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Extension of maps defined on many fibres

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A nuestro compañero y amigo Fernando Serrano

Abstract

Let S be a fibred surface. We prove that the existence of morphisms from non countably many fibres to curves implies, up to base change, the existence of a rational map from S to another surface fibred over the same base reflecting the properties of the original morphisms. Under some conditions of unicity base change is not needed and one recovers exactly the initial maps.

Introduction

Let S be a surface and let $\pi : S \longrightarrow B$ be a fibration of curves of genus $g \geq 2$. Assume that for non countably many $t \in B$, the fibre F_t is endowed with a non-constant morphism $\varphi_t : F_t \longrightarrow D_t$ into a smooth curve. The goal of this note is to show that the existence of these maps φ_t implies the existence of another fibration

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$T \longrightarrow B$ and of a rational map over B from S to T reflecting the properties of many φ_t. In fact we recover the original maps φ_t only for non countably many values of t (even in the case that one applies this under the hypothesis of existence of φ_t for a general t, one can not get better results as simple examples show).

To obtain the surface T we shall need base change in general. However under some hypothesis of unicity this base change can be avoided.

We consider three cases: first we assume that the maps φ_t are automorphisms (Theorem 1.2); secondly we suppose that $1 \leq g(D_t) < g$ (Theorem 2.4), and finally we study linear series (Theorem 3.1). We also obtain a similar result for abelian schemes with abelian subvarieties in the general fibres (Theorem 2.5).

We use basically two standard techniques: relative Hilbert schemes and relative Brill-Noether loci. We work over the field of the complex numbers, although most of the results and constructions in sections 1 and 2 can be done for any non countable algebraically closed field.

These kind of problems appear naturally in the study of fibrations (see e.g. [3]), as Fernando Serrano pointed out to the first author. We consider this work as a consequence of his encouragement.

We would like to thank G. Welters for giving us some useful suggestions.

Notation. By a fibration we mean a non constant morphism from a smooth complete algebraic surface to a smooth complete algebraic curve with connected fibres. We say that a fibration is of genus g if the smooth fibres have geometric genus g. We say that the fibration is isotrivial if all smooth fibres are isomorphic.

If $\pi : S \longrightarrow B$ is a fibration and $B' \longrightarrow B$ is a non constant map from another smooth curve, we denote by $S_{B'}$ the surface $S \times_B B'$.

1. Glueing automorphisms

Let $\pi : S \longrightarrow B$ be a fibration and denote by F_t the fibre of π in $t \in B$. The aim of this section is to prove that the existence of automorphisms of many fibres of π induces, up to base change, the existence of a birational automorphism of S. To prove this, we shall need some standard facts on Hilbert schemes that we recall now.

We fix a relatively very ample sheaf $\mathcal{O}_X(1)$ on $X := S \times_B S \longrightarrow B$. Following Grothendieck ([5]), we can consider the scheme $\underline{\mathcal{A}ut}_{S/B}$ as an open subscheme of $\underline{\mathcal{H}ilb}_{X/B}$ representing the functor

$$\begin{aligned} Aut: \ B-\text{schemes} &\longrightarrow \text{Groups} \\ T &\longmapsto \mathrm{Aut}_T(S_T). \end{aligned}$$

Then, giving a section σ of the natural map $\underline{\mathcal{A}ut}_{S/B} \longrightarrow B$ corresponds, via the identification

$$\mathrm{Hom}_B(B, \underline{\mathcal{A}ut}_{S/B}) = Aut_B(S),$$

to an automorphism Φ of S over B such that for $t \in B$, $\Phi_{|F_t} = \sigma(t) \in \mathrm{Aut}(F_t)$. We recall that $\underline{\mathcal{A}ut}_{S/B}$ is a group-scheme over B and, as in the case of Hilbert schemes, decomposes as a disjoint union of schemes $\underline{\mathcal{A}ut}^{p(t)}_{S/B}$ obtained by fixing the Hilbert polynomial $p(t) \in \mathbb{Q}[t]$.

Choosing a suitable Hilbert polynomial and considering only elements of m-torsion one constructs a B-scheme $\underline{\mathcal{A}ut}^{m,r}_{S/B}$ parametrizing the automorphisms of the fibres of π of order m and with r fixed points.

