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On the motive of a quotient variety*

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En memoria de nuestro compañero y amigo Ferran Serrano

Abstract

We show that the motive of the quotient of a scheme by a finite group coincides with the invariant submotive.

Introduction

Let k be a field and $\mathcal{V}_k$ the category of smooth projective varieties over k. In the 60's Grothendieck proved there exists a category $\mathcal{M}_k$ called the category of motives and a functor $h : \mathcal{V}_k \longrightarrow \mathcal{M}_k$ that factorises the different Weil cohomology theories, such as the ℓ-adic, singular or de Rham cohomologies.

If X is a smooth projective variety over k acted on by a finite group G, the quotient variety X/G is no longer necessarily smooth so, a priori, the Grothendieck motive of X/G is not defined, however one can reasonably define $h(X/G)$ to be the G-invariant part of $h(X)$, which is an object of $\mathcal{M}_k$, this definition is consistent

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with the realisation functors and Chow groups. Recently, in the case $\operatorname{char} k = 0$, Guillén and Navarro Aznar have given in [4] an extension of the functor h to arbitrary schemes taking values in the homotopy category of complexes of motives, $Ho\,C^b\mathcal{M}_k$, in particular it provides with another possible definition of $h(X/G)$. The main result of this note is that these two definitions coincide. In particular if we call *pure* the objects in the essential image of the functor $\mathcal{M}_k \longrightarrow Ho\,C^b\mathcal{M}_k$, then $h(X/G)$ is pure as expected.

Contents We start in Section 1 by extending the theory of Chow motives to the category of varieties that are quotients of a smooth projective varieties by finite groups. In the next section we present a variant of the extension principle in [4], and as a corollary we obtain the equality $h(X/G) = h(X)^G$ in the introduction. In Section 3, given a finite group G, we prove there exist functors from the category of G-schemes to a certain category of G-motives, and that a morphism of groups defines restriction functors on both categories of schemes and motives. Given a morphism of finite groups in Section 4 we define induction functors on both categories of schemes and of motives. We prove that these induction functors are adjoint of the restriction functors. Section 5 is devoted to the proof of the isomorphism $h_c(X/G) \simeq h_c(X)^G$ for an arbitrary scheme X. This equality is consequence of a more general, functorial statement. In the last section we present some applications of the ideas developed so far. We start considering decompositions of a G-motive induced by an irreducible representation of G. At the level of the Grothendieck group, $K_0\mathcal{M}_k$, these decompositions were considered by Denef and Loeser in [1], we can prove an assertion conjectured in *loc. cit.* (Corollary 6.3). We end this section studying the concept of a motive with coefficients in a local system of motives with finite monodromy.

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1. The pure motive of quotient varieties

In this section we generalise the theory of Chow motives to varieties which are quotients of smooth projective varieties by finite groups. For this recall how the category of motives, $\mathcal{M}_k$, is constructed (see [6], [7]): one first considers the category of correspondences, $\mathcal{CV}_k$, defined by

$$Ob(\mathcal{CV}_k) = Ob(\mathcal{V}_k)$$
$$\operatorname{Hom}_{\mathcal{CV}_k}(X,Y) = CH_{\mathbb{Q}}^{\dim X}(X \times Y)\,,$$

where CH^n stands for the group of algebraic cycles of codimension n modulo rational equivalence and $CH^n_{\mathbb{Q}}$ is CH^n tensored by $\mathbb{Q}$. Composition of correspondences is given by

$$f \circ g = p_{XZ*}\left(p^*_{XY} f \cdot p^*_{YZ} g\right) \tag{1}$$

for $f \in \mathrm{Hom}_{\mathcal{CV}_k}(X,Y)$ and $g \in \mathrm{Hom}_{\mathcal{CV}_k}(Y,Z)$, where p_{XY}, p_{YZ} and p_{XZ} are the projections $p_{XY} : X \times Y \times Z \longrightarrow X \times Y, \dots$. This makes $\mathcal{CV}_k$ into an additive tensor category. The category of effective motives, $\mathcal{M}_k$, is the pseudoabelian closure of $\mathcal{CV}_k$. There is a natural contravariant functor $h : \mathcal{V}_k \longrightarrow \mathcal{M}_k$ defined by $h(X) = (X, \Delta_X)$.

Formula (1) makes sense and verifies the axioms of a category thanks to the following facts:

1. If X is a smooth projective variety $CH^*(X)$ has a natural ring structure given by intersection theory.
2. Given X, Y smooth projective varieties and a morphism $f : X \longrightarrow Y$ there is natural ring morphism

$$f^* : CH^*(Y) \longrightarrow CH^*(X)$$

and a natural morphism of $CH^*(Y)$-modules

$$f_* : CH^*(X) \longrightarrow CH^{*-\dim X + \dim Y}(Y)$$

($CH^*(X)$ is a $CH^*(Y)$-module via f^*).

This is not available in general for arbitrary varieties, but in [2], 17.4.10, it is proved to hold for varieties of the type X/G, with X smooth and projective and G a finite group, provided we tensor these groups by $\mathbb{Q}$.

Let $\mathcal{V}'_k$ be the category of varieties of the type X/G, with $X \in Ob\,\mathcal{V}_k$ and G a finite group. The previous paragraph shows that the definition of $\mathcal{CV}_k$ still makes sense using $\mathcal{V}'_k$ and one obtains the category of correspondences $\mathcal{CV}'_k$. Let $\mathcal{M}'_k$ the pseudoabelian closure of $\mathcal{CV}'_k$: its objects are of the form (X,p), with $X \in Ob\,\mathcal{V}'_k$ and $p \in \mathrm{End}_{\mathcal{CV}'_k}(X)$ a projector, i.e. $p^2 = p$. Morphisms in $\mathcal{M}'_k$ are defined, as in $\mathcal{M}_k$, by

$$\mathrm{Hom}_{\mathcal{M}'_k}\big((X,p),(Y,q)\big) = q \circ \mathrm{Hom}_{\mathcal{CV}'_k}(X,Y) \circ p\,.$$

Also in this case there is a natural contravariant functor $h' : \mathcal{V}'_k \longrightarrow \mathcal{M}'_k$.

As in the case of motives in $\mathcal{M}_k$, by defining

$$CH^*_{\mathbb{Q}}(X,p) = \operatorname{Im}\big(p_* : CH^*_{\mathbb{Q}}(X) \longrightarrow CH^*_{\mathbb{Q}}(X)\big)$$

we obtain a functor $CH^*_{\mathbb{Q}} : \mathcal{M}'_k \longrightarrow \mathbf{VEC}_{\mathbb{Q}}$ that factorises $CH^*_{\mathbb{Q}} : \mathcal{V}'_k \longrightarrow \mathbf{VEC}_{\mathbb{Q}}$, here $\mathbf{VEC}_{\mathbb{Q}}$ stands for the category of vector spaces over $\mathbb{Q}$ of *arbitrary* dimension. The following is a version of part of Manin's identity principle ([6]) for the category $\mathcal{M}'_k$.

Lemma 1.1

A morphism $f \in \operatorname{Hom}_{\mathcal{M}'_k}(M,N)$, is an isomorphism if, and only if, the induced morphisms in Chow groups

$$CH^*_{\mathbb{Q}}(f \otimes Id) : CH^*_{\mathbb{Q}}\big(M \otimes h'(S)\big) \longrightarrow CH^*_{\mathbb{Q}}\big(N \otimes h'(S)\big)$$

are isomorphisms for any $S \in Ob\,\mathcal{V}'_k$.

