semistable by the polarization that has weight* a_i *in the components of* C_0 *and weight* 0 *in the additional components. Every element of the moduli space admits representatives which are vector bundles on a suitable curve.*

The last statement of the Theorem is easy to prove. Assume that the stalk of the torsion-free at the node is isomorphic to a direct sum of a copies of the maximal ideal and $n-a$ copies of the local ring of the node. Introduce a rational component separating the node. Take on this component $\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a}$. Glue $[\mathcal{O}(1)^a]_{P_1^1}$ to V_1 and $[\mathcal{O}(1)^a]_{P_2^1}$ to V_2. Glue then the rest so that mod $\mathcal{O}(1)^a$ one obtains the original gluing. With definition (4.1),this new vector bundle is equivalent to the torsion-free sheaf we started with.

In order to prove the remaining statements in (4.2), we shall need a few preliminary Lemmas.

Lemma 4.3

Let E and E' be related as in (4.1). Then the space of sections of E can be identified to the space of sections of E' in a canonical way (by means of the restriction map).

Proof. We follow the notations that we introduced in (4.1). Moreover, for a section σ, we shall denote by $\bar{\sigma}$ class in either F_{P_1}/V_1 or F_{P_2}/V_2. Then a section of the bundle E is given by a section s of F such that $\varphi(\bar{s}_1) = \bar{s}_2$. Let us show that we can construct from s a unique section of E'. This amounts to saying that there is a unique section t of $\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a}$ such that $\varphi_1(\bar{s}'_1) = \bar{t}^1_1$ and $\varphi_2(\bar{s}'_2) = \bar{t}^1_2$ where $\bar{\sigma}'$ denotes class of a section σ in either F_{P_1}/V'_1 or F_{P_2}/V'_2 and $\bar{\sigma}^1$ denotes class in either $T_{P^1_1}/V^1_1$ or $T_{P^1_2}/V^1_2$. Define $t_1 = \varphi_1(\bar{s}'_1)$ and $t_2 = \varphi_2(\bar{s}'_2)$ in $(\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a})_{P^1_1}/V^1_1$ and $(\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a})_{P^1_2}/V^1_2$ respectively. Consider now the natural map f from the space of sections of the restriction of the torsion-free sheaf to the rational curve to the quotients of the fibers at the two points. Consider the composition

$$H^0(\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a}) \xrightarrow{f} (\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a})_{P^1_1}/V^1_1 \oplus (\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a})_{P^1_2}/V^1_2 \xrightarrow{g} F_{P_1}/V_1 \, .$$

Here the map g has components $(\varphi\pi_1\varphi_1^{-1}, -\pi_2\varphi_2^{-1})$ where π_1 and π_2 are natural projections. From the commutativity of the diagram in (4.1) $gf = 0$. Hence f factors through a map f' with image in $\ker g$. The condition $\varphi(\bar{s}_1) = \bar{s}_2$ is equivalent to (t_1, t_2) being in the kernel of g. Then, the existence of t amounts to the surjectivity of f'.

Denote by v'_1 and v'_2 the dimensions of V'_1 and V'_2 respectively. From ii) and iii) in (4.1), F_{P_2}/V_2 is a vector space of dimension $n - \dim V_2 = n - a - v'_1 - v'_2$. The vector space $(\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a})_{P^1_1}/V^1_1 \oplus (\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a})_{P^1_2}/V^1_2$ has dimension $2n - v'_1 - v'_2$. Therefore, $\ker g$ is a vector space of dimension $n + a = h^0(\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a})$. Hence, f' is surjective if and only if it is injective.

Write $H^0(\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a}) = H^0(\mathcal{O}(1)^a) \oplus H^0(\mathcal{O}^{n-a})$. A section of $\mathcal{O}(1)^a$ is completely determined by its fiber at two points. Hence, the map from $H^0(\mathcal{O}(1)^a)$ to $(\mathcal{O}(1)^a)_{P_1^1} \oplus (\mathcal{O}(1)^a)_{P_2^1}$ is injective.

Note also that a section of $\mathcal{O}^{n-a}$ is completely determined by the stalk at one point. By assumption, V_1^1 and V_2^1 do not intersect $\mathcal{O}(1)^a_{P_1^1}$ and $\mathcal{O}(1)^a_{P_2^1}$ respectively and $\psi(V_1^1)$ does not intersect V_2^1. Hence the map induced by f

$$\begin{aligned} H^0(\mathcal{O}^{n-a}) \to & [(\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a})_{P_1^1}/(\mathcal{O}(1)a)_{P_1^1} \oplus V_1^1)] \\ & \bigoplus [(\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a})_{P_2^1}/(\mathcal{O}(1)^a)_{P_2^1} \oplus V_2^1)] \end{aligned}$$

is injective too. Therefore, f' is injective and this proves the lemma. $\square$