Definition 1.1. Let F be a smooth curve of genus $g \geq 2$ and let φ be an automorphism of F. Assume $\varphi \neq \mathrm{Id}$. Consider the curve $G := F/ < \varphi >$ and the map $f : F \longrightarrow G$. If n is the order of φ we call the *type of* φ to the following data:

$$\Lambda = \{(k, a_k)\}_{k|n,\, k \neq 1}$$

where a_k is the number of ramification points of f of index k. Equivalently one can give $\Lambda' = \{(k, b_k)\}_{k|n,\, k \neq 1}$ where b_k is the number of fixed points of $\varphi^{n/k}$ (notice that

$$b_k = \sum_{k|k'|n} a_{k'}).$$

Observe that one can consider the B-scheme of automorphisms of type Λ defining:

$$\underline{\mathcal{A}ut}^{\Lambda}_{S/B} := \bigcap_{k|n, k\neq 1} H_k^{-1}(\underline{\mathcal{A}ut}^{n/k, b_k}_{S/B}),$$

where $H_k : \underline{\mathcal{A}ut}_{S/B} \longrightarrow \underline{\mathcal{A}ut}_{S/B}$ sends x to x^k.

The main result of this section is the following:

Theorem 1.2

Let $\pi : S \longrightarrow B$ be a fibration of genus $g \geq 2$. Assume that for non countably many $t \in B$, $\mathrm{Aut}(F_t) \neq$Id. Then:
a) There exist a type Λ, a base change $B' \longrightarrow B$ and a birational automorphism Φ' of the non singular model $\tilde{S}$ of $S_{B'}$ such that the restriction of Φ' to a general fibre of $\tilde{S} \longrightarrow B'$ is an automorphism of type Λ.
b) The type Λ of a) can be chosen previously, provided that for a non countably many $t \in B$ the fibre F_t has an automorphism of type Λ.
c) If, furthermore, for a non countably many $t \in B$, the automorphism of type Λ is unique, then base change is not needed.

Proof. As $g \geq 2$, the fibres of $\underline{\mathcal{A}ut}_{S/B} \longrightarrow B$ are finite. Since B is a curve, there exists an irreducible component B' of $\underline{\mathcal{A}ut}_{S/B}$, dominating B. We can assume B' to be complete; otherwise the construction that follows will extend in a natural way to a compactification of B'. Since $\underline{\mathcal{A}ut}_{S/B}$ is a finite union of subschemes $\underline{\mathcal{A}ut}^{\Lambda}_{S/B}$, an infinite number of points (and hence a Zariski open set) of B' correspond to automorphisms of the same type Λ. Then

$$\underline{\mathcal{A}ut}_{S_{B'}/B'} = \underline{\mathcal{A}ut}_{S/B} \times_B B' \longrightarrow B',$$

has an obvious section which produces a relative automorphism Φ' of $S_{B'}$. Then Φ' determines a birational automorphism of $\tilde{S}$. This proves a).

Observe that the same argument works for b) just fixing Λ and $\underline{\mathcal{A}ut}^{\Lambda}_{S/B}$ from the beginning. Finally, under the hypothesis of c), $\underline{\mathcal{A}ut}^{\Lambda}_{S/B} \longrightarrow B$ is generically one-to-one and has a section. $\square$

Remark 1.3. In [3], the first author gives an example of a bielliptic fibration of genus 5 for which the general fibre has two different bielliptic involutions and such that a non-trivial base change is needed in order to glue them into a global birational involution. So, in general, c) does not hold without the hypothesis of unicity.

2. Glueing morphisms of curves

In this section we consider a fibration $\pi : S \longrightarrow B$ such that for non countably many $t \in B$, the fibres F_t have a map onto a non-rational smooth curve. The aim is to produce, perhaps after base change, a rational map over B, $S - - \rightarrow T$, such that for non countably many fibres we recover the original morphisms, perhaps (if $T \longrightarrow B$ is an elliptic fibration) up to automorphisms on the image curve. The main point of the construction is to observe that a morphism from F_t onto a curve of genus ≥ 1 induces an endomorphism of $J(F_t)$, the Jacobian variety of F. Then we prove that endomorphisms on many fibres of an abelian scheme produce an endomorphism of the abelian scheme and from this the result follows quickly. As a by-product we find that the existence of non-trivial abelian subvarieties on the fibres of an abelian scheme implies the existence of a non-trivial abelian subscheme.