Proof. The proof is the same as in $\mathcal{M}_k$:

By Yoneda's lemma the functor

$$\begin{aligned} \mathcal{M}'_k &\longrightarrow (\mathcal{M}'_k, \mathbf{Sets}) \\ M &\longmapsto \Big(N \mapsto \operatorname{Hom}_{\mathcal{M}'_k}(N,M)\Big) \end{aligned}$$

is fully faithful. This implies that $M \longrightarrow N$ is an isomorphism if, and only if, for any $P \in Ob\,\mathcal{M}'_k$ the induced morphism

$$\operatorname{Hom}_{\mathcal{M}'_k}(P,M) \longrightarrow \operatorname{Hom}_{\mathcal{M}'_k}(P,N)$$

is an isomorphism. The lemma results now from the fact that $\operatorname{Hom}_{\mathcal{M}'_k}(P,M) = CH^0_{\mathbb{Q}}(M^\vee \otimes P)$ and that any motive is a direct factor of a motive of the type $h'(S)$ for $S \in Ob\,\mathcal{V}'_k$ (see [7]). □

From the definition we get a commutative diagram of functors

$$\begin{array}{ccccc} \mathcal{V}_k & \longrightarrow & \mathcal{C}\mathcal{V}_k & \longrightarrow & \mathcal{M}_k \\ \downarrow & & \downarrow & & \downarrow \Phi \\ \mathcal{V}'_k & \longrightarrow & \mathcal{C}\mathcal{V}'_k & \longrightarrow & \mathcal{M}'_k \end{array}$$

Proposition 1.2

The functor Φ is an equivalence of categories.

Proof. It is clear that Φ is fully faithful, for morphisms are defined in the same way in $\mathcal{M}_k$ and in $\mathcal{M}'_k$, thus it remains to see that Φ is essentially surjective. Let $(X',p') \in Ob\mathcal{M}'_k$, there exists a finite group G acting on a smooth projective variety X, such that $X' \simeq X/G$. Let $\pi : X \longrightarrow X'$ be the quotient morphism and $p = (\pi\times\pi)^* p'$, we claim π^* induces an isomorphism

$$(X',p') \xrightarrow[\simeq]{\pi^*} \left(X, \frac{p}{|G|} \circ \frac{1}{|G|}\sum[g]\right)$$

for this it is enough to see that the diagram

$$\begin{array}{ccc} h'X' & \xrightarrow{\pi^*} & \left(X, \frac{1}{|G|}\sum[g]\right) \\ {\scriptstyle p'}\Big\downarrow & & \Big\downarrow{\scriptstyle \frac{p}{|G|}} \\ h'X' & \xrightarrow{\pi^*} & \left(X, \frac{1}{|G|}\sum[g]\right) \end{array}$$

commutes and that its horizontal arrows are isomorphisms. By [7] (1.10) we have

$$\begin{aligned} \pi^* \circ p' &= (Id_{X'} \times \pi)^* p' \\ p \circ \pi^* &= (\pi \times Id_X)_* p = (\pi \times Id_X)_* (\pi \ \times \pi)^* p' \\ &= |G| \cdot (Id_{X'} \times \pi)^* p' \end{aligned}$$

this proves that the previous diagram commutes. By [2] 1.7.6 the morphism

$$(\pi \times Id_S)^* : CH^*_{\mathbb{Q}}(X' \times S) \longrightarrow CH^*_{\mathbb{Q}}(X \times S)^G$$

is an isomorphism for any $S \in Ob\mathcal{V}'_k$, so from Manin's identity principle, extended to $\mathcal{M}'_k$ in Lemma 1.1, it follows that

$$h'X' \xrightarrow{\pi^*} \left(X, \frac{1}{|G|}\sum[g]\right)$$

is an isomorphism. □

Choose an inverse to Φ, this gives an extension of $h : \mathcal{V}_k \longrightarrow \mathcal{M}_k$ to a functor

$$h' : \mathcal{V}'_k \longrightarrow \mathcal{M}_k$$

such that $h'(X/G) \simeq h(X)^G$ for any smooth projective variety, X, acted on by a finite group G.

In some of the considerations that will arise later in the note it is convenient to use motives with coefficients in fields larger than $\mathbb{Q}$. This is the content of the following definition.

Definition. The category of Chow motives over k with coefficients in an extension field E of $\mathbb{Q}$, $\mathcal{M}_{k,E}$, is defined to be the pseudoabelian closure of the category of E-correspondences, $\mathcal{CV}_{k,E}$, defined by

$$\begin{aligned} Ob\,\mathcal{CV}_{k,E} &= Ob\,\mathcal{CV}_k \\ \mathrm{Hom}_{\mathcal{CV}_{k,E}}(X,Y) &= \mathrm{Hom}_{\mathcal{CV}_k}(X,Y)\otimes_{\mathbb{Q}} E\,. \end{aligned}$$

Note that there is a faithful functor $\mathcal{M}_k \longrightarrow \mathcal{M}_{k,E}$ that may not be essentially surjective.

The results in this note extend immediately to this new category of motives.

2. The extension principle

In [4] an extension criterion has been devised that allows to extend a functor defined on a category of schemes $\mathbf{M}$ with values in a descent category $(\mathbf{D}, E, \mathbf{s})$ to a larger category of schemes $\mathbf{W}$ that contains $\mathbf{M}$ as a full subcategory. We shall briefly recall the set up in which this criterion applies, and refer to [4] for further details.

First recall that a category of descent is a triple $(\mathbf{D}, E, \mathbf{s})$ where $\mathbf{D}$ is a category, E is a class of morphisms of $\mathbf{D}$ and $\mathbf{s}$ is a functor that associates to any cubical diagram in $\mathbf{D}$, $X_\bullet : \Box \longrightarrow \mathbf{D}$, an object in $\mathbf{D}$ called its *simple*, $\mathbf{s}_\Box X_\bullet$. This triple is required to verify some axioms ([4] (1.5)). The descent category we consider in this note is the following: let $\mathcal{A}$ be an additive category, take $\mathbf{D} = C^b\mathcal{A}$ to be the category of bounded complexes in $\mathcal{A}$, E the homotopy equivalences and $\mathbf{s}$ the functor that takes a multiple complex to its associated simple complex. In [4] (1.10) it is proved that this is a descent category.

Let $\mathbf{W} = \mathbf{Sch_k}$ be the category of separated schemes of finite type over a field k of zero characteristic and $\mathbf{M} = \mathbf{Reg_k}$ the full subcategory of smooth schemes. Define an *acyclic diagram* in $\mathbf{W}$ to be a cartesian diagram of the type

(2)
$$\begin{array}{ccc} \widetilde{Y} & \hookrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & X \end{array}$$

in which the vertical arrows are proper, horizontal arrows are closed immersions and the vertical arrow on the right induces an isomorphism $\widetilde{X} - \widetilde{Y} \simeq X - Y$. An *elementary acyclic diagram* will be a blow up diagram in $\mathbf{M}$.

The main theorem in [4] is:

Theorem 2.1 ([4] (2.2))

Given F a functor from $\mathbf{M}$ to a descent category

$$F : \mathbf{M} \longrightarrow (\mathbf{D}, E, \mathbf{s})$$

such that

(F1) $F(\emptyset) = 0$.

(F2) *The natural morphism $F(X \sqcup Y) \longrightarrow F(X) \times F(Y)$ is an isomorphism.*

(F3) *For any elementary acyclic diagram, $X_\bullet$, the object $\mathbf{s}F(X_\bullet)$ is acyclic.*

There exists an essentially unique extension of F to a functor

$$F : \mathbf{W} \longrightarrow Ho\,\mathbf{D}$$

such that the following descent property holds.

