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Elliptic surfaces with a nef line bundle of genus two

Antonio Lanteri and Cristina Turrini

Dipartimento di Matematica "F. Enriques", Università

Via C. Saldini 50, I-20133 Milano, Italy

E-mail: lanteri@mat.unimi.it
turrini@mat.unimi.it

To the memory of Fernando Serrano

Abstract

Complex projective elliptic surfaces endowed with a numerically effective line bundle of arithmetic genus two are studied and partially classified. A key role is played by elliptic quasi-bundles, where some ideas developed by Serrano in order to study ample line bundles apply to this more general situation.

Introduction

Polarized surfaces with sectional genus 2 have been studied and classified in [2], [5], [17], [7]. Elliptic surfaces play an interesting an delicate role in this classification; in particular, papers by Serrano [17] and Fujita [7] pointed out the relevance of certain relatively minimal elliptic surfaces, named elliptic quasi-bundles, arising in this setting. On the other hand Maeda [16] recently started to study projective surfaces endowed with a nef line bundle of low arithmetic genus g. In particular he obtained a classification for $g \leq 1$, generalizing what was known for surfaces polarized by an ample line bundle. In this paper we consider elliptic projective surfaces endowed with a nef line bundle with $g = 2$. We would like to stress that some ideas of Serrano (partially extended in [13]) developed to study ample line

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bundles on relatively minimal elliptic surfaces also apply in this wider setting. Note that elliptic surfaces already occur in Maeda' s classification for $g = 1$, but in that case one can say very little. Actually any normal semipolarized surface of sectional genus 1 is birationally equivalent to a pair (S, L) where S is a smooth minimal elliptic surface and L is a nef line bundle satisfying $L^2 = LK_S = 0$ ([16], Theorem 3, (1)). For instance, any linear combination of fibres of S satisfies these conditions. On the contrary, when $g = 2$ we can say much more; actually in this case we show that $(L^2, LK_S) = (1, 1)$ or $(0, 2)$. In particular the equality $LK_S = 0$ cannot occur.

Here is an outline of our approach. As a first step we prove that up to contracting all (-1)-curves not intersecting L, either S is minimal or (S, L) admits a simple reduction (S', L') where S' is minimal and L' is still a nef line bundle of genus two satisfying $L'K_{S'} = 1$. Then we analyze minimal surfaces by means of the canonical bundle formula. Due to the fact that $LK_S > 0$ we have that L has positive intersection with the fibres of the elliptic fibration $\psi : S \to C$. This allows us to apply the same procedure working in the case of ample line bundles and determine all the numerical invariants of S: the geometric genus p_g, the irregularity q, the genus $g(C)$ of the base curve C and the multiplicities of the multiple fibres of ψ. The results are summarized in several tables along the paper. In particular we provide examples of nef non ample line bundles of genus 2, but, unfortunately, we have no concrete example satisfying $LK_S = 1$. In fact in this subcase we get just few possibilities in addition to those occurring when L is an ample line bundle. Moreover it turns out that in most cases a minimal elliptic surface S allowing a nef line bundle L with $g = 2$ is an elliptic quasi-bundle over a smooth curve C of genus ≤ 1. When $C = \mathbb{P}^1$ there exists a structure theorem for $\mathrm{Num}(S)$ and then the values of L^2 and LK_S allow us to also determine explicitly the numerical class of L. In particular it turns out that for elliptic quasi-bundles over $\mathbb{P}^1$ if $LK_S = 1$ then L nef implies L ample.

The paper is organized as follows. In Section 1 we collect some background material. Section 2 is devoted to the first step of the classification. In Section 3 we discuss the case of minimal surfaces with no multiple fibres. Minimal elliptic surfaces with multiple fibres are studied in Sections 4 and 5, the latter being devoted to the special case of elliptic quasi-bundles. The paper is concluded by an Appendix where a result in [12] is improved, showing that the lowest possible genus of an ample and spanned line bundle on an elliptic surface is 4.

1. Background material

We work over the complex number field $\mathbb{C}$. We use standard notation and terminology in algebraic geometry: in particular we denote additively the tensor products of line bundles and we use the symbol $\equiv$ to denote the numerical equivalence. Following a current abuse we do not distinguish between line bundles and invertible sheaves.

As we said in the Introduction we are interested in the classification of elliptic projective surfaces endowed with a nef line bundle of arithmetic genus 2. To make our set-up clear let us fix some more terminology. Let X be a projective surface and $\mathcal{L}$ a line bundle on X. If $\mathcal{L}$ is nef, i. e. $\mathcal{L}C \geq 0$ for all integral curves C on X, we call the pair $(X, \mathcal{L})$ a *semipolarized surface*. Two semipolarized surfaces $(X_1, \mathcal{L}_1)$, $(X_2, \mathcal{L}_2)$ are said to be *birationally equivalent* if there exist a projective surface Y and birational morphisms $f_i : Y \to X_i$ $(i = 1, 2)$ such that $f_1^*\mathcal{L}_1 = f_2^*\mathcal{L}_2$. If X is normal, the *sectional genus* $g(X, \mathcal{L})$ of the semipolarized normal surface $(X, \mathcal{L})$ is defined by the formula $2g(X, \mathcal{L}) - 2 = (\omega_X + \mathcal{L})\mathcal{L}$, where ω_X denotes the canonical sheaf of X. Let $(X, \mathcal{L})$ be a semipolarized normal surface with $g(X, \mathcal{L}) > 0$. Then, by [16], Theorem 1, there exist a smooth surface S and a nef line bundle L on S such that the semipolarized surface (S, L) is birationally equivalent to $(X, \mathcal{L})$ and $K_S + L$ is nef, where K_S is the canonical bundle of S. Moreover, by [6], Lemma 1.8, (see also [16], Lemma 3.1) we have $g(S, L) = g(X, \mathcal{L})$. In particular this shows that to study semipolarized normal surfaces of sectional genus 2 up to birational equivalence it is enough to consider smooth semipolarized surfaces (S, L) where

$$2 = 2g(S, L) - 2 = (K_S + L)L = K_S L + L^2. \tag{1.0.1}$$

So from now on the word *surface* will mean smooth projective surface. In particular, in this paper we deal with properly elliptic surfaces, i. e. $\kappa(S) = 1$.

Hereafter we recall some general properties of properly elliptic surfaces over a smooth curve C and especially of elliptic quasi-bundles, which we need throughout the paper.