2.1. Let $\pi : \mathcal{A} \longrightarrow \mathcal{U}$ be an abelian scheme and let σ be the zero section. The theory of Hilbert schemes ensure the existence of a $\mathcal{U}$-scheme $\mathcal{E}nd_{\mathcal{U}}(\mathcal{A})$ parametrizing the endomorphisms of the fibres of π. By taking the kernel of the following map of group schemes over $\mathcal{U}$,

$$\mathcal{E}nd_{\mathcal{U}}(\mathcal{A}) \longrightarrow \mathcal{E}nd_{\mathcal{U}}(\mathcal{A}) \times \mathcal{U} \xrightarrow{\mathrm{Id}\times\sigma} \mathcal{E}nd_{\mathcal{U}}(\mathcal{A}) \times \mathcal{A} \xrightarrow{\text{evaluation}} \mathcal{A}$$

one obtains the existence of a $\mathcal{U}$-scheme $\underline{\mathcal{E}nd_{\mathcal{U}}}(\mathcal{A})$ parametrizing the endomorphisms of the fibres of π as abelian varieties. By construction one identifies the global sections of this scheme with the group of endomorphisms of $\mathcal{A}$ over $\mathcal{U}$ as abelian scheme.

To stay the main theorem we shall need the following definition:

DEFINITION 2.2. Let $f : C \longrightarrow D$ be a non-constant map of complete smooth curves. We say that f is *indecomposable* if it does not exist a factorization of f through a cyclic étale covering of D of degree $n \geq 2$.

Now the Proposition 4.3 in [4], p. 337 reads:

2.3. Let $f : C \longrightarrow D$ as above. The map $f^* : JD \longrightarrow JC$ is injective if and only if f is indecomposable.

The main result of this section is the following theorem:

Theorem 2.4

Let $\pi : S \longrightarrow B$ be a fibration of curves of genus g and fibres F_t, $t \in B$. Assume that for non countably many $t \in B$ there exist a non-constant map $\varphi_t : F_t \longrightarrow D_t$ on a curve of positive genus $q < g$. Then the following statements hold:
a) There exist a base change $B' \longrightarrow B$, an integer q, $0 < q < g$, a fibration $T' \longrightarrow B'$ of curves of genus q, and a rational map $\Psi : S_{B'} - - - \rightarrow T'$ over B'.
b) For non countably many $t \in B$ one has $\Psi_t = \varphi_t$ (up to automorphisms of D_t if $q_0 = 1$).
c) If the map φ_t is unique for non countably many $t \in B$, then base change is not needed.

Proof. After base change and taking a suitable open subset $\mathcal{U}$ of B we can assume the existence of a diagram

$$\begin{array}{ccc} S_{|\mathcal{U}} & \xrightarrow{\varepsilon} & \mathcal{J} \\ {\scriptstyle \pi^0}\downarrow & & \downarrow \\ \mathcal{U} & = & \mathcal{U}, \end{array}$$

where $\mathcal{J} \longrightarrow \mathcal{U}$ is the Jacobian fibration. The map ε is defined with the aid of a section of π^0 and fibre to fibre gives an inclusion of each F_t in its Jacobian variety.

Let us denote by $\mathcal{E}J$ the $\mathcal{U}$-scheme $\underline{\mathcal{E}nd_{\mathcal{U}}}\mathcal{J}$ introduced in (2.1). It is well-known that an abelian variety has countably many endomorphisms, hence fibres of

$\mathcal{EJ} \longrightarrow \mathcal{U}$ are countable. Define $\rho_t : JF_t \longrightarrow JF_t$ the endomorphism $\varphi_t^* \circ \mathrm{Nm}_{\varphi_t}$, where Nm_{φ_t} is the norm map

$$\begin{aligned} \mathrm{Nm}_{\varphi_t} : JF_t &\longrightarrow JD_t \\ \left[\sum n_i P_i\right] &\longmapsto \left[\sum n_i \varphi_t(P_i)\right]. \end{aligned}$$

By hypothesis one has an irreducible component $W \subset \mathcal{E}J$ dominating $\mathcal{U}$. By using again base change and the functoriality of $\mathcal{E}J$, we can assume the existence of a section of $\mathcal{E}J \longrightarrow \mathcal{U}$, providing an endomorphism λ of the $\mathcal{U}$-scheme $\mathcal{J}$.