(D) *For any acyclic diagram $X_\bullet$, the object $\mathbf{s}F(X_\bullet)$ is defined and is acyclic.*

The proof of this theorem relies, via the theory of cubical hyperresolutions of [5], on the following two results of Hironaka:

(H) For any object X in $\mathbf{W}$ there is an acyclic diagram as in (2) in which $\widetilde{X} \in Ob\mathbf{M}$ and $\dim Y, \dim \widetilde{Y}' < \dim X$.

(CH) For any acyclic diagram as (2) with X, $\widetilde{X} \in Ob\mathbf{M}$ there is a sequence of elementary diagrams $\widetilde{X}_0 \longrightarrow \widetilde{X}_1 \longrightarrow \cdots \longrightarrow X$ that factors via a morphism $\widetilde{X}_0 \longrightarrow \widetilde{X}$.

As well as the following obvious fact:

(0) All the objects in $\mathbf{W}$ of dimension zero are in $\mathbf{M}$.

As remarked in [4] (2.5), these results, and hence the extension theorem, are available for other instances of pairs $(\mathbf{M}, \mathbf{W})$.

For example, if $\mathbf{M} = \mathcal{V}_k$ is the category of smooth projective varieties and $\mathbf{W} = \mathbf{Sch}_{k,\mathbf{comp}}$ is the category of schemes proper over k then (H), (CH) and (0) hold and Theorem 2.1 is valid for these categories. This example is of particular relevance for the theory of motives, for the functor $h : \mathcal{V}_k \longrightarrow \mathcal{M}_k \longrightarrow C^b\mathcal{M}_k$ clearly verifies (F1) and (F2) whereas (F3) is a consequence of Manin ([6], [4] (6.1)). This yields an essentially unique extension $h : \mathbf{Sch}_{k,\mathbf{comp}} \longrightarrow Ho\, C^b\mathcal{M}_k$.

We want to present a version of Theorem 2.1 for the categories $\mathbf{M} = \mathcal{V}_k'$ and $\mathbf{W} = \mathbf{Sch}_{k,\mathbf{comp}}$, for this we need to define acyclic diagrams and elementary acyclic diagrams in this set up. Define an acyclic diagram in $\mathbf{Sch}_{k,\mathbf{comp}}$ in the same way as in $\mathbf{Sch}_k$, that is, as a cartesian diagram of the type

$$\begin{array}{ccc} \widetilde{Y} & \hookrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ \widetilde{Y} & \hookrightarrow & X \end{array}$$

in which the vertical maps are proper, horizontal maps are closed immersions and the right vertical map induces an isomorphism $\widetilde{X} - \widetilde{Y} \simeq X - Y$. But now an elementary acyclic diagram in $\mathcal{V}_k'$ will be a diagram, $X_\bullet'$, as the previous one obtained by taking the quotient by a finite group G acting on a blow up diagram of smooth projective varieties $X_\bullet$, that is $X_\bullet' = X_\bullet/G$.

Our result is

Theorem 2.2

If F is a functor from $\mathcal{V}_k'$ to a descent category $(\mathbf{D}, E, \mathbf{s})$ such that (F1), (F2) and (F3) hold, then there exists an essentially unique extension of F to a functor $F : \mathbf{Sch}_{k,\mathbf{comp}} \longrightarrow Ho\,\mathbf{D}$ satisfying (D).

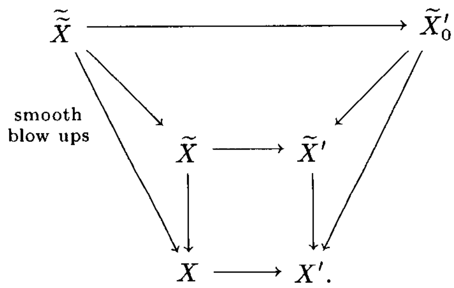
Proof. The proof is along the lines of the proof of Theorem 2.1 and its variations in [4]. We need to check properties (H), (CH) and (0) in the present context.

(0) is trivial, whereas (H) is as before a consequence of the theorem of resolution of singularities of Hironaka.

For (CH) we need to prove that any birational proper morphism $\widetilde{X}' \longrightarrow X'$ in $\mathcal{V}_k'$ is dominated by a sequence of elementary acyclic diagrams. The variety X' will be of the form X/G, let $\pi : X \longrightarrow X'$ be the quotient map. If $\widetilde{X}'$ is the blow up of X' along the sheaf of ideals $\mathcal{I}$ then G acts on the sheaf of ideals $\pi^*\mathcal{I}$ and if $\widetilde{X}$ is the blow up of X along $\pi^*\mathcal{I}$ we have a cartesian diagram

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \widetilde{X}' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ \pi\ } & X'\,. \end{array}$$

Now, by the G-equivariant Chow-Hironaka lemma applied to a G-equivariant resolution of $\widetilde{X}$, the morphism $\widetilde{X} \longrightarrow X$ is dominated by a sequence of blow ups with G-invariant smooth centres. Taking quotients by G yields a variety $\widetilde{X}'_0 \in Ob\mathcal{V}'_k$ which is obtained by a sequence of elementary diagrams from X' and that dominates the initial birational proper morphism $\widetilde{X}' \longrightarrow X'$,



This proves (CH). □

Corollary 2.3

The functor $h' : \mathcal{V}'_k \longrightarrow \mathcal{M}_k$ defined in Section 1, has an essentially unique extension to a functor $h' : \mathbf{Sch}_{k,\mathbf{comp}} \longrightarrow HoC^b\mathcal{M}_k$, such that (D) holds.

Proof. This is an application of the previous theorem. The functor h trivially verifies (F1) and (F2). To prove (F3) consider an elementary acyclic diagram $X'_\bullet$,

$$\begin{array}{ccc} \widetilde{Y}' & \hookrightarrow & \widetilde{X}' \\ \downarrow & & \downarrow \\ Y' & \hookrightarrow & X', \end{array}$$

thus there is an elementary acyclic diagram $X_\bullet$ in $\mathcal{V}_k$

$$\begin{array}{ccc} \widetilde{Y} & \hookrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & X \end{array}$$

with a G-action and $X_\bullet = X'_\bullet/G$. The description of the motive of a blow up by Manin ([6], [4] (6.1)) gives that the complex $\mathbf{s}h(X_\bullet)$

$$0 \longrightarrow h(X) \longrightarrow h(\widetilde{X}) \oplus h(Y) \longrightarrow h(\widetilde{Y}) \longrightarrow 0$$

is acyclic. This exact sequence is split and G-equivariant. As taking G-invariants is an additive functor we see that $\mathbf{s}h'(X'_\bullet)$ is acyclic. □

As the functor $h' : \mathcal{V}'_k \longrightarrow \mathcal{M}_k$ restricted to $\mathcal{V}_k$ coincides with $h : \mathcal{V}_k \longrightarrow \mathcal{M}_k$, we have two extensions of $h : \mathcal{V}_k \longrightarrow \mathcal{M}_k$ to $\mathbf{Sch}_{k,\mathbf{comp}}$ that verify (D): the one in [4] $h : \mathbf{Sch}_{k,\mathbf{comp}} \longrightarrow Ho\, C^b\mathcal{M}_k$ and the one in the previous corollary h'. By the unicity of such extensions we have $h \simeq h'$. In particular:

Corollary 2.4

If X is a smooth projective variety acted on by a finite group G then

$$h(X/G) \simeq h(X)^G.$$

3. Motives for G-schemes

Given categories G and $\mathcal{C}$, $(G, \mathcal{C})$ will note the category of functors $G \longrightarrow \mathcal{C}$. If G is a group, we also write G for the category with one object and G as morphisms. Then $(G, \mathcal{C})$ is the category of objects in $\mathcal{C}$ with a G-action.