(1.1) Let $\psi : S \to C$ be a relatively minimal elliptic surface over a smooth curve C of genus $g(C)$. We assume that $\kappa(S) = 1$: hence ψ is the unique elliptic fibration of S (e. g. see [10], Proposition 7); moreover $\chi(\mathcal{O}_S) \geq 0$ by the Castelnuovo-De Franchis theorem. Let $F_0 = \sum_{j=1}^{s} n_j B_j$ be a singular fibre of ψ, where the B_j's are the irreducible components and the n_j's denote their multiplicities. We say that F_0 is a *multiple fibre* of ψ of *multiplicity* m if $m :=$ g.c.d.$\{n_j\} \geq 2$. In this case $F_0 = mf$, where f is an effective divisor on S such that $f^2 = 0$. We recall that $\psi : S \to C$ is said to be an elliptic quasi-bundle if all smooth fibres of ψ are isomorphic to each other and every singular fibre is a multiple of a smooth irreducible curve. Let $q(S)$ denote the irregularity of S; the following facts hold (see [18], (1.5), (1.6), see also [19], Section 4).

(1.1.1) If $\chi(\mathcal{O}_S) > 0$ then $q(S) = g(C)$.

(1.1.2) If $\chi(\mathcal{O}_S) = 0$ then there are two possibilities: either ψ has trivial monodromy, in which case $q(S) = g(C) + 1$, or $q(S) = g(C)$.

(1.1.3) $\chi(\mathcal{O}_S) = 0$ if and only if $\psi : S \to C$ is an elliptic quasi-bundle.

(1.1.4) Let $m_1, \dots, m_t$ be the multiplicities of the multiple fibres of ψ. If $q(S) = g(C) + 1$ then either $t = 0$ or $t \geq 2$ and for every i the integer m_i divides the l.c.m.$\{m_1, \dots, \hat{m}_i, \dots, m_t\}$, where $\hat{}$ means suppression.

We will refer to (1.1.4) as the Katsura-Ueno divisibility property ([11], Corollary 4.1), though this statement, which is slightly more general, is due to Serrano ([17], Proposition 1.3).

We also recall the canonical bundle formula ([1], p. 161). If the multiple fibres of $\psi : S \to C$ are $F_i = m_i f_i$, $i = 1, \dots, t$, then

$$K_S = \psi^*(K_C + N) + \sum_{i=1}^{t}(m_i - 1)f_i, \tag{1.1.5}$$

where N is a line bundle on C of degree $\deg N = \chi(\mathcal{O}_S)$.

(1.2) The structure of elliptic quasi-bundles satisfying $q(S) = g(C) + 1$ is very well understood ([17], Theorem 1.2) (see also the review in [7], Section 1). There exist a smooth curve B of genus $g(B) \geq 2$, a smooth curve E of genus 1 and a finite abelian group $G := \mathbb{Z}_m \times \mathbb{Z}_n$ acting faithfully on B and E, and by translations on E, such that $S \cong (B \times E)/G$ where G acts diagonally on the product. Moreover $C \cong B/G$ and under these identifications ψ is the morphism $S \to B/G$ induced by the first projection of $B \times E$. We denote by F the general fibre of ψ and by D the general fibre of $S \to E/G$. Note that $F \cong E$ and $D \cong B$. We also have $DF = \gamma$, the order of G.

Moreover, in the special case when $C \cong \mathbb{P}^1$ we have the following

Theorem 1.3 ([13])

Let $\psi : S \to \mathbb{P}^1$ be an elliptic quasi-bundle as above; let $\mu = \mathrm{l.c.m.}\{m_i\}$ be the least common multiple of the multiplicities of the fibres and let γ be the order of G. Then $\mathrm{Num}(S)$ *is generated by the classes of*

$$\frac{1}{\mu}F \qquad \textit{and} \qquad \frac{\mu}{\gamma}D + \frac{\delta}{2\mu}F,$$

where $\delta = 0$ or 1 according to whether $(2g(B) - 2)\frac{\mu}{\gamma}$ is even or odd. Furthermore a line bundle on S of type (a, b), (i.e. whose class is the linear combination of these generators with integral coefficients a and b) is ample if and only if

$$2a + \delta b > 0 \qquad \text{and} \qquad b > 0.$$

We conclude this Section with a very useful remark. Recall that if L is a nef line bundle on a surface, then $L^2 \geq 0$; as usual we say that L is big if the above inequality is strict.

Lemma 1.4

Let S be a properly elliptic surface and let $L \in \mathrm{Pic}(S)$ be a nef and big line bundle. Then $LK_S > 0$.

Proof. Since S is properly elliptic, a suitably high positive multiple of K_S is effective, so that $LK_S \geq 0$, L being nef. By contradiction, assume that $LK_S = 0$. Then L is effective. Actually, since L is big, $(K_S - L)L = -L^2 < 0$, hence $K_S - L$ cannot be effective and then by Serre duality we get $h^2(L) = 0$. Thus, since $\chi(\mathcal{O}_S) \geq 0$, the Riemann-Roch theorem gives

$$h^0(L) = h^1(L) + \chi(\mathcal{O}_S) + \frac{1}{2}\,(L^2 - LK_S) \geq \frac{1}{2}\,L^2.$$

Hence $h^0(L) > 0$, since L is big. Now let $\eta : S \to S_0$ be a birational morphism from S to the relatively minimal model S_0. Then $K_S = \eta^* K_{S_0} + \mathcal{E}$, where $\mathcal{E}$ is an effective divisor contracted by η to a 0-dimensional subset. We have

$$0 = LK_S = L\eta^* K_{S_0} + L\mathcal{E},$$

both summands being nonnegative since both $\mathcal{E}$ and mK_{S_0} for $m >> 0$ are effective. Therefore $L\eta^* K_{S_0} = 0$. Since K_{S_0} is numerically equivalent to a positive rational multiple of a fibre of S_0 we thus conclude that also $LF = 0$, for F a fibre of S. So any effective divisor $D \in |L|$ is contained in a union of fibres of S. We can thus write $D = D_1 + \ldots + D_r$, where the D_i's $(i = 1, \ldots, r)$ are effective divisors contained in distinct fibres of S, whence $D_i D_j = 0$ for $i \neq j$. Moreover $D_i^2 \leq 0$ for every i, by Zariski's lemma ([1], p. 90). But this implies that $D^2 \leq 0$, contradicting the bigness of L. $\square$

2. Reduction to the relatively minimal model

Let (S, L) be a properly elliptic semipolarized surface with $g(S, L) = 2$; then both summands in the right hand of (1.0.1) are non negative since L is nef. A priori this gives the following three possibilities for (LK_S, L^2): (2,0), (1,1), (0,2); however Lemma 1.4 rules out the last one. So we have

Fact. 2.1

Let (S, L) be a properly elliptic semipolarized surface with $g(S, L) = 2$. Then $(LK_S, L^2) = (2, 0)$ or $(1, 1)$.