Lemma 4.4

If E and E' are related as in (4.1), *then one of them is (semi)stable for a polarisation a_i if and only if the other is for the polarisation which coincides with a_i on the old components and is zero on the new rational component.*

Proof. Assume E is stable. Let G' be a subsheaf of E'. Then G' is obtained from subsheaves H of F and J of $\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a}$ by identifying the fibers at P_1 and P_2 by means of the restrictions of the identifications in (4.1). From the commutativity of the diagram in $(4,1)iv), H_{P_1}/V_1 \cap H_{P_1}$ is identified to $H_{P_2}/V_2 \cap H_{P_2}$. Hence it gives rise to a subsheaf G of E. From the semistability of $E, \chi(G)/(\sum a_i n_i)(\leq) < \chi(E)/n$. We can write

$$\chi(G) = \chi(H) - \dim G_P = \chi(H) - (\dim H_{P_1} - \dim V_1 \cap H_{P_1})$$

and

$$\begin{aligned} \chi(G') &= \chi(H) + \chi(J) - \dim G'_{P_1} - \dim G'_{P_2} \\ &= \chi(H) + \chi(J) - (\dim H_{P_1} - \dim V_1' \cap H_{P_1}) - (\dim J_{P_2^1} - \dim V_2^1 \cap J_{P_2^1}). \end{aligned}$$

Notice that J is a subsheaf of $\mathcal{O}(1)^a \oplus \mathcal{O}^{n-a}$. Hence

$$\chi(J) \leq rk(J \cap \mathcal{O}(1)^a)_{P_1^1} + rkJ_{P_2^1} + \dim[(J)_{P_1^1} \cap \psi^{-1}(V_2^1)] - \dim[(J)_{P_2^1} \cap V_2^1].$$

Also

$$\begin{aligned} & \dim V_1 \cap H_{P_1} = \dim \varphi_1^{-1}(V_2^1 + \mathcal{O}(1)^a_{P_1^1}) \cap H_{P_1} \\ \geq\ & \dim((V_1' \cap H_{P_1}) + \dim \mathcal{O}(1)^a_{P_1^1} \cap \varphi_1(H_{P_1^1}) + \dim \psi^{-1}(V_2^1) \cap \varphi_1(H_{P_1^1})\,. \end{aligned}$$

Then,

$$\begin{aligned}\chi(G') \leq \chi(H) &- \dim H_{P_1} + \dim V_1' \cap H_{P_1} + (rk(J \cap \mathcal{O}(1)^a)_{P_1^1} \\ &- \dim \mathcal{O}(1)^a_{P_1^1} \cap \varphi_1(H_{P_1^1})) + (\dim(J)_{P_2^1} \cap \psi^{-1}(V_2^1) \\ &- \dim \psi^{-1}(V_2^1) \cap \varphi_1(H_{P_1^1})) \leq \chi(H) - \dim H_{P_1} + \dim V_1 \cap H_{P_1} \\ &= \chi(G).\end{aligned}$$

It remains to show (with the notations of (4.4)) that if $\sum a_i n_i = 1$ and $G' \neq E', \chi(G') < \chi(E')$. If $G \neq E$, this follows from $\chi(G') \leq \chi(G)$ and the semistability of E. If $G = E$, then $J \neq \mathcal{O}(1)^a \bigoplus \mathcal{O}^{n-a}$. Hence $\chi(J) < \chi(\mathcal{O}(1)^a \bigoplus \mathcal{O}^{n-a})$ and $\chi(G') < \chi(E')$.

Assume now that the sheaf E' on the curve with a rational component is stable. We want to show that E is stable too.

Let G be a subsheaf of E. Then G is obtained from a subsheaf H of F by means of the identification of $H_{P_1}/V_1 \cap H_{P_1}$ and $H_{P_2}/V_2 \cap H_{P_2}$ induced by the identification of the fibers of F. We now choose a subsheaf J of $\mathcal{O}(1)^a \bigoplus \mathcal{O}^{n-a}$ in the following way.

Choose $\dim H_{P_1} - \dim V_1 \cap H_{P_1}$ copies of $\mathcal{O}$ which glue with both $H_{P_1}/V_1' \cap H_{P_1}$ and $H_{P_2}/V_2' \cap H_{P_2}$. This is possible because of the compatibility of the gluings in (4.1) iv).

Choose

$$b = \dim \psi[\varphi_1(H_{P_1} \cap (\mathcal{O}(1)^a)_{P_1^1})] \cap \phi_2(H_{P_2}) \cap \mathcal{O}(1)^a_{P_2^1}$$

copies of $\mathcal{O}(1)$ that glue with both $\phi_1(H_{P_1}) \cap \mathcal{O}(1)^a_{P_1^1}$ and $\varphi_2(H_{P_2}) \cap \mathcal{O}(1)^a_{P_2^1}$.