In this section we reformulate some results of [1] so as to associate to every G-scheme objects in $Ho\,(G, C^b\mathcal{M}_k)$, $h(X)$ and $h_c(X)$.

We first make some remarks on quotients. In the following we will be dealing with the quotient of a scheme X by a finite group G, but such a quotient need not exist as a scheme. In fact a necessary and sufficient condition for it to exist is that the orbit by G of any point is contained in an affine open subset of X. We shall assume this to be the case. However, it follows from results of Artin and Knutson that X/G always exists as an algebraic space, so by results of Gillet and Vistoli on the intersection theory of algebraic spaces, this assumption is not essential. In view of this we shall note the category of G-schemes with the above restriction by $(G, \mathbf{Sch}_k)$. $(G, \mathbf{Sch}^c_k)$ will note the category with the same objects but restricting the morphisms to proper morphisms.

3.1. The G-motive of a G-scheme

In this subsection we define the G-motive associated to a G-scheme, in order to make sense of the following theorems it is important to note that the category $(G, \mathcal{M}_k)$ is additive, therefore by [4] $C^b(G, \mathcal{M}_k) = (G, C^b\mathcal{M}_k)$ is a descent category.

Theorem 3.1

There is an essentially unique extension of $h : (G, \mathcal{V}_k) \longrightarrow (G, \mathcal{M}_k)$ to a functor

$$h_c : (G, \mathbf{Sch}^c_k) \longrightarrow Ho\,(G, C^b\mathcal{M}_k)$$

such that

(D) *For any G-invariant acyclic diagram $X_\bullet$ the object $\mathbf{s}h_c(X_\bullet)$ is defined and is acyclic.*

(E) *Given a closed G-subscheme Y of a G-scheme X, we have a natural isomorphism $h_c(X - Y) \simeq \mathbf{s}\,(h_c(X) \longrightarrow h_c(Y))$.*

Proof. The proof is another application of Theorem 2.2 in [4], see the Appendix to [1]. □

Theorem 3.2

There is an essentially unique extension of $h : (G, \mathcal{V}_k) \longrightarrow (G, \mathcal{M}_k)$ *to a functor*

$$h : (G, \mathbf{Sch}_k) \longrightarrow Ho\,(G, C^b\mathcal{M}_k)$$

such that

(D) *For any G-invariant acyclic diagram $X_\bullet$ the object $\mathbf{s}h(X_\bullet)$ is defined and is acyclic.*

(E) *Given a smooth projective variety X and D a divisor with normal crossings in X that is the union of smooth divisors and such that G acts on $(X, X - D)$, there is a natural isomorphism $h(X - D) \simeq \mathbf{s}\,(h_*(S_\bullet(D) \longrightarrow X))$ (see [4]).*

Proof. The proof is as in [4], see the Appendix to [1]. □

3.2. The restriction functor

Let $\psi : G \longrightarrow G'$ be a morphism of finite groups. We shall call *restriction via* ψ and note Res_ψ any of the following natural functors

$$(G', \mathbf{Sch}_k) \longrightarrow (G, \mathbf{Sch}_k) \qquad (G', C^b\mathcal{M}_k) \longrightarrow (G, C^b\mathbf{Sch}_k)$$

$$(G', \mathbf{Sch}_k^c) \longrightarrow (G, \mathbf{Sch}_k^c) \qquad Ho\,(G', C^b\mathcal{M}_k) \rightarrow Ho\,(G, C^b\mathbf{Sch}_k)\,.$$

By the unicity of functors h and h_c above, it is easy to check that that the following diagrams commute

$$\begin{array}{ccc} (G', \mathbf{Sch}_k) & \xrightarrow{h} & Ho\,(G', C^b\mathcal{M}_k) \\ {\scriptstyle \mathrm{Res}_\psi}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_\psi} \\ (G, \mathbf{Sch}_k) & \xrightarrow{h} & Ho\,(G, C^b\mathcal{M}_k) \end{array} \qquad \begin{array}{ccc} (G', \mathbf{Sch}_k^c) & \xrightarrow{h_c} & Ho\,(G', C^b\mathcal{M}_k) \\ {\scriptstyle \mathrm{Res}_\psi}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}_\psi} \\ (G, \mathbf{Sch}_k^c) & \xrightarrow{h_c} & Ho\,(G, C^b\mathcal{M}_k)\,. \end{array}$$

4. Induction and Frobenius reciprocity

In this section we shall use the constant finite group scheme associated to a finite group G, $\operatorname{Spec} k^G$. For simplicity of notation we still note G this finite group scheme. The motive of this scheme shall be noted $\mathbb{1}[G]$, it is an algebra object in the category $\mathcal{M}_k$, i.e. there are morphisms $\mathbb{1}[G] \otimes \mathbb{1}[G] \longrightarrow \mathbb{1}[G]$ and $\mathbb{1} \longrightarrow \mathbb{1}[G]$ that verify the usual axioms of an associative unitary algebra. The group G then acts on $\mathbb{1}[G]$ on the left and on the right, these two actions will be called the right and left regular representations.

We shall construct for any morphism of finite groups $\psi : G \longrightarrow G'$ a functor Ind_ψ which is a left adjoint of the functor Res_ψ.

Definition. Let $\psi : G \longrightarrow G'$ be a morphism of finite groups and $X \in Ob(G, \mathbf{Sch}_k)$. Then G acts on $G' \times X$ by $g \cdot (g', x) = \big(g'\psi(g^{-1}), gx\big)$ and G' acts on $G' \times X$ by $g \cdot (g', x) = (gg', x)$. These two actions commute and we define

$$\operatorname{Ind}_\psi X = \frac{G' \times X}{G}$$

with its natural G'-action. This defines functors

$$(G, \mathbf{Sch}_k) \xrightarrow{\operatorname{Ind}_\psi} (G', \mathbf{Sch}_k), \qquad (G, \mathbf{Sch}_k^c) \xrightarrow{\operatorname{Ind}_\psi} (G', \mathbf{Sch}_k^c)\,.$$

Recall that in a $\mathbb{Q}$-linear pseudoabelian category, $\mathcal{A}$, for any object M in $(G, \mathcal{A})$, the image of the projector $\frac{1}{|G|}\sum_{g\in G}[g]$ defines an object M^G, of $\mathcal{A}$ called the G-invariant part of M. In this way we obtain an additive functor $(\,\cdot\,)^G : (G, \mathcal{A}) \longrightarrow \mathcal{A}$.

Definition. Let $\psi : G \longrightarrow G'$ be a morphism of finite groups and $M \in Ob(G, \mathcal{M}_k)$. Let G act on $\mathbb{1}[G'] \otimes M$ via the inverse of the right regular representation on $\mathbb{1}[G']$ and the given action on M and let G' act on $\mathbb{1}[G'] \otimes M$ via the left regular representation on $\mathbb{1}[G']$ and trivially on M. These two actions commute and we define

$$\operatorname{Ind}_\psi M = \left(\mathbb{1}[G'] \otimes M\right)^G$$

with its natural G'-action. This defines an additive functor

$$\operatorname{Ind}_\psi : (G, \mathcal{M}_k) \longrightarrow (G', \mathcal{M}_k).$$

Note that Ind_ψ being an additive functor, it automatically gives rise to functors

$$(G, C^b\mathcal{M}_k) \xrightarrow{\operatorname{Ind}_\psi} (G', C^b\mathcal{M}_k) \qquad Ho(G, C^b\mathcal{M}_k) \xrightarrow{\operatorname{Ind}_\psi} Ho(G', C^b\mathcal{M}_k)$$

EXAMPLES:

1. If $\psi : G \longrightarrow \{1\}$ is the trivial morphism, then the functor Ind_ψ is $X \mapsto X/G$ in $(G, \mathbf{Sch}_k)$ and $M \mapsto M^G$ in $(G, \mathcal{M}_k)$.
2. If ψ is a quotient morphism $\psi : G \longrightarrow G/H = G'$, then Ind_ψ takes $X \in Ob(G, \mathbf{Sch}_k)$ to X/H with its residual G/H-action and $M \in Ob\,Ho\,(G, \mathcal{M}_k)$ to M^H with its residual G/H-action.