Define T' to be a desingularization of the closure of the image of

$$S_{|U} \xrightarrow{\varepsilon} \mathcal{J} \xrightarrow{\lambda} \mathcal{J}$$

in some compactification of $\mathcal{J}$. By construction, one has the rational map we were looking for and part a) is proved.

Observe that for non countably many $t \in B$, we recover the map $F_t \longrightarrow \varphi_t^*(D_t)$. If φ_t is an indecomposable map for non countably many t, then b) is clear from the fact (2.3). Otherwise, we can write for a non countably many $t \in B'$ $\varphi_t = \alpha_t \circ \beta_t$, where β_t is indecomposable and $\alpha_t : D'_t \longrightarrow D_t$ is an étale cyclic covering of degree $n_t \geq 2$. For non countably many t the degree n_t is constant and hence we can assume constant the genus of the curves D'_t. Notice that the morphisms α_t are determined by automorphisms on the curves D'_t. Therefore we can apply the indecomposable case proved above to glue the maps β_t and then the part b) of the Theorem 1.2 finishes de proof of b) (perhaps after a new base change).

Assume now that φ_t is unique for non countably many $t \in B$. In particular the curves D_t have not automorphisms (this forces $q \geq 2$) and the automorphism of F_t permute the fibres of φ_t.

As in b) we suppose first that the φ_t is indecomposable. The results of a) and b) give a base change $B' \longrightarrow B$, a fibration T' of curves of genus q over B' and a rational map from $S_{B'}$ to T'. The first step is to observe that there exists a fibration $T \longrightarrow B$ such that T' is obtained from T by base change (at least on an open set of B'). Indeed, we consider the image B_0 of B' in the moduli space of curves of genus q. If $\dim B_0 = 0$, then the fibration $T' \longrightarrow B'$ is isotrivial and (doing again base change) we can assume that T' is the product $B' \times D$. In this case one simply defines T to be $B \times D$.

If $\dim B_0 = 1$, then one can construct a universal family of curves over an open set of B_0 (recall that D_t has not automorphisms). Fix a point $t \in B$ such that φ_t is unique and denote by $t_1, \ldots, t_r$ the preimages in B'. Since the curves F_{t_i} are isomorphic and by the unicity, one obtains that the fibres of $T' \longrightarrow B'$ at $t_1, \ldots, t_r$ are isomorphic. From this one easily proves that the modular morphism from B' to B_0 factorizes through the morphism $B' \longrightarrow B$. The pull-back of the universal family over B_0 to B allows to construct a surface T with a fibration over an open set of B. One can assume as usual that T is fibred over B.

Now, by the existence of such a fibration $T \longrightarrow B$ and the hypothesis of unicity, one checks that the graph of the rational map from S' to T' descents to a graph of a rational map from S to T.

As in part b) we divide the proof of the general case into two parts. One glue first the indecomposable maps and after one uses the part c) of Theorem 1.2. Observe that the curves D'_t (with the notations of b)) have a unique automorphism of this type due to the unicity of φ_t. $\square$

Consider now a polarized abelian variety (A, L) of dimension a and let B be an abelian subvariety of dimension b. Let $\hat{A}$ be the dual abelian variety of A (i.e. the Picard variety of A) and call $\lambda : A \longrightarrow \hat{A}$ to the isogeny induced by the polarization. Consider the map

$$\alpha : A \xrightarrow{\lambda} \hat{A} \xrightarrow{j} \hat{B}$$

where j is the dual of the inclusion $B \subset A$. It is easy to see that the variety $P := \mathrm{Ker}\,(\alpha)^0 \subset A$ is an abelian subvariety of dimension $a - b$ and such that $I := B \cap P$ is finite. In other words, the addition map $s : B \times P \longrightarrow A$ is an isogeny. Let $r \in \mathbb{N}$ such that $rI = 0$. Then for any pair of integers (m, n) such that $r|(m-n)$ the endomorphism

$$\begin{array}{ccc} B \times P & \longrightarrow & B \times P \\ (x, y) & \longmapsto & (mx, ny) \end{array}$$

produces an endomorphism $\psi_{m,n}$ of A. Observe that $\psi_{r,0}(A) = B$ and $\psi_{0,r}(A) = P$. The same arguments used above allow to prove the following theorem:

Theorem 2.5

Let $\mathcal{A} \longrightarrow \mathcal{U}$ an abelian scheme such that $\dim\mathcal{U}$=1. Assume that for non countably many $t \in \mathcal{U}$ the abelian variety A_t has a non-trivial abelian subvariety B_t of dimension b_t. Then there exist a base change $\mathcal{U}' \longrightarrow \mathcal{U}$ and a constant b such that $\mathcal{A}_{\mathcal{U}'}$ has an abelian subscheme $\mathcal{B} \longrightarrow \mathcal{U}'$ of relative dimension a and $(\mathcal{B})_t = B_t$ for non countably many $t \in \mathcal{U}'$.