Choose $\dim(H_{P_1}) \cap \mathcal{O}(1)^a_{P_1^1}) - b$ copies of $\mathcal{O}(1)(-P_2^1)$ which together with the above glue with $\varphi_1(H_{P_1}) \cap \mathcal{O}(1)^a_{P_1^1}$ and similarly choose $\dim \varphi_2(H_{P_2}) \cap \mathcal{O}(1)^a_{P_2^1} - b$ copies of $\mathcal{O}(1)^a(-P_1^1)$.

Consider the subspace of $[\varphi_1(H_{P_1}) \cap \psi^{-1}(V_2^1)]/\mathcal{O}(1)^a$. Choose a subbundle J_1 of T made of copies of $\mathcal{O}$ that glues with this space at P_1^1. Similarly, choose a subbundle J_2 of T made of copies of $\mathcal{O}$ that glues with $[\varphi_2(H_{P_2}) \cap \psi(V_1^1)]/\mathcal{O}(1)^a$ at P_2^1. Take on $\mathbf{P^1}$ the direct sum J of all the bundles chosen in this way.This can be glued to H to produce a torsion-free subsheaf G of E'. One can check then that G' has the same Euler-Poincaré characteristic as G. By the stability of E', the result follows. □

Lemma 4.5

There exists a d_0 such that every sheaf E of depth one a_i-semistable of constant rank n such that $\mu(E) \geq d_0$ satisfies E is generated by global sections and $h^1(X, E) = 0$.

Proof. (Compare [7] p. 156, Proposition 16). Assume $h^1(E) \neq 0$. Then, there exists a non-zero map from E to K. The Euler-Poincaré characteristic of the image is bounded by an integer m which depends on K only. Denote by H the kernel of this map. Then $\chi(H) > \chi(E) - m$. Let $\alpha = \min\{a_i | a_i \neq 0\}$. If H differs from E outside a component with $a_i \neq 0$ $\sum a_i rk(H_{|Ci}) \leq n - \alpha$. Then, by stability of E, $[\chi(E) - m]/(n - \alpha) \leq \chi(E)/n$. So, $\chi(E) < nm/\alpha$ and this is impossible if $\chi(E)$ is large enough. Therefore H only differs from E on the union of the rational components that have $a_i = 0$. From Lemma (1.5), the restriction of E to such a chain $\tilde{C}$ is a direct sum of line bundles of degree 0 and 1. Therefore, we have a non-zero map $L_1 \oplus \ldots \oplus L_n \to w|_{\tilde{C}}$ which vanishes at the two end points. As the degree of $w|_{\tilde{C}}$ is zero, this is impossible.

The condition for E to be generated by global sections is that $h^1(E(-x)) = 0$ for every x. As before, we find that $h^1(E_{|C-\tilde{C}}) = 0$. Moreover, for a rational component, the restriction of $E(-x)$ is either a direct sum of copies of $\mathcal{O}$ and $\mathcal{O}(-1)$ or a direct sum of copies of $\mathcal{O}$ and $\mathcal{O}(1)$ depending on whether the point belongs or not to the component. Hence h^1 is zero. □

Proof of (4.2). Fix a curve C_0 and a polarisation a as in (4.2). Fix a rank n and an Euler-Poincare characteristic χ. Consider the set of curves C obtained from C_0 by adding a few chains of rational components separating a node in C_0. Consider a polarisation in C as in (4.4). Choose an invertible sheaf $\mathcal{O}(1)$ on C such that $\deg \mathcal{O}(1)_{|C_i} / \deg \mathcal{O}(1) = a_i$. Define the polynomial $P(m) = \chi(E(m))$. Let $Q(C)$ be Grothendieck's scheme of quotients of $\mathcal{O} \otimes k^\chi$ with Hilbert polynomial P. Let F be the universal sheaf on Q. Denote by R the open set of points q such that F_q is torsion free and the map from k^χ to $H^0(F_q)$ is an isomorphism. Denote by R^s and R^{ss} the subset of R consisting of semistable and stable points.

For each component C_i of C, choose an integer $N_i >> 0$ such that $N_i / \sum N_i = a_i$. Choose N_i points on C_i in general position.

Denote by $G(a, \chi)$ the Grassmannian of quotients of dimension a of k^χ. Let

$$Z = \prod_i G(n, \chi)^{N_i}.$$

Consider the map $t : R \to Z$ given by sending a sheaf F_q to its fibers at the chosen points on each component. From Lemma (4.3) above, t identifies all the point in an equivalence class by the relation (4.1). From [7] Theorem 19 i) p. 158, the map from the set of equivalence classes is injective. From Lemma (4.4) above and Theorem 19 ii) and iii) in [7], the set of stable (semistable) points in R coincides with the inverse image of the set of stable (semistable points in the Grassmannian by the action of the linear group. Hence, the set of stable sheaves on the variable curve modulo the equivalence relation in (4.1) is naturally identified to the moduli space of (semi)stable sheaves on the curve C_0. □

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