Now suppose that S is not relatively minimal. Then there exists a (-1)-curve E contained in a fibre of S; let $\sigma : S \to S'$ be the contraction of E and let $p = \sigma(E)$.

Remark 2.2. If L is a nef line bundle on S, then there exists a nef line bundle $L' \in \mathrm{Pic}(S')$ such that

$$L = \sigma^* L' - rE, \qquad \text{where} \qquad r = LE \geq 0. \tag{2.2.1}$$

Proof. Since $\mathrm{Pic}(S) \cong \sigma^*\mathrm{Pic}(S') \oplus \mathbb{Z}$, the second summand being generated by E, there exists $L' \in \mathrm{Pic}(S')$ satisfying (2.2.1); of course $LE \geq 0$ since L is nef. So we just need to prove that L' is nef. To see this let $C' \subset S'$ be any irreducible curve. Then $\sigma^* C' = C + mE$, where $C = \sigma^{-1}(C')$ is the proper transform of C' and $m = \mathrm{mult}_p(C') \geq 0$. We thus get

$$LC = (\sigma^* L' - rE)(\sigma^* C' - mE) = L'C' - rm \leq L'C'.$$

So L' is nef, due to the nefness of L. □

Let (S, L) and (S', L') be as above. By adapting the terminology of adjunction theory for ample line bundles to our setting we say that (S', L') is the *simple reduction of* (S, L) if $r = 1$. Note that in this case $g(S', L') = g(S, L)$.

Our aim in this Section is to use birational equivalence to reduce the study of our pairs (S, L) to the case when S is a relatively minimal surface. To do this we need the following

DEFINITION 2.3. Let $E \subset S$ be a (-1)-curve. We say that E is *relevant* (*unrelevant*) if $LE > 0$ ($LE = 0$). According to Fukuma ([8], Definition 1.9) we say that (S, L) is *L-minimal* if all (-1)-curves in S are relevant.

By using (2.2) it is easy to check the following

Fact. 2.4

For any semipolarized surface (S, L) there exist a birational morphism $\eta : S \to S_0$ onto a surface S_0 and a nef line bundle $L_0 \in Pic(S_0)$ such that $L = \eta^* L_0$ and (S_0, L_0) is L_0-minimal.

Note that $(K_S L, L^2) = (K_{S_0} L_0, L_0^2)$.

So, since we are working up to birational equivalence, we can confine ourselves to consider pairs (S, L) which are L-minimal.

Theorem 2.5

Let (S, L) be a properly elliptic semipolarized surface of sectional genus 2. If (S, L) is L-minimal then either

i) S is relatively minimal and $(LK_S, L^2) = (2, 0)$ or $(1, 1)$, or

ii) (S, L) has a simple reduction (S_0, L_0), where S_0 is relatively minimal and $(L_0 K_{S_0}, L_0^2) = (1, 1)$.

Proof. If S is relatively minimal then the assertion follows from Fact 2.1. If S is not relatively minimal, then there exists a (-1)-curve E contained in a fibre of S. Let $\sigma : S \to S'$ be the contraction of E, set $p = \sigma(E)$ and consider the line bundle L' on S' such that $L = \sigma^* L' - rE$ as in Remark 2.2. Since (S, L) is L-minimal we have $r = LE > 0$. Hence from

$$0 \leq L^2 = (\sigma^* L' - rE)^2 = L'^2 - r^2$$

we see that L' is nef and big. Moreover

$$LK_S = (\sigma^* L' - rE)(\sigma^* K_{S'} + E) = L' K_{S'} + r. \tag{2.5.1}$$

Note that the first summand in the right hand of (2.5.1) is positive in view of Lemma 1.4 applied to (S', L'). Since $r > 0$, recalling Fact 2.1 we get from (2.5.1) that

$$LK_S = 2 \qquad \text{and} \qquad L' K_{S'} = r = 1. \tag{2.5.2}$$

This shows that (S', L') is the simple reduction of (S, L). Now assume that S' is not relatively minimal and let $E' \subset S'$ be a (-1)-curve. Note that since $L' K_{S'} = 1$, E' cannot be a relevant (-1)-curve; otherwise by repeating the argument leading to (2.5.2) we would get $L' K_{S'} = 2$, a contradiction. Thus

$$L' E' = 0. \tag{2.5.3}$$

(2.5.4) *Claim.* E' does not contain the point p.

To prove the claim suppose, by contradiction, that $p \in E'$. Then $\sigma^* E' = \tilde{E}' + E$, where $\tilde{E}' = \sigma^{-1}(E')$. Let $\tau : S' \to S''$ be the contraction of E'. By Remark 2.2 there exists a nef line bundle $L'' \in \mathrm{Pic}(S'')$ such that $L' = \tau^* L''$, in view of (2.5.3). So $L = \sigma^* L' - E = \sigma^*(\tau^* L'') - E = (\tau \circ \sigma)^* L'' - E$ and then $L''^2 = L'^2 = L^2 + 1 > 0$. So L'' is nef and big and then $L'' K_{S''} > 0$, by Lemma 1.4. Recalling (2.5.2) we thus get

$$\begin{aligned} 2 = LK_S &= L(\sigma^* K_{S'} + E) \\ &= L\big(\sigma^*(\tau^* K_{S''} + E') + E\big) \\ &= L\big((\tau \circ \sigma)^* K_{S''}\big) + L\tilde{E}' + 2LE \\ &= \big((\tau \circ \sigma)^* L'' - E\big)\,\big((\tau \circ \sigma)^* K_{S''}\big) + L\tilde{E}' + 2LE \\ &= L'' K_{S''} + L\tilde{E}' + 2LE \geq 3, \end{aligned}$$

a contradiction. This proves the claim. Coming back to S, (2.5.4) shows that both E and $\tilde{E}'$ are exceptional curves on S and $E\tilde{E}' = 0$. This means that $\tilde{E}' = \sigma^* E'$. But then we get

$$L\tilde{E}' = (\sigma^* L' - E)\sigma^* E' = L'E' = 0$$

by (2.5.3), which contradicts the L-minimality of (S, L). It thus follows that S' is relatively minimal. $\square$