Proof. We fix a relative polarization on $\mathcal{A}$. As above, the existence of an abelian subvariety in A_t induces the existence of an endomorphism $\psi_{r_t,0}$ of A_t. Arguing as in Theorem 2.4 we glue, up to base change, these endomorphisms to obtain an endomorphism of $\mathcal{A}$ over $\mathcal{U}$. The image of this endomorphism gives the abelian subscheme. □

Remark 2.6. One easily checks that base change is not needed if for non countably many $t \in \mathcal{U}$ there is a unique abelian subvariety of a given dimension b. If there is more than one subvariety this is not true: consider the fibration of bielliptic curves with two bielliptic maps constructed in [3]. The corresponding Jacobian fibration (on an open set of the base) gives a counterexample.

3. Glueing linear series

Let $\pi : S \longrightarrow B$ be a fibration such that for non countably many t, the fibre F_t is d-gonal (i.e. F_t possesses a base point free g^1_d). As in previous sections we want to extend, after base change, the corresponding morphisms $F_t \longrightarrow \mathbb{P}^1$. More precisely, we want to prove:

Theorem 3.1

Let $\pi : S \longrightarrow B$ be a fibration such that for non countably many $t \in B$, F_t is d-gonal; then
a) there exist a base change $B' \longrightarrow B$, a ruled surface R' over B' and a rational map $\Phi : S' - - - \rightarrow R'$ over B' such that $\deg \Phi = d$.
b) If for non countably many $t \in B$, F_t has a unique g^1_d (hence complete), then base change is not needed.

Remark 3.2. This theorem is classical when $d = 2$ (cf., e.g., [10]). Recall that, for hyperelliptic curves, all base point free g^1_d are obtained by composing the hyperelliptic map with a Segre embedding of degree $d/2$ and then projecting. From this one obtains immediately the theorem for hyperelliptic curves.

Proof. We consider the cases $d \geq g+1$, $d = g$ and $d \leq g$ separately.

Assume $d \geq g+1$. After a base change we obtain a d-section D of π. By Riemann-Roch $h^0(F_t, D_t) \geq 2$ for all smooth F_t. We can choose a rank 2 vector bundle $\mathcal{E} \subset \pi_* \mathcal{O}_S(D) \otimes A$ generically generated by two global sections, where $A \in Pic(B)$ is of degree big enough. The natural map $\pi^*(\mathcal{E}) \longrightarrow \mathcal{O}_S(D) \otimes \pi^*(A)$ induces a rational map $\Phi : S - - - \rightarrow R = \mathbb{P}(\mathcal{E})$ over B such that $\Phi^* \mathcal{O}_{\mathbb{P}}(1)$ is the image of $\pi^*\mathcal{E} \longrightarrow \mathcal{O}_S(D) \otimes \pi^*(A)$. Fixing previously that D_{t_0} is base point free for some $t_0 \in B$ (it is possible by hypothesis) and that the two sections generating $\mathcal{E}$ have no base point at t_0, we can conclude that such image has no horizontal base component. Hence Φ has degree d and we are done (see, for example, [7] for details on these kind of constructions).

Assume $d = g$. Note that the same proof of the case above works if we have a d-section D such that for $t \in B$ general $h^0(F_t, D_t) \geq 2$ and for some $t_0 \in B$, D_{t_0} is a base point free g^1_d. Take D' a $(g-2)$-section of π (after base change if necessary) such that $K_{F_{t_0}} - D'_{t_0}$ is base point free. Then, if $A \in \operatorname{Pic} B$ is ample enough, take a global section $D \in |K_S - D' + \pi^*(A)|$. For $t \in B$ general we have $h^0(F_t, D_t) = h^0(F_t, D'_t) + 1 \geq 2$ by Riemann-Roch.

Assume now that $d \leq g-1$. By Remark 3.2 we can assume that the generic fibre is not hyperelliptic.