Whenever we are given finite groups G and G' and it is clear from the context what the morphism $\psi : G \longrightarrow G'$ is, we shall use the notation $\mathrm{Res}_G^{G'}$ and $\mathrm{Ind}_G^{G'}$ for Res_ψ and Ind_ψ respectively. Examples are: G or G' are trivial, G a subgroup of G' or G' a quotient of G.

The following propositions give analogues of the Frobenius reciprocity law in this context.

Proposition 4.1

The functor $\mathrm{Ind}_\psi : (G', \mathbf{Sch}_k) \longrightarrow (G, \mathbf{Sch}_k)$ *is a left adjoint of* Res_ψ*, that is, there is a natural bijection*

$$\mathrm{Hom}_{(G,\mathbf{Sch}_k)}(X, \mathrm{Res}_\psi Y) \xrightarrow{\simeq} \mathrm{Hom}_{(G',\mathbf{Sch}_k)}(\mathrm{Ind}_\psi X, Y)\,.$$

Proof. Let $f : X \longrightarrow \mathrm{Res}_\psi Y$ be a morphism of G-schemes. Then $f'(g', x) = g' \cdot f(x)$ defines a G'-equivariant morphism $f' : G' \times X \longrightarrow Y$. But $f'\left(g'\psi(g^{-1}), gx\right) = f'(g', x)$, therefore f' factors via $\mathrm{Ind}_\psi X$ giving a $h \in \mathrm{Hom}_{(G',\mathbf{Sch}_k)}\,(\mathrm{Ind}_\psi X, Y)$. This defines a map

$$\mathrm{Hom}_{(G,\mathbf{Sch}_k)}(X, \mathrm{Res}_\psi Y) \longrightarrow \mathrm{Hom}_{(G',\mathbf{Sch}_k)}(\mathrm{Ind}_\psi X, Y)\,.$$

To prove it is an isomorphism we construct its inverse. Let $h : \mathrm{Ind}_\psi X \longrightarrow Y$ be a morphism of G'-schemes, then $f(x) = h(1, x)$ defines a map $X \longrightarrow Y$. As for any $g \in G$ we have $f(gx) = h(1, gx) = h(\psi(g), x) = h(\psi(g) \cdot (1, x)) = \psi(g)f(x)$ we see that f defines a G-equivariant morphism $f \in \mathrm{Hom}_{(G,\mathbf{Sch}_k)}\,(X, \mathrm{Res}_\psi)$. It is easy to check that this gives the inverse of the previous map. $\square$

Proposition 4.2

Let $\psi : G \longrightarrow G'$ *be a morphism of finite groups. The functors* $\mathrm{Ind}_\psi : (G, C^b\mathcal{M}_k) \longrightarrow (G', C^b\mathcal{M}_k)$ *and* $\mathrm{Ind}_\psi : Ho(G, C^b\mathcal{M}_k) \longrightarrow Ho(G', C^b\mathcal{M}_k)$ *are left adjoint to* Res_ψ*.*

Proof. We shall prove the statement for the categories $(G, \mathcal{M}_k)$ and $(G', \mathcal{M}_k)$, from the naturality it follows that the morphisms we construct commute with the differentials and the statement for the categories $(G, C^b\mathcal{M}_k)$ and $(G', C^b\mathcal{M}_k)$ follows.

Let $M \in Ob(G, \mathcal{M}_k)$ and $N \in Ob(G', \mathcal{M}_k)$.

We first remark that a G'-action on an object N of $\mathcal{M}_k$ is equivalent to a structure of left module over the algebra $\mathbb{1}[G']$, that is a morphism

$$m : \mathbb{1}[G'] \otimes N \longrightarrow N$$

verifying the axioms of a module over an algebra. In particular if $r : G' \longrightarrow \mathrm{Aut}(\mathbb{1}[G'])$ (resp. $\ell : G' \longrightarrow \mathrm{Aut}(\mathbb{1}[G'])$) is the right (resp. left) regular representation and $g \in G'$ then the diagrams

$$\begin{CD} \mathbb{1}[G'] \otimes N @>m>> N \\ @V{r(g)\otimes[g^{-1}]}VV @VV{Id}V \\ \mathbb{1}[G'] \otimes N @>m>> N \end{CD} \qquad\qquad \begin{CD} \mathbb{1}[G'] \otimes N @>m>> N \\ @V{\ell(g)\otimes[1]}VV @VV{[g]}V \\ \mathbb{1}[G'] \otimes N @>m>> N \end{CD} \tag{3}$$

commute.

To prove the statement in the proposition let $f : M \longrightarrow \mathrm{Res}_\psi N$ be a morphism in $(G, \mathcal{M}_k)$. Let h be the following morphism

$$\mathbb{1}[G'] \otimes M \xrightarrow{Id\otimes f} \mathbb{1}[G'] \otimes N \xrightarrow{m} N$$

This is G'-equivariant because for every $g' \in G'$ we have a diagram

$$\begin{CD} \mathbb{1}[G'] \otimes M @>{Id\otimes f}>> \mathbb{1}[G'] \otimes N @>m>> N \\ @V{\ell(g')\otimes Id}VV @V{\ell(g')\otimes Id}VV @V{[g']}VV \\ \mathbb{1}[G'] \otimes M @>{Id\otimes f}>> \mathbb{1}[G'] \otimes N @>m>> N \end{CD}$$

that is commutative: the first square is obviously commutative whereas the second is (3).

The morphism h is also G-invariant because for every $g \in G$ the diagram

$$\begin{CD} \mathbb{1}[G'] \otimes M @>{Id\otimes f}>> \mathbb{1}[G'] \otimes N @>m>> N \\ @V{r(\psi(g^{-1}))\otimes[g]}VV @V{r(\psi(g^{-1}))\otimes[\psi(g)]}VV @V{Id}VV \\ \mathbb{1}[G'] \otimes M @>{Id\otimes f}>> \mathbb{1}[G'] \otimes N @>m>> N \end{CD}$$

is commutative: the first square commutes because $M \longrightarrow \mathrm{Res}_\psi N$ is G-equivariant whereas the second commutes by (3).

Therefore $h \circ [g] = h$ and h factors via $(\mathbb{1}[G'] \otimes M)^G$. This defines a map

$$\mathrm{Hom}_{(G,\mathcal{M}_k)}(M, \mathrm{Res}_\psi N) \longrightarrow \mathrm{Hom}_{(G',\mathcal{M}_k)}(\mathrm{Ind}_\psi M, N).$$

To prove this is bijective we give its inverse. Let $h : \mathrm{Ind}_\psi M \longrightarrow N$ be a G'-morphism, define f to be the composition

$$M \xrightarrow{\simeq} (\mathbb{1}[G] \otimes M)^G \longrightarrow (\mathbb{1}[G'] \otimes M)^G \xrightarrow{h} N.$$

Then f defines an element of $\mathrm{Hom}_{(G,\mathcal{M}_k)}(M, \mathrm{Res}_\psi N)$. It is easy to see that this provides the inverse of the previous map.