Remark 2.6. Let (S, L) be as in Theorem 2.5, case ii), and let $p = \sigma(E)$ be the point of S_0 at which $\sigma : S \to S_0$ contracts the (-1)-curve E satisfying the condition $LE = 1$. If $C \subset S_0$ is a curve having a point of multiplicity $m \geq 1$ at p, then

$$L\sigma^{-1}(C) = (\sigma^* L_0 - E)(\sigma^*(C) - mE) = L_0 C - m.$$

So, starting from the semipolarized surface (S_0, L_0) and blowing-up the point p we see that L is nef if and only if $\varepsilon(L_0, p) \geq 1$, where

$$\varepsilon(L_0, p) = \inf_{C \ni p} \ \frac{L_0 C}{\mathrm{mult}_p(C)}$$

is the Seshadri constant of L_0 at p ([4], Section 6). For instance, assume that $L_0 F = 1$, F being the general fibre of the elliptic fibration ψ of S_0. If p is a singular point of a reduced fibre of ψ then $\varepsilon(L_0, p) \leq \frac{1}{2}$, hence the corresponding line bundle L is not nef.

What we proved in this Section is summed-up by the following

Corollary 2.7

Let (S, L) be a properly elliptic semipolarized surface of sectional genus 2. Then there exist a birational morphism $f : S \to \hat{S}$ and a nef line bundle $\hat{L} \in \mathrm{Pic}(\hat{S})$ such that $L = f^*\hat{L}$ and $(\hat{S}, \hat{L})$ is a semipolarized surface of sectional genus 2 satisfying one of the following conditions:

(1) $\hat{S}$ is relatively minimal and $(\hat{L}K_{\hat{S}}, \hat{L}^2) = (2, 0)$ or $(1, 1)$, or

(2) $(\hat{S}, \hat{L})$ has a simple reduction (S_0, L_0), where S_0 is relatively minimal and $(L_0K_{S_0}, L_0^2) = (1, 1)$.

Conversely, let $\hat{L}$ be a nef line bundle of genus 2 on an elliptic surface $\hat{S}$;

(a) for any birational morphism $f : S \to \hat{S}$, $f^*\hat{L}$ is a nef line bundle of genus 2 on S while

(b) if $\hat{S}$ is relatively minimal, $\hat{L}^2 = \hat{L}K_{\hat{S}} = 1$, $\sigma : S \to \hat{S}$ is the blowing-up at a point $p \in \hat{S}$ and $\varepsilon(\hat{L}, p) \geq 1$, then $\sigma^*\hat{L} - E$ is a nef line bundle of genus 2 on S, where $E = \sigma^{-1}(p)$.

This reduces our problem to investigating semipolarized surfaces (S, L) of sectional genus 2 where S is a relatively minimal elliptic surface and, in view of Fact 2.1, L satisfies

$$(LK_S, L^2) = (1, 1) \quad \text{or} \quad (2, 0). \tag{2.8}$$

This is what we are doing in the next sections.

3. Elliptic surfaces without multiple fibres

Let S be relatively minimal. Since K_S is a positive rational multiple of the general fibre F, from $LK_S > 0$ we see that $LF > 0$. If $\psi : S \to C$ has no multiple fibres then it may happen that $LF = 1$, even if some fibre is reducible; this is really a different feature with respect to the case of ample line bundles. Actually, when there are no multiple fibres (1.1.5) becomes

$$K_S = \psi^*(K_C + N) \tag{3.0.1}$$

where N is a line bundle on C of degree $\deg N = \chi(\mathcal{O}_S) \geq 0$. For shortness set $g = g(C)$, $\chi = \chi(\mathcal{O}_S)$, $q = q(S)$ and $p_g = p_g(S)$; thus $K_S \equiv (2g - 2 + \chi)F$. Now look at (2.8).

If $LK_S = 1$ then $LF = 1$ and we get $(g, \chi) = (0, 3)$, or $(1, 1)$. In both cases $q = g$, by (1.1.1), whence $p_g = \chi + g - 1 = 2$ or 1 respectively.

If $LK_S = 2$ there are two possibilities according to whether $LF = 1$ or 2. If $LF = 2$, then

$$(g, \chi) = (0, 3), (1, 1), \quad q = g \quad \text{and} \quad p_g = 2 \quad \text{or} \quad 1$$

accordingly, as before. If $LF = 1$, then

$$(g, \chi) = (0, 4), (1, 2), (2, 0).$$

In the first two cases we have $q = g$ by (1.1.1) again, and then $p_g = \chi + g - 1 = 3$ or 2. In the third case S is quasi-bundle by (1.1.3), but since there are no multiple fibres it is in fact a holomorphic fiber bundle. Hence $q = g + 1 = 3$, by (1.1.2) and then $p_g = 2$.

The above discussion is summarized in Tables 3.1 and 3.2.

Table 3.1

no multiple fibres: case $LK_S = 1$				
g	χ	q	p_g	FL
0	3	0	2	1
1	1	1	1	1

Table 3.2

no multiple fibres: case $LK_S = 2$				
g	χ	q	p_g	FL
0	3	0	2	2
1	1	1	1	2
0	4	0	3	1
1	2	1	2	1
2	0	3	2	1

Both cases occurring when $LF = 2$ are effective, as shown by the following

EXAMPLE 3.1: Let C be a smooth curve of genus $\gamma \leq 1$. Consider the product $\Sigma := C \times \mathbb{P}^1$ and let σ and f denote the fibres of the projections of Σ on the factors $\mathbb{P}^1$ and C respectively. Since the line bundle $[4\sigma + (6 - 4\gamma)f] \in 2\mathrm{Pic}(\Sigma)$ is ample and spanned, its linear system contains a smooth divisor Δ. Let $\pi : X_\gamma \to \Sigma$ be the double cover branched along Δ. By the ramification formula we see that $K_{X_\gamma} = \pi^* f$.