We consider the Brill-Noether loci $W^r_d(F)$ of a fixed smooth d-gonal fibre F. Since $W^1_d(F)$ is not contained in $W^2_d(F)$ (cf. [1], p. 182) we can assume that the linear series g^1_d is complete. Given a complete $L \in W^1_d(F)$, the projectivized tangent cone $W_{F,L}$ of $W^0_d(F)$ at L is a minimal degree variety in $\mathbb{P}^{g-1} = \mathbb{P}(H^0(F, K_F)^*)$ of dimension $d-1$ ruled by $(d-2)$-planes generated by the images of the divisors of $|L|$ by the canonical embedding (cf. [1], p. 241). The singular locus of $W_{F,L}$ is the linear variety $H_{F,L} = \mathbb{P}(T_L W^1_d(F))$ intersection of such $(d-2)$-planes. Let us denote by $e(L)-1$ the dimension of $H_{F,L}$ and call $\tilde{W}_{F,L} = Bl_{H_{F,L}} W_{F,L}$. Observe that $\tilde{W}_{F,L}$ is a smooth rational scroll ruled by $(d-2)$-planes and with an endowed map $\beta_{F,L} : \tilde{W}_{F,L} \longrightarrow \mathbb{P}^1$. If L is base point free we have $F \hookrightarrow \tilde{W}_{F,L}$ and the composition of this inclusion with the map $\beta_{F,L}$ determines L.

Recall that, being $W_{F,L}$ a scroll of dimension $(d-1)$, $W_{F,L}$ has more than one system of $(d-2)$-planes if and only if $W_{F,L}$ is a rank 4 quadric (see [6], pp. 49, 51) and in this case it has two systems. We have then that $d = g-1$ and $W_{F,L} = W_{F,K_F-L}$, $L \neq K_F - L$. Finally we note that every $(d-2)-$plane contained in $W_{F,L}$ must be a fibre of one of the rullings.

In order to consider the above constructions relatively, we apply base change. Then we obtain the existence of enough sections and this fills-up the hypothesis in [9]. Hence, there exists a variety $W^r_d(\pi)$ over B, such that, for smooth F_t, $W^r_d(F_t) \cong W^r_d(\pi)_t$. By hypothesis, $\alpha : W^1_d(\pi) \longrightarrow B$ is dominant.

Since the construction of $W^1_d(\pi)$ is functorial, base change guarantees the existence of a section η of α such that $\eta(t)$ is, for t general, a complete base point free g^1_d over F_t. Let us call D to the image of η and W_D to the projectivized tangent cone of $W^0_d(\pi)$ at D. Note that W_D contains the relative canonical image of $S - - - \rightarrow \mathbb{P}_B(\pi_*\omega_{S/B})$ (see [7]).

We can consider that, after some blow-ups, $\mu : S \longrightarrow W_D$ is a morphism (generically of degree one onto the image). Let

$$\begin{aligned}
&\mathbb{G} = \mathrm{Grass}_B^{d-2}\big(\mathbb{P}(\pi_*\omega_{S/B})\big),\\
&B^0 = \{t \in B \mid \pi \text{ is smooth at } t\},\\
&\varphi : W_D \longrightarrow B\\
&\mathcal{U} = \big\{(p,[R]) \in W_D \times_B \mathbb{G} \mid p \in R,\ R \subset W_D,\ \varphi(p) \in B^0\big\},\\
&\alpha_1 : \mathcal{U} \longrightarrow W_D,\ \alpha_2 : \mathcal{U} \longrightarrow \mathbb{G} \text{ the natural projections,}\\
&M = \alpha_2(U) \text{ and}\\
&\tilde{S} = S \times_W \mathcal{U}.
\end{aligned}$$

Note that α_1 is birational if $d < g-1$ and generically of degree ≤ 2 if $d = g-1$. Moreover, if $d < g-1$ and $t \in B^0$ then $\alpha_1^{-1}(W_t) \cong \tilde{W}_{F_t,\eta(t)}$. The map $\alpha_2 : \mathcal{U} \longrightarrow M$ is generically a $\mathbb{P}^{d-2}$-bundle.

Let N be an irreducible horizontal component of a relative multihyperplane section of W_D over B of dimension two. Note that N is a ruled surface, and meets every $(d-2)$-plane of a general fibre of W_D exactly at one point. Then, if $\tilde{N}$ is the pull-back of N in $\mathcal{U}$, we have that $\alpha_{2|\tilde{N}} : \tilde{N} \longrightarrow M$ is birational. Then M is a (possibly singular) ruled surface over B if $\deg \alpha_1 = 1$ or is a ruled surface over a double cover of B if $\deg \alpha_1 = 2$.