The proof that Ind_ψ is adjoint to Res_ψ in the localised categories follows from the fact that the bijections given above take morphisms homotopic to zero to morphisms homotopic to zero. □

Proposition 4.3

Let $\psi : G \longrightarrow G'$ be a morphism of finite groups, $M \in Ob(G, C^b\mathcal{M}_k)$ and $N \in Ob(G', C^b\mathcal{M}_k)$. Then we have the following projection formula

$$\mathrm{Ind}_\psi(\mathrm{Res}_\psi N \otimes M) \simeq N \otimes \mathrm{Ind}_\psi M.$$

Proof. Adjunction gives a natural morphism of complexes of motives

$$\Pi_{M,N} : \mathrm{Ind}_\psi(\mathrm{Res}_\psi N \otimes M) \longrightarrow N \otimes \mathrm{Ind}_\psi M.$$

As the degree n component of $\Pi_{M,N}$ is $\sum_{i+j=n} \Pi_{M_i,N_j}$ we can reduce to the case M and N are complexes concentrated in degree zero.

To see it is an isomorphism let E be a finite Galois extension of $\mathbb{Q}$ in which all the irreducible complex representations of G and G' are defined. Applying the functor $\mathcal{M}_k \longrightarrow \mathcal{M}_{k,E}$ we see it is enough to check that $\Pi_{M,N}$ is an isomorphism in this extended category for its unique inverse $\Pi_{M,N}^{-1}$ will be invariant under the action of the Galois group $Gal(E|\mathbb{Q})$ and hence will be defined over $\mathbb{Q}$.

In $\mathcal{M}_{k,E}$ we can apply Getzler's Peter-Weyl theorem ([3], Theorem 3.2) and reduce the proof to the categories of representations of G and G' on finite dimensional vector spaces over E where the statement is well known. □

Proposition 4.4

Let $\psi : G \longrightarrow G'$ be a morphism of finite groups, $X \in Ob(G, \mathbf{Sch}_k)$ and $Y \in Ob(G', \mathbf{Sch}_k)$. Then we have the following projection formula

$$\mathrm{Ind}_\psi\left(\mathrm{Res}_\psi Y \times X\right) \simeq Y \times \mathrm{Ind}_\psi X$$

Proof. As in the proof of the previous proposition, adjunction provides us with a natural morphism

$$\Pi_{X,Y} : \mathrm{Ind}_\psi\left(\mathrm{Res}_\psi Y \times X\right) \simeq Y \times \mathrm{Ind}_\psi X$$

which in concrete terms is the map

$$\begin{aligned} \frac{G' \times \mathrm{Res}_\psi Y \times X}{G} &\longrightarrow Y \times \frac{G' \times X}{G} \\ (g', y, x) &\longmapsto (g'y, g', x)\,, \end{aligned}$$

note that this makes sense for it maps $(g'\psi(g^{-1}), \psi(g)y, gx)$ to $(g'y, g'\psi(g^{-1}), gx)$ and this is $(g'y, g \cdot (g', x))$.

We show that $\Pi_{X,Y}$ is an isomorphism by exhibiting its inverse.
Put $\Sigma_{X,Y}(y, g', x) = (g', {g'}^{-1}y, x)$, this defines a morphism

$$\Sigma_{X,Y} : Y \times \frac{G' \times X}{G} \longrightarrow \frac{G' \times \mathrm{Res}_\psi Y \times X}{G}$$

because $\Sigma_{X,Y}(y, g'\psi(g^{-1}), gx) = g(g', {g'}^{-1}, x)$. It is straightforward to check that $\Pi_{X,Y} \circ \Sigma_{X,Y} = Id$ and $\Sigma_{X,Y} \circ \Pi_{X,Y} = Id$. □

5. The motive of a quotient scheme

In this section we extend the isomorphism $h(X)^G \simeq h(X/G)$ in Corollary 2.4 to the case where X is an arbitrary separated scheme of finite type over k. This is a consequence of the following statement.

Theorem 5.1

Let $\psi : G \longrightarrow G'$ be a morphism of finite groups, then the diagram

$$\begin{array}{ccc} (G, \mathbf{Sch}_k^c) & \xrightarrow{h_c} & Ho\,(G, C^b\mathcal{M}_k) \\ \Big\downarrow{\scriptstyle \mathrm{Ind}_\psi} & & \Big\downarrow{\scriptstyle \mathrm{Ind}_\psi} \\ (G, \mathbf{Sch}_k^c) & \xrightarrow{h_c} & Ho\,(G, C^b\mathcal{M}_k) \end{array}$$

commutes. In other words, for every G-scheme X there is a natural isomorphism

$$h_c(\mathrm{Ind}_\psi X) \simeq \mathrm{Ind}_\psi(h_c X).$$

Proof. This is another application of the extension principle of [4].

Let X be a G-scheme and consider the morphism induced by the quotient map $\pi^* : G' \times X \longrightarrow \frac{G' \times X}{G}$:

$$\pi^* : h_c\left(\frac{G' \times X}{G}\right) \longrightarrow \left(\mathbb{1}[G'] \otimes h_c X\right)^G . \tag{4}$$

This defines a morphism of functors

$$h_c \mathrm{Ind}_\psi \longrightarrow \mathrm{Ind}_\psi h_c \,.$$

If X is smooth and projective acted on by G then so is $G' \times X$ and Corollary 2.4 implies that (4) is an isomorphism in this case. Therefore the functors under consideration are isomorphic when restricted to $(G, \mathcal{V}_k)$.

This common restriction clearly verifies (F1) and (F2), whereas (F3) can proved as in the proof of Corollary 2.3. By an obvious variant of Theorem 3.1 there exists a unique extension to $(G, \mathbf{Sch}_k)$ such that (D) and (E) hold, so it is clearly enough to see that both $\mathrm{Ind}_\psi h_c$ and $h_c \mathrm{Ind}_\psi$ satisfy (D) and (E). This is a consequence of the fact that Ind_ψ takes acyclic diagrams to acyclic diagrams. □

Corollary 5.2

Given a finite group G, a normal subgroup H of G and $X \in Ob(G, \mathbf{Sch}_k)$ we have a natural isomorphism in $Ho\,(G/H, C^b\mathcal{M}_k)$

$$h_c(X)^H \;\simeq\; h_c(X/H).$$

Proof. Apply the previous theorem to the quotient morphism $\psi : G \longrightarrow G' = G/H$. □

Corollary 5.3

Given a finite group G and $X \in Ob(G, \mathbf{Sch}_k)$ there is a canonical isomorphism

$$h_c(X)^G \;\simeq\; h_c(X/G).$$

6. Applications

6.1. Isotypical decompositions

In this subsection we shall consider decompositions of a G-motive induced by a representation of G. Let E be a finitely generated extension of $\mathbb{Q}$.

Definition. Let α be an effective character of a finite group G defined over E, and let $G \longrightarrow GL(V_\alpha)$ be the corresponding representation and $V_\alpha^\vee$ the dual representation. For an object M of $(G, C^b\mathcal{M}_{k,E})$ we define

$$M_\alpha = \mathrm{Ind}_G^1(V_\alpha^\vee \otimes M) \in Ob\, C^b\mathcal{M}_{k,E}.$$

This defines an additive functor $(\,\cdot\,)_\alpha : (G, C^b\mathcal{M}_{k,E}) \longrightarrow C^b\mathcal{M}_{k,E}$ that localises to a functor $(\,\cdot\,)_\alpha : Ho\,(G, C^b\mathcal{M}_{k,E}) \longrightarrow Ho\, C^b\mathcal{M}_{k,E}$. For a G-scheme, X, put

$$h(X,\alpha) = (hX)_\alpha$$
$$h_c(X,\alpha) = (h_cX)_\alpha\,.$$

Let $X_E(G)$ note the group of characters of G defined over E. We can prove the following results of Denef and Loeser.