So X_γ is a smooth minimal surface with $\kappa(X_\gamma) = 1$, the elliptic fibration $\psi : X_\gamma \to C$ being obtained by composing π with the first projection of Σ. Moreover, since

$$h^i(K_{X_\gamma}) = h^i(\pi_* K_{X_\gamma}) = h^i(K_\Sigma) + h^i(f),$$

for $i = 0, 1$ we get $(p_g(X_\gamma), q(X_\gamma)) = (2, 0)$ or $(1, 1)$ respectively, according to whether $\gamma = 0$ or 1. Now the line bundle $L_b := \pi^*[\sigma + bf]$ is nef for any integer $b \geq 0$. Note that it is in fact spanned if $\gamma = 0$ and in addition it is ample for $b \geq 1$. We have

$$2g(L_b) - 2 = (K_{X_\gamma} + L_b)L_b = \pi^*(\sigma + (b+1)f)\pi^*(\sigma + bf) = 2(2b + 1),$$

hence $g(L_b) = 2b + 2$. Since $L_b K_{X_\gamma} = 2$, the pair (X_γ, L_0) gives examples as in the first and second cases of Table 3.2 (according to the values of γ).

As to the cases with $LF = 1$, an obvious example of the last one in Table 3.2 is the following: take $S := C \times F$, the product of a smooth curve C of genus 2 and an elliptic curve F, with the line bundle L corresponding to a fibre C of the second projection. This is essentially the only example since we have

Proposition 3.2

If (S, L) is as in the last case of Table 3.2 *then $S \cong C \times F$, ψ being the first projection and $L \equiv C$, a fibre of the second projection.*

Proof. Set $\mathcal{L} = L + F$; then $\mathcal{L}^2 = 2LF = 2$, hence $\mathcal{L}$ is nef and big. Moreover, since $\mathcal{L}K_S = LK_S = 2$, we get $g(\mathcal{L}) = 3 = q$. Note also that S is minimal, being relatively minimal with $g = 2$. Then, by a result of Fukuma ([8], Theorem 3.1), we conclude that $S \cong C \times F$ with $\mathcal{L} \equiv C + F$. This gives the assertion. □

In the remaining cases listed for $LF = 1$ we have no examples. Note that $\chi > 0$ in these cases; hence ψ has some singular fibres, possibly reducible, according to Kodaira's table ([1], p. 150). However, if we assume that all fibres of ψ are irreducible, then the numerical possibilities listed above reduce and we can provide some description of L.

Proposition 3.3

Let (S, L) be as above with S relatively minimal with no multiple fibres and assume that all fibres are irreducible. Suppose furthermore that $LF = 1$. Then there are just two possibilities:

(1) if $LK_S = 1$ then $g = 1$ and $L \equiv Z + F$, where Z is a section with $Z^2 = -1$;

(2) if $LK_S = 2$ then (S, L) is as in Proposition 3.2.

Proof. We adapt an argument by Fujita ([7], (1.2), p. 156). Due to the assumptions, every fibre is irreducible, reduced and of arithmetic genus 1; since $LF = 1$ we thus get $h^0(L_F) = 1$ for every fibre F and so $\mathcal{F} := \psi_* L$ is an invertible sheaf. Moreover the scheme theoretic support Z of the Cokernel of the natural homomorphism $\psi^* \mathcal{F} \to L$ is a section of ψ. Then $L = Z + \psi^* \mathcal{F}$. Note that $\psi^* \mathcal{F} \equiv eF$, for some integer e, which we want to determine. So $L \equiv Z + eF$. Since $\psi_{|Z} : Z \to C$ is an isomorphism, recalling (3.0.1) we get

$$K_S Z = \psi^*(K_C + N)Z \geq \psi^* K_C Z = \deg \psi_{|Z}^* K_C = 2g(Z) - 2.$$

Hence the genus formula gives $Z^2 = 2g(Z) - 2 - K_S Z \leq 0$. On the other hand, since L is nef, we have

$$0 \leq LZ = (Z + eF)Z = Z^2 + eFZ = Z^2 + e,$$

hence

$$e \geq -Z^2 \geq 0.$$

Now, if $LK_S = 1$ then also $ZK_S = (L - eF)K_S = LK_S = 1$ so that Z^2 is odd; moreover from $1 = L^2 = L(Z + eF) = LZ + eLF$ we conclude that $e = 1$, giving $-1 \leq Z^2 \leq 0$, hence $Z^2 = -1$. Finally the genus formula gives $g = g(Z) = 1 + \frac{1}{2}(Z^2 + ZK_S) = 1$. Similarly, if $LK_S = 2$, then also $ZK_S = 2$; moreover from $0 = L^2 = L(Z + eF) = LZ + eLF$ we see that $e = 0$, which implies $Z^2 = 0$. Finally the genus formula again gives $g = g(Z) = 2$. Hence (S, L) is as in the last case of Table 3.2 and Proposition 3.2 applies. $\square$

We have no examples as in (1) of Proposition 3.3. Note that in this case $LZ = (Z + F)Z = 0$; so if this case should really occur it would provide an example of a nef non-ample line bundle L of genus two with $LK_S = 1$.

4. Elliptic surfaces with multiple fibres

Now assume that $\psi : S \to C$ has $t > 0$ multiple fibres $F_i = m_i f_i$, $i = 1, \ldots, t$. As we already said, the fact that $LK_S > 0$ implies $LF > 0$ for any fibre F of S; this in turn implies that $Lf_i > 0$ for every i. Hence for the general fibre F we have $LF \geq \mu$, where $\mu :=$ l.c.m.$\{m_i\}$ is the least common multiple of the multiplicities. This allows us to apply the same argument as in the case of ample line bundles ([13], see also [3]) in order to investigate the genus $g = g(C)$, the numerical characters $q = q(S)$,

$p_g = p_g(S)$ of S and the possible multiplicities of the multiple fibres. Hereafter we give some detail.

Order the multiplicities so that $2 \le m_1 \le m_2 \le \ldots \le m_t$ and put for shortness $l = LK_S$; so $l = 1$ or 2 in view of (2.8). Recalling (1.1.5) we have

$$l = \Big(2g - 2 + \chi + \sum_{i=1}^{t} \frac{m_i - 1}{m_i}\Big) FL. \tag{4.0.1}$$

Since $FL \ge m_t$ and $\sum_{i=1}^{t-1} \frac{1}{m_i} \le \frac{t-1}{2}$, (4.0.1) gives

$$l \ge \Big(2g - 2 + \chi + t - \sum_{i=1}^{t-1} \frac{1}{m_i} - \frac{1}{m_t}\Big) m_t \ge \Big(2g - 2 + \chi + \frac{t+1}{2}\Big) m_t - 1\,;$$

hence we get

$$2(l+1) \ge (4g + 2\chi + t - 3) m_t \tag{4.0.2}$$

which, for given l, g, χ and t, supplies an upper bound for the higher multiplicity m_t, except when $(4g + 2\chi + t - 3) \le 0$ (implying $g = 0$ and either $\chi = 0$ and $t \le 3$ or $\chi = t = 1$); we will deal with this case separately.