More precisely, take $\overline{M}$ a horizontal irreducible component of a desingularization of M and let $\overline{M} \longrightarrow \overline{B} \longrightarrow B$ the Stein factorization of $\overline{M} \longrightarrow B$. If we pull-back $\tilde{S} \longrightarrow \mathcal{U} \longrightarrow M$ to $\overline{S} \longrightarrow \overline{\mathcal{U}} \longrightarrow \overline{M}$ we have a rational map $\psi : \overline{S} - - - \rightarrow \overline{M}$ over a ruled surface over $\overline{B}$. If $\deg \alpha_1 = 1$ then $\overline{B} = B$ and for $t \in B^0$, the map ψ_{F_t} corresponds to $\eta(t)$. If the map $\overline{B} \longrightarrow B$ has degree 2 and $\bar{t}_1$ and $\bar{t}_2$ are the preimages of $t \in B^0$, then $\psi_{\overline{F}_{\bar{t}_i}}$ corresponds to $\eta(t)$ or $K_{F_t} - \eta(t)$. Finally note that, by construction, $\overline{S}$ is birational to $S \times_B \overline{B}$.

In order to prove b) we only have to prove that the existence of $\tilde{W}_{B'}$ over a base extension B' of π implies the existence of $\tilde{W}$ over B when the base point free linear series is unique.

Let δ be a base change

$$\begin{tikzcd}
\tilde{S} \arrow[r, "\gamma'"] & S' \arrow[r, "\gamma"] \arrow[d, "\pi'"'] & S \arrow[d, "\pi"] \\
 & B' \arrow[r, "\delta"] & B,
\end{tikzcd}$$

and let $\tilde{S}$ be a minimal desingularization of S'. Denote by $\tilde{\pi}$ to $\pi'\gamma'$. Then, from $\omega_S \hookrightarrow \omega_S \otimes \gamma_* \mathcal{O}_{S'}$, applying $\delta^* \pi_*$, base change and ramification formula for $\gamma\gamma'$, one obtains

$$0 \longrightarrow \delta^*(\pi_* \omega_S) \longrightarrow \tilde{\pi}_*(\omega_{\tilde{S}}).$$

Since both are locally free sheaves of the same rank we get a birational map given by a sequence of elementary transformations on suitable fibres

$$\mathbb{P}_{B'}(\tilde{\pi}_* \omega_{\tilde{S}}) - - - \to \mathbb{P}_{B'}(\delta^* \pi_* \omega_S),$$

which produces

$$\rho : \mathbb{P}_{B'}(\tilde{\pi}_* \omega_{\tilde{S}/B'}) \cong \mathbb{P}_{B'}(\tilde{\pi}_* \omega_{\tilde{S}}) - - - \to \mathbb{P}_{B'}(\delta^* \pi_* \omega_S) \longrightarrow \mathbb{P}_B(\pi_* \omega_S) \cong \mathbb{P}_B(\pi_* \omega_{S/B}).$$

This map ρ is linear on fibres and restricts to the natural map from the relative canonical image of $\tilde{\pi} : \tilde{S} \longrightarrow B'$ onto the relative canonical image of $\pi : S \longrightarrow B$. Fix a general $t \in B$ and consider $\delta^{-1}(t) = \{t_1, \ldots, t_k\}$. Then $\rho(\tilde{W}_{t_i})$ is a variety of minimal degree containing the canonical image of F_t, ruled by $(d-2)$-planes producing the linear series on F_t. By the unicity we have that all the images $\rho(\tilde{W}_{t_i})$ agree. Hence $(\rho(\tilde{W}_{B'}))_t = \tilde{W}_{F_t}$ for general $t \in B$. Then $\rho(\tilde{W}_{B'})$ is the variety $\tilde{W}$ we were seeking. Note that, if the g^1_d is unique we are not in the case where $W_{F,L}$ is a rank 4 quadric and then no new base change is needed in the proof of a). $\square$

Remark 3.3. Observe that the hypothesis of having a unique g^1_d is general for small values of d. Indeed, according to [2], Theorem 2.6 a general d-gonal curve with $2 \leq d < \frac{g}{2} + 1$ has a unique g^1_d.

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