Theorem 6.1 ([1] Theorem 1.3.1)

There is a unique function

$$\chi_c : Ob\,(G, \mathbf{Sch}_k) \times X_E(G) \longrightarrow K_0\mathcal{M}_{k,E}$$

such that

1. *χ_c is a group morphism in the second variable.*
2. *For a smooth projective G-variety, X, and an irreducible character $\alpha \in X_E(G)$ of dimension n_α we have that $n_\alpha\chi_c(X,\alpha)$ is the class of the motive $\left(X, \frac{n_\alpha}{|G|}\sum_{g\in G}\alpha(g^{-1})[g]\right)$ in $K_0\mathcal{M}_{k,E}$.*
3. *If Y is a closed subscheme of X then $\chi_c(X-Y,\alpha) = \chi_c(X,\alpha) - \chi_c(Y,\alpha)$ for any $\alpha \in X_E(G)$.*
4. *If all irreducible complex representations of G are defined over E, then we have $\chi_c(X) = \sum_\alpha n_\alpha\chi_c(X,\alpha)$ where α runs over the irreducible characters of G over E and n_α indicates the dimension of α.*

Proof. Let $[\,\cdot\,]$ note the class of an object in K_0.

For $\alpha \in X_E(G)$ effective, define $\chi_c(X,\alpha) = [h_c(X,\alpha)]$. This is linear in α hence it extends to an additive function on $X_E(G)$.

In order to prove the second condition note that for X a smooth projective G-variety, and $\alpha \in X_E(G)$ an irreducible character of dimension n_α, we have a morphism in $\mathcal{M}_{k,E}$

$$\left(X, \frac{n_\alpha}{|G|}\sum_{g\in G}\alpha(g^{-1})[g]\right) \hookrightarrow hX \twoheadrightarrow h(X,\alpha)\otimes V_\alpha$$

which induces isomorphism in Chow groups by the theory of representations of finite groups. Manin's identity principle yields the result.

To prove the third condition let α be an effective character and Y a closed G-invariant subscheme of a G-scheme X, then, by property (E) in Theorem 3.1, we have an isomorphism in $Ho\,(G, C^b\mathcal{M}_{k,E})$,

$$h_c(X-Y) \simeq \mathbf{s}\,(h_cX \longrightarrow h_cY)\,.$$

Applying the functor $(\,\cdot\,)_\alpha$ we get

$$h_c(X-Y,\alpha) \simeq \mathbf{s}\big(h_c(X,\alpha) \longrightarrow h_c(Y,\alpha)\big),$$

taking the class in $K_0\mathcal{M}_{k,E}$ we see that $[h_c(X-Y,\alpha)] = [h_c(X,\alpha)] - [h_c(Y,\alpha)]$, as required. The general statement follows from linearity on the second argument.

The last statement is a consequence of Getzler's Peter-Weyl theorem ([3], Theorem 3.2), which, in this context, asserts that $h(X) \simeq \oplus_\alpha h(X,\alpha)\otimes V_\alpha$ where α runs over the irreducible characters. □

Theorem 6.2 ([1] Theorem 1.3.2)

There is a unique function

$$\chi : Ob\,(G, \mathbf{Sch}_k) \times X_E(G) \longrightarrow K_0\mathcal{M}_{k,E}$$

such that

1. *χ is a group morphism in the second variable.*
2. *For a smooth projective G-variety, X, and an irreducible character $\alpha \in X_E(G)$ of dimension n_α, we have that $n_\alpha\chi(X,\alpha)$ is the class of the motive $\Big(X, \frac{n_\alpha}{|G|}\sum_{g\in G}\alpha(g^{-1})[g]\Big)$.*

3. If

$$\begin{array}{ccc} \widetilde{Y} & \hookrightarrow & \widetilde{X} \\ \downarrow & & \downarrow \\ Y & \hookrightarrow & X \end{array}$$

is an acyclic diagram in $(G, \mathbf{Sch}_k)$ *and* $\alpha \in X_E(G)$ *then* $\chi(X, \alpha) = \chi(\widetilde{X}, \alpha) + \chi(Y, \alpha) - \chi(\widetilde{Y}, \alpha)$.

4. If D *is a smooth* G*-invariant divisor of a smooth* G*-scheme* X *and* $\alpha \in X_E(G)$. *Then* $\chi(X - D, \alpha) = \chi(X, \alpha) - \chi(D, \alpha)(-1)$.

5. If all complex representations of G *are defined over* E, *then we have* $\chi(X) = \sum_\alpha n_\alpha \chi(X, \alpha)$ *where* α *runs over the irreducible characters of* G *over* E *and* n_α *indicates the dimension of* α.

Proof. Define, for an effective $\alpha \in X_E(G)$, $\chi(X, \alpha) = [h(X, \alpha)]$. This is linear in α hence it extends to an additive function on $X_E(G)$.

Properties 2 and 5 are proved exactly in the same way as in the preceding theorem. The remaining is a consequence of (D) and (E) in Theorem 3.2. □

The following result was conjectured by Denef and Loeser ([1], Assertion 1.5.2).

Proposition 6.3

Let $\psi : G \longrightarrow G'$ *be a morphism of finite groups,* α *a character of* G' *and* X *an object of* $(G, \mathbf{Sch}_k)$. *We have*

$$\chi_c(\mathrm{Ind}_\psi X, \alpha) \; = \; \chi_c(X, \mathrm{Res}_\psi \alpha).$$

Proof. By linearity we may assume α to be effective. By the proof of Theorem 6.1 $\chi_c(\mathrm{Ind}_\psi X, \alpha)$ is the class of the motive $h_c(\mathrm{Ind}_\psi X, \alpha)$ which is by definition

$$\mathrm{Ind}^1_{G'} \left(V_\alpha^\vee \otimes h_c \mathrm{Ind}_\psi X \right).$$

But, in Theorem 5.1, we have seen that $h_c(\mathrm{Ind}_\psi X) \simeq \mathrm{Ind}_\psi h_c(X)$ so this is

$$\mathrm{Ind}^1_{G'} \left(V_\alpha^\vee \otimes \mathrm{Ind}_\psi h_c X \right).$$

By the projection formula in Proposition 4.3 this equals

$$\mathrm{Ind}^1_{G'} \big(\mathrm{Ind}_\psi \left(\mathrm{Res}_\psi V_\alpha^\vee \otimes h_c X \right) \big).$$

As Ind is functorial for group morphisms this equals

$$\mathrm{Ind}^1_G \left(V^\vee_{\mathrm{Res}_\psi \alpha} \otimes h_c X \right) = h_c(X, \mathrm{Res}_\psi \alpha).$$

Again, by the proof of Theorem 6.1, the class of this in $K_0 \mathcal{M}_{k,E}$ is $\chi_c(X, \mathrm{Res}_\psi \alpha)$, this proves the proposition. □

6.2. Motives with coefficients in a local system of motives with finite monodromy

Recall that a local system, L, of vector spaces over a differentiable manifold X is a vector bundle in which the transition matrices can be taken to be constant. This is equivalent to giving a representation $\pi_1(X,x) \longrightarrow GL(L_x)$ called the monodromy representation.

Let X be a scheme over k and E a field of characteristic zero. We define a local system, L, of E-motives over X to be a representation,

$$\rho : \pi_1(X,\overline{x}) \longrightarrow \mathrm{Aut}_{\mathcal{M}_{k,E}}(M)$$

where $\overline{x}$ is a geometric point of X and M is a motive which we note by $L_{\overline{x}}$.