Assuming $(4g + 2\chi + t - 3) > 0$, we now show how to bound g, χ and t. From (4.0.1), since $FL \ge 2$, we get

$$2g - 2 + \chi + \sum_{i=1}^{t} \frac{m_i - 1}{m_i} \le \frac{l}{2} \tag{4.0.3}$$

which implies $g \le 1$ (indeed, if $l = 1$ and $\chi > 0$, it also implies $g = 0$). Hence we have only to consider the following possibilities: $(l, g) = (1, 0), (2, 0), (1, 1)$, or $(2, 1)$. Inequality (4.0.3) yields

$$\chi \le \sum_{i=1}^{t} \frac{1}{m_i} - 2g + 2 - t + \frac{l}{2} \le \frac{t}{2} - 2g + 2 - t + \frac{l}{2} = \frac{l + 4 - 4g - t}{2},$$

hence

$$\text{if} \quad (l,g) = (1,0), \quad \text{then} \quad t \le 5 \quad \text{and} \quad \chi \le \frac{5-t}{2};$$

$$\text{if} \quad (l,g) = (2,0), \quad \text{then} \quad t \le 6 \quad \text{and} \quad \chi \le \frac{6-t}{2};$$

$$\text{if} \quad (l,g) = (1,1), \quad \text{then} \quad t = 1 \quad \text{and} \quad \chi = 0;$$

$$\text{if} \quad (l,g) = (2,1), \quad \text{then} \quad t \le 2 \quad \text{and} \quad \chi = 0.$$

Having an upper bound on t, (4.0.1) gives upper bounds for all the m_i's stronger than (4.0.2), as one can see by proceeding in decreasing order on the m_i's (e. g. see [3], p. 227). This procedure leads to a finite number of possibilities for the sequence of integers $(l, g, \chi, t, m_1, \dots, m_t)$. For them one has to check (4.0.1), and, in case $q = g + 1$, the Katsura-Ueno divisibility property (1.1.4). This can be done quite easily so that we finally get a maximal list of possible numerical characters.

Finally assume $(4g + 2\chi + t - 3) \leq 0$. As we have already noticed, this implies $g = 0$ and either $\chi = 0$ and $t \leq 3$ or $\chi = t = 1$. Case $g = 0, \chi = t = 1$ is easily excluded by (4.0.1). Cases $g = 0, \chi = 0, t = 1, 2, 3$ can be handled with an ad hoc argument. Just to give an example, assume $l = 1, g = 0, \chi = 0$ and $t = 3$. From (4.0.1) one gets $1 - \frac{1}{m_1} - \frac{1}{m_2} - \frac{1}{m_3} \leq \frac{1}{2}$, hence $\frac{1}{2} \leq \frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \leq \frac{3}{m_1}$, which implies $m_1 \leq 6$. For each value of $m_1 = 2, 3, 4, 5, 6$ one applies similar arguments to bound m_2 and m_3 as well. So, even in this case, we get a finite number of possible sequences, for which conditions (4.0.1) and (1.1.4) have to be checked.

All this leads to a maximal list of possible characters for elliptic surfaces with multiple fibres. Hereafter we give the result in the non quasi-bundle case, i.e. when $\chi > 0$, postponing the case of quasi-bundles to Section 5.

Table 4.1

non quasi-bundles: case $LK_S = 1$					
g	χ	q	p_g	FL	$(m_1, \dots, m_t)$
0	2	0	1	2	(2)
0	1	0	0	6	$(2,3)$
0	1	0	1	4	$(2,4)$
0	1	0	0	3	$(3,3)$
0	1	0	0	2	$(2,2,2)$

Table 4.2

non quasi-bundles: case $LK_S = 2$					
g	χ	q	p_g	FL	$(m_1, \dots, m_t)$
0	2	0	1	4	(2)
0	2	0	1	3	(3)
0	2	0	1	2	$(2,2)$
0	1	0	1	12	$(2,3)$
0	1	0	1	8	$(2,4)$
0	1	0	0	6	$(2,6)$

Remark 4.1. Note that the first line in Table 4.1 doesn't appear in the same situation when L is ample ([17], Theorem 2.1). The invariants

$$g = 0, \chi = 2, t = 1, m_1 = 2 \quad \text{and} \quad LF = 2$$

come out from the discussion above. Since $\chi = 2$ we thus get $q = 0$ by (1.1.1), hence $p_g = 1$ and in fact $K_S = f$, where $2f$ is the unique fibre of multiplicity 2. In this case, since L is nef and big we have $h^i(K_S + L) = 0$ for $i = 1, 2$. So, looking at the exact cohomology sequence

$$0 \to H^0(L) = H^0(K_S + L - f) \to H^0(K_S + L) \xrightarrow{r} H^0\big((K_S + L)_f\big) \to H^1(L) \to 0,$$

we see that $h^1(L) \leq 1$ since $(K_S + L)f = Lf = 1$ and $g(f) = 1$; moreover equality holds if and only if r is trivial, which cannot happen because $h^0(K_S + L) = p_g + g(L) - q = 3$ and so $|K_S + L|$ contains divisors intersecting f. We thus conclude that $h^1(L) = 0$, hence $h^0(L) = 2$. So $|L|$ is a pencil having a single base point, say x, since $L^2 = 1$. Note that x is forced to lie on f, since $Lf = 1$ and $g(f) = 1$. Now let y be a point on f distinct from x. Then the divisor $D \in |L - y| = |L - x - y|$ has the form $D = f + R$ where R is an effective divisor in $|L - f|$. Of course this situation cannot occur if L is ample ([2], Lemma 2.1), because since $L^2 = 1$ every element in $|L|$ has to be irreducible and reduced. However it is not contradictory in our case, L simply being a nef line bundle: it only implies that $h^0(L - f) > 0$, which is compatible with the nefness of L since $(L - f)L = 1 - 1 = 0$. Note also that $R^2 = (L - f)^2 = -1$. Since $RK_S = (L - f)f = 1$, this gives $g(R) = 1$.

5. Quasi-bundles with multiple fibres

Here we assume that $t > 0$ and $\psi : S \to C$ is an elliptic quasi-bundle. The discussion made in Section 4, in case $g > 0$ gives the following table.

Table 5.1

quasi-bundles: case $g > 0$						
LK_S	g	χ	q	p_g	FL	$(m_1, \ldots, m_t)$
1	1	0	1	0	2	(2)
2	1	0	1	0	4	(2)
2	1	0	1	0	3	(3)
2	1	0	1	0	2	$(2, 2)$

In the first three cases of Table 5.1 the equality $q = 1$ follows from (1.1.4) since there is a single multiple fibre. On the other hand the fact that $q = 1$ in the last case follows from [7], (2.3). Examples as in the first cases have been constructed by Fujita ([7], (2.6)-(2.8)) in the setting of ample line bundles.

When $C \cong \mathbb{P}^1$ we get the following cases which for convenience are grouped into two tables according to the values of LK_S.

Table 5.2

quasi-bundles: case $g = 0$, $LK_S = 1$				
χ	q	p_g	FL	$(m_1, \dots, m_t)$
0	1	0	6	$(2, 6, 6)$
0	1	0	4	$(4, 4, 4)$
0	1	0	2	$(2, 2, 2, 2, 2)$

Table 5.3

quasi-bundles: case $g = 0$, $LK_S = 2$					
type	χ	q	p_g	FL	$(m_1, \dots, m_t)$
a)	0	1	0	10	$(2, 5, 10)$
b)	0	1	0	12	$(2, 6, 6)$
c)	0	1	0	8	$(2, 8, 8)$
d)	0	1	0	6	$(3, 6, 6)$
e)	0	1	0	8	$(4, 4, 4)$
f)	0	1	0	5	$(5, 5, 5)$
g)	0	1	0	6	$(2, 2, 3, 3)$
h)	0	1	0	4	$(2, 2, 4, 4)$
i)	0	1	0	3	$(3, 3, 3, 3)$
j)	0	1	0	2	$(2, 2, 2, 2, 2)$
k)	0	1	0	2	$(2, 2, 2, 2, 2, 2)$

When $C \cong \mathbb{P}^1$, we can improve the above results also describing the numerical class of the nef line bundle L in terms of the generators of $\mathrm{Num}(S)$.

To do this recall that $S \cong (B \times E)/G$ with B, E and G as in (1.2) and that $\psi : S \to B/G = \mathbb{P}^1$ is the morphism induced by the first projection of $B \times E$. As in (1.2) denote by F the general fibre of ψ and by D the general fibre of the Albanese map $S \to E/G$ and recall that $F \cong E$ and $D \cong B$.

Now, in view of Theorem 1.3 we can assume that L is of type (a, b) for suitable integers a and b; hence

$$L \equiv \frac{b\mu}{\gamma} D + \frac{2a + \delta b}{2\mu} F.$$

To compute intersections on S recall that $D^2 = F^2 = 0$, while $DF = \gamma$, the order of G. Note that in every case in Tables 5.2, 5.3 we can write $K_S \equiv \frac{s}{\mu} F$ for some positive integer s. A close inspection of these tables with the help of (1.1.5) shows that $s = 1$ in the three cases of Table 5.2, which also appear as types (b), (e), (j) in Table 5.3, while $s = 2$ in all the remaining cases. We have

$$LK_S = bs, \tag{5.0.1}$$

hence $b > 0$, since $LK_S > 0$.

Now recall (2.8). First let $LK_S = 1$. Then we have

$$1 = L^2 = (2a + \delta b) b \tag{5.0.2}$$

and in view of Theorem 1.3 we conclude that L is ample. Moreover, in the three cases of Table 5.2, $g(B)$ and the structure of the group G are known: actually $\big(g(B), G\big) = (2, \mathbb{Z}_2 \times \mathbb{Z}_6)$, $(3, \mathbb{Z}_4 \times \mathbb{Z}_4)$, and $(2, \mathbb{Z}_2 \times \mathbb{Z}_2)$ respectively ([17], Theorem 2.2, (iv)). Then it is immediate to check that $\delta = 1$ in these cases. As a consequence (5.0.2) immediately gives $(a, b) = (0, 1)$. This shows the following

Proposition 5.1

Let S be an elliptic quasi-bundle over $\mathbb{P}^1$ and $L \in \mathrm{Pic}(S)$ as in Table 5.2 with $g(L) = 2$. Then the following conditions are equivalent:
(1) L is nef;
(2) L is ample;
(3) $L \equiv \frac{\mu}{\gamma} D + \frac{1}{2\mu} F$.

Note that (3) is equivalent to the expression provided by Serrano ([17], Theorem 2.2, (iii)) (see also [7], p. 158).

Finally let $LK_S = 2$. Recalling that $b > 0$, the condition $0 = L^2 = (2a + \delta b) b$ implies

$$2a + \delta b = 0. \tag{5.0.3}$$

There are two possibilities according to the values of δ. Let $\delta = 1$; thus $b = -2a$ and so (5.0.1) gives $(s, a, b) = (1, -1, 2)$. As we observed before the only types in Table 5.3 corresponding to $s = 1$ are (b), (e), (j) (the same as in Table 5.2) and in these cases, as we said, $\delta = 1$. This shows in particular that $\delta = 1$ if and only if $s = 1$, if and only if S is as in (b), (e), (j) of Table 5.3 and $L \equiv \frac{2\mu}{\gamma} D$. Let $\delta = 0$. Then (5.0.3) and (5.0.1) give $(s, a, b) = (2, 0, 1)$. So S is as in the remaining eight cases of Table 5.3 and $L \equiv \frac{\mu}{\gamma} D$. Conversely, note that in every case L is actually nef, so being D. We thus get

Proposition 5.2

Let (S, L) be a semipolarized elliptic quasi-bundle over $\mathbb{P}^1$ as in Table 5.3 with $g(L) = 2$. Then $L \equiv \frac{2\mu}{\gamma} D$ in cases (b), (e), (j), *while $L \equiv \frac{\mu}{\gamma} D$ in the remaining cases.*

To conclude we give an explicit example in which we are able to supply $g(B)$ and the group G. In fact, contrary to what happens for types (b), (e), (j), in the remaining cases of Table 5.3 there is only a partial description of the possible groups G giving rise to S [13]. However, following closely ([17], Example 2.6), we can construct the following