Our aim of this paragraph is to define the motive of X with coefficients in a local system of motives L. We are able to do this when the monodromy map, ρ, is continuous with the natural topologies on both groups (profinite in $\pi_1(X,\overline{x})$ and discrete in $\mathrm{Aut}_{\mathcal{M}_{k,E}}(L_{\overline{x}})$), that is when the image of ρ is finite.

We start by giving some examples of local systems of motives.

Definition. Let $f : X \longrightarrow Y$ be a morphism of k-schemes and let $\pi_1(f) : \pi_1(X,\overline{x}) \longrightarrow \pi_1(Y,\overline{y})$ be the associated morphism in the fundamental groups.

1. If L is a local system over Y associated to a representation ρ of $\pi_1(Y,\overline{y})$. Define f^*L to be the local system associated to the representation $\mathrm{Res}_{\pi_1(f)}(\rho)$ of $\pi_1(X,\overline{x})$.
2. If L is a local system over X associated to a representation ρ of $\pi_1(X,\overline{x})$. If $\pi_1(f)$ has finite cokernel define f_*L to be the local system associated to the representation $\mathrm{Ind}_{\pi_1(f)}(\rho)$ of $\pi_1(Y,\overline{y})$.

Examples: ([1])

1. Let $\mathbb{G}_m = \mathrm{Spec}\, k[t,t^{-1}]$ be the multiplicative group scheme over k. The fundamental group of $\mathbb{G}_m$ is an extension of $Gal(\overline{k}|k)$ by $\mu_\infty(\overline{k}) = \varprojlim \mu_d(\overline{k})$,

$$1 \longrightarrow \mu_\infty(\overline{k}) \longrightarrow \pi_1(\mathbb{G}_m, 1) \longrightarrow Gal(\overline{k}|k) \longrightarrow 1\,.$$

   A k-point of $\mathbb{G}_m$ determines a section of the surjection $\pi_1(\mathbb{G}_m,1) \longrightarrow Gal(\overline{k}|k)$ hence an action of the Galois group on $\mu_\infty(\overline{k})$. If we assume $\mu_d(k) = \mu_d(\overline{k})$, then $Gal(\overline{k}|k)$ acts trivially on $\mu_d(\overline{k})$ and a character

$$\alpha : \mu_d(k) \longrightarrow GL_1 k$$

induces a representation

$$\pi_1(\mathbb{G}_m, 1) \longrightarrow \mu_d(k) \xrightarrow{\alpha} GL_1 k = \mathrm{Aut}_{\mathcal{M}_k}(\mathbb{1})\,.$$

We call the local systems over $\mathbb{G}_m$ arising in this way Kummer sheaves or Kummer local systems and use the notation L_α for the Kummer sheaf defined above.

2. Let X be a scheme over k, a Kummer sheaf over X is the pullback, f^*L_α, of a Kummer sheaf on $\mathbb{G}_m$ via a morphism $f : X \longrightarrow \mathbb{G}_m$.

Let G be the image of the monodromy map and let $\pi : \widetilde{X} \longrightarrow X$ be the étale cover determined by the morphism $\pi_1(X, \overline{x}) \longrightarrow G$. We then define.

Definition. The motive of X with coefficients in the local system of motives L is defined to be the following objects of $Ho\,C^b\mathcal{M}_{k,E}$.

$$h(X, L) = \left(h\widetilde{X} \otimes L_{\overline{k}}\right)^G,$$
$$h_c(X, L) = \left(h_c\widetilde{X} \otimes L_{\overline{k}}\right)^G.$$

Note that if X is smooth and projective then for any such local system L over X, $h(X, L) = h_c(X, L)$ is a pure motive.

Example: Let X, f and α be as in the previous example, to this data in there is associated in [1] §1 an element of $K_0\mathcal{M}_{k,E}$, $[X, f^*L_\alpha]$. With our notations we see that this is just the class of $h_c(X, f^*L_\alpha)$, the motive with compact supports of X with coefficients in the Kummer local system f^*L_α.

The following elementary properties are motivic versions of well known results for the cohomology with coefficients in a local system.

Proposition 6.4

1. *Let L and L' be local systems over a scheme X, there are natural isomorphisms $h(X, L \oplus L') \simeq h(X, L) \oplus h(X, L')$ and $h_c(X, L \oplus L') \simeq h_c(X, L) \oplus h_c(X, L')$.*
2. *Let X be a smooth scheme and L a local system over X, there is a natural isomorphism $h(X, L) \simeq h_c(X, L^\vee)^\vee(-\dim X)$.*

Proof. We treat the case without supports, the proof is the same for the motive with compact supports. For 1 note that if L, L' are associated to representations ρ, ρ' respectively. Then $L \oplus L'$ is associated to the representation $\rho \oplus \rho'$. This has image H contained in $G \times G'$, where $G = \mathrm{Im}\rho$ and $G' = \mathrm{Im}\rho'$. We have a diagram of étale coverings of X,

$$\begin{array}{ccccc} & & \widetilde{X}_H & & \\ & \swarrow & & \searrow & \\ \widetilde{X}_G & & \Big\downarrow & & \widetilde{X}_{G'} \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

But by elementary properties of the functor $(\,\cdot\,)^G = \mathrm{Ind}_1^G$ we have

$$\begin{aligned} h(X, L \oplus L') &= \left(h\widetilde{X}_H \otimes (L_{\overline{x}} \oplus L'_{\overline{x}})\right)^H \\ &\simeq \left(h\widetilde{X}_H \otimes L_{\overline{x}} \oplus h\widetilde{X}_H \otimes L'_{\overline{x}}\right)^H \\ &\simeq \left(h\widetilde{X}_G \otimes L_{\overline{x}}\right)^G \oplus \left(h\widetilde{X}_{G'} \otimes L_{\overline{x}}\right)^{G'} \\ &\simeq h(X, L) \oplus h(X, L')\,. \end{aligned}$$

To prove 2 note that the quotients of $\pi_1(X, \overline{x})$ determined by L and $L^\vee$ coincide. If $\widetilde{X}$ is the Galois covering determined by this quotient, $\pi_1(X) \longrightarrow G$, by definition we have $h(X, L) = \left(h(\widetilde{X}) \otimes L_{\overline{x}}\right)^G$ and $h_c(X, L^\vee) = \left(h_c(\widetilde{X}) \otimes L^\vee_{\overline{x}}\right)^G$. It is clearly enough to prove that there is an isomorphism $h(\widetilde{X}) \simeq h_c(\widetilde{X})^\vee(-\dim X)$ in $Ho(G, C^b\mathcal{M}_k)$.

By applying the G-equivariant Chow lemma we can assume $\widetilde{X}$ is quasiprojective. Let $\overline{X}$ be a smooth G-equivariant projective completion of $\widetilde{X}$ such that $D = \overline{X} - \widetilde{X}$ is a divisor with normal crossings union of smooth divisors. By (E) in Theorem 3.2 we have an isomorphism

$$h(\widetilde{X}) \simeq \mathbf{s}\left(h_*\left(S_\bullet D \longrightarrow \overline{X}\right)\right)(-\dim X)\,.$$

The functor h_* is $h_c(\,\cdot\,)^\vee$, and the previous gives an isomorphism

$$h(\widetilde{X}) \simeq \mathbf{s}\left(h_c D \longleftarrow h_c\overline{X}\right)^\vee(-\dim X)\,.$$

By (E) in Theorem 3.1 this is isomorphic to $h_c(\widetilde{X})^\vee(-\dim X)$. This proves 2. □

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