EXAMPLE 5.3: Let $C = \mathbb{P}^1$ and consider six distinct points $p_1, \ldots, p_6 \in C$. Let $\rho_1 : \Gamma_1 \to C$, $\rho_2 : \Gamma_2 \to C$ be two double covers, branched at p_1, p_2, p_3, p_4 and at p_3, p_4, p_5, p_6 respectively, such that the two elliptic curves Γ_1 and Γ_2 are not isomorphic. Let B be the normalization of the fibre product $\Gamma_1 \times_C \Gamma_2$. We claim that B is a smooth irreducible curve of genus $g(B) = 3$. Of course B is smooth, hence, were B reducible, it would be disconnected. Since both maps $\theta_i : B \to \Gamma_i$ induced by the projections have degree 2 then B would consist of two connected components, each one being isomorphic to Γ_1 and Γ_2 at the same time. But this is clearly impossible since $\Gamma_1 \not\cong \Gamma_2$. Finally look at the map $\theta_1 : B \to \Gamma_1$. Taking the normalization separates the branches at the singular points of $\Gamma_1 \times_C \Gamma_2$, which lie on the fibres over p_3 and p_4; hence θ_1 is branched only at the 4 points of Γ_1 constituting the fibres of ρ_1 over p_5 and p_6. Thus the Riemann-Hurwitz formula gives $g(B) = 3$. Now the actions of $\mathbb{Z}_2$ on Γ_1 and Γ_2 yielding C define an action of $G := \mathbb{Z}_2 \times \mathbb{Z}_2$ on the fibre product, which can be lifted to B, giving $B/G = C$. Now take any elliptic curve E and make G act on it faithfully by translations. Consider the diagonal action of G on $B \times E$ and set $S := (B \times E)/G$. Denote by D and F the general fibres of the morphisms $S \to E/G$ and $S \to B/G = C$ respectively. Then $S \to B/G = C$ is an elliptic quasi-bundle with six multiple fibres with multiplicity 2 and $K_S \equiv F$; hence according to Theorem 1.3 $\frac{1}{2}D$ is a divisor, its class being a generator of $\mathrm{Num}(S)$. In any case, for sake of completeness, here we give a direct proof of this. Note that $\frac{1}{2}F$ is a divisor whose class in $\mathrm{Num}(S)$ is not further divisible ([17], Proposition 1.4). By Poincaré duality and since $p_g = 0$ we infer that there exists a line bundle $M \in \mathrm{Pic}(S)$ such that $\frac{1}{2}FM = 1$. Put $M \equiv aD + bF$ with $a, b \in \mathbb{Q}$. Since $DF = 4$, we get $a = \frac{1}{2}$, and so, $M^2 + MK_S = 4b + 2$ being even, we see that $2b$ is an integer. Hence $\frac{1}{2}D = M - 2b(\frac{1}{2}F)$ is a divisor. Then the pair $(S, L := \frac{1}{2}D)$ gives an example of type (k) in Table 5.3.

Appendix. The lowest genus of an ample and spanned line bundle

Antonio Lanteri

Stimulated by a renewed interest on the genus of ample and spanned line bundles (e. g. see [9], Appendix), I take this opportunity for improving a result in [12] following an idea essentially due to Serrano [13].

Proposition A.1

Let L be an ample and spanned line bundle on a properly elliptic surface S. Then $g(L) \geq 4$.

Proof. First of all we can assume that

$$L^2 \geq 3. \tag{A.1.1}$$

This follows e. g. from ([14], Lemma 0.6.1) and the fact that if S is a double cover of $\mathbb{P}^2$ with $\kappa(S) > 0$, then K_S is ample, which implies $\kappa(S) = 2$, a contradiction. Since $LK_S > 0$ we thus see from the genus formula that $g(L) \geq 3$. So we have only to show that case $g(L) = 3$ does not occur. Note that if $g(L) = 3$, in view of (A.1.1) the genus formula gives $LK_S = 1$, and then the same argument as in the proof of Lemma 1.4 shows that S is a minimal surface. Surfaces polarized by an ample and spanned line bundle of genus three have been studied in [12]. The possible pairs (S, L) with $\kappa(S) = 1$ are described in [12], Proposition 3.3. In fact those listed in (3.3.1) there do not occur in view of the Katsura-Ueno divisibility property (1.1.4). So the surface S can only be as follows [12], (3.3.2):

(A.1.2) S is minimal, $p_g(S) = q(S) = 0$, the elliptic fibration $\psi : S \to C$ has $C = \mathbb{P}^1$ as basis and exactly two multiple fibres of multiplicities 2 and 3.

Let F and $F_0 = 3f$ be the general fibre and the fibre of multiplicity 3 of ψ respectively. By the canonical bundle formula we get $K_S \equiv \frac{1}{6}F \equiv \frac{1}{2}f$. Hence condition $LK_S = 1$ gives $Lf = 2$. Assume that $h^0(L - f) > 0$; then there exists an effective divisor $R \in |L - f|$. Since L is ample and spanned, from the equality $LR = L^2 - Lf = 1$ we see that R is a smooth rational curve. On the other hand, since $1 = LR = R^2 + Rf = R^2 + (R + f)f = R^2 + Lf = R^2 + 2$ we conclude that R is a (-1)-curve, contradicting the minimality of S. This shows that $h^0(L - f) = 0$. Hence from the cohomology sequence induced by

$$0 \to L - f \to L \to L_f \to 0$$

we get

$$h^0(L) \leq h^0(L_f). \tag{A.1.3}$$

Now, since L is ample and $Lf = 2$, according to [1], p. 151, f can only be either smooth elliptic, rational with a node, or consisting of two (-2)-curves intersecting at two distinct points. In all cases, using the Riemann-Roch theorem for the embedded curve $f \subset S$ ([1], p. 51), we easily see that

$$h^0(L_f) = \deg L_f + \chi(\mathcal{O}_f) + h^1(L_f) = 2. \tag{A.1.4}$$

But (A.1.3) combined with (A.1.4) gives a contradiction since $h^0(L) \geq 3$, L being ample and spanned. So case (A.1.2) does not occur. $\square$

Recall that if L is simply an ample line bundle on S then $g(L) \geq 2$, while if L is very ample then $g(L) \geq 6$ ([15], §4). Both bounds are effective; also the bound provided by (A.1) is effective, as shown by the pair (X_0, L_1) in Example 3.1.

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