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On the k-regularity of some projective manifolds

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In memory of F. Serrano

Abstract

The conjecture on the (*degree* – *codimension* + 1) - regularity of projective varieties is proved for smooth linearly normal polarized varieties (X, L) with L very ample, for low values of $\Delta(X, L) =$ *degree* – *codimension* –1. Results concerning the projective normality of some classes of special varieties including scrolls over curves of genus 2 and quadric fibrations over elliptic curves, are proved.

1. Introduction

A complex projective variety $X \subset \mathbb{P}^N$ is k-regular in the sense of Castelnuovo-Mumford if $h^i(\mathcal{I}_X(k-i)) = 0$ for all $i \geq 1$ where $\mathcal{I}_X$ is the ideal sheaf of X. If X is k-regular then the minimal generators of its homogeneous ideal have degree less than or equal to k. A long standing conjecture, known to us as the Eisenbud Goto conjecture, states that an n-dimensional variety $X \subset \mathbb{P}^N$ of degree $\deg X = d$ is $(d-(N-n)+1)$-regular. Gruson Lazarsfeld and Peskine [14] established the conjecture for curves, Lazarsfeld [23] for smooth surfaces and Ran [29] for threefolds with

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high enough codimension. A nice historical account of the conjecture and further results can be found in [22]. In section 3 the conjecture is proved for all smooth varieties X embedded by the complete linear system associated with a very ample line bundle L such that $\Delta(X, L) \leq 5$ where $\Delta(X, L) = \dim X + \deg X - h^0(L)$. Notice also that in recent times computer algebra systems like Macaulay have made possible the explicit construction and study of examples of algebraic varieties starting from minimal generators of the homogeneous ideal of the variety. A priori information on the k-regularity of a variety is therefore useful for these constructions.

Strictly related to the notion of k-regularity is the notion of k-normality of a projective variety. A variety $X \subset \mathbb{P}^N$ is k-normal if hypersurfaces of degree k cut a complete linear system on X or, equivalently, if $h^1(\mathcal{I}_X(k)) = 0$. If X is k-regular it is clearly $(k-1)$-normal. X is said to be projectively normal if it is k-normal for all $k \geq 1$.

As a by-product of the proof of the above result the projective normality of a class of surfaces of degree nine in $\mathbb{P}^5$ which was left as an open question in [7] is established in Lemma 3.9. The non existence of a class of scrolls of degree 10, left as an open problem in [10], is also established in Remark 3.13.

In section 4 we deal with the projective normality of scrolls $X = \mathbb{P}(E)$ over a curve of genus 2 embedded by the complete linear system associated with the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$, assumed to be very ample. Two-dimensional such scrolls are shown to be always projectively normal except for a class S of non 2-normal surfaces of degree eight in $\mathbb{P}^5$ studied in detail in [2]. Three-dimensional scrolls $X = \mathbb{P}(E)$ of degree $\deg X \geq 13$ are then shown to be projectively normal if and only if E does not admit a quotient $E \to \mathcal{E} \to 0$ where $P(\mathcal{E})$ belongs to the class S of non quadratically normal surfaces mentioned above.

In section 5, building on the work of Homma [16], [17] and Purnaprajna and Gallego [28], criteria for the projective normality of three-dimensional quadric bundles over elliptic curves are given, improving some results contained in [8].

2. General results and preliminaries

2.1 Notation

The notation used in this work is mostly standard from Algebraic Geometry. Good references are [15] and [13]. The ground field is always the field $\mathbb{C}$ of complex numbers. Unless otherwise stated all varieties are supposed to be projective. $\mathbb{P}^N$ denotes the N-dimensional complex projective space. Given a projective n-dimensional variety X, $\mathcal{O}_X$ denotes its structure sheaf and $Pic(X)$ denotes the group of line bundles over X. Line bundles, vector bundles and Cartier divisors are denoted by capital letters as $L, M, \mathcal{M} \dots$. Locally free sheaves of rank one, line bundles and Cartier divisors are used interchangeably as customary.

Let $L, M \in Pic(X)$, let E be a vector bundle of rank r on X, let $\mathcal{F}$ be a coherent sheaf on X and let $Y \subset X$ be a subvariety of X. Then the following notation is used:

LM the intersection of divisors L and M

L^n the degree of L,

$|L|$ the complete linear system of effective divisors associated with L,

L_Y or $L_{|Y}$ the restriction of L to Y,

$L \sim M$ linear equivalence of divisors

$L \equiv M$ numerical equivalence of divisors

Num(X) the group of line bundles on X modulo numerical equivalence

$\mathbb{P}(E)$ the projectivized bundle of E, see [15]

$H^i(X, \mathcal{F})$ the i^{th} cohomology vector space with coefficients in $\mathcal{F}$,

$h^i(X, \mathcal{F})$ the dimension of $H^i(X, \mathcal{F})$, here and immediately above X is sometimes omitted when no confusion arises.

If C denotes a smooth projective curve of genus g, and E a vector bundle over C of deg $E = c_1(E) = d$ and rk$E = r$, we need the following standard definitions:

E is *normalized* if $h^0(E) \neq 0$ and $h^0(E \otimes \mathcal{L}) = 0$ for any invertible sheaf $\mathcal{L}$ over C with deg $\mathcal{L} < 0$.

E has slope $\mu(E) = \frac{d}{r}$.

E is *semistable* if and only if for every proper subbundle S, $\mu(S) \leq \mu(E)$. It is *stable* if and only if the inequality is strict.

The Harder-Narasimhan filtration of E is the unique filtration:

$$0 = E_0 \subset E_1 \subset .... \subset E_s = E$$

such that $\frac{E_i}{E_{i-1}}$ is semistable for all i, and $\mu_i(E) = \mu\big(\frac{E_i}{E_{i-1}}\big)$ is a strictly decreasing function of i.

A few definitions from [8] needed in the sequel are recalled.

Let $0 = E_0 \subset E_1 \subset .... \subset E_s = E$ be the Harder-Narasimhan filtration of a vector bundle E over C. Then

$\mu^-(E) = \mu_s(E) = \mu\big(\frac{E_s}{E_{s-1}}\big)$

$\mu^+(E) = \mu_1(E) = \mu(E_1)$

or alternatively

$\mu^+(E) = \max \{ \mu(S) | 0 \to S \to E \}$

$\mu^-(E) = \min \{ \mu(Q) | E \to Q \to 0 \}$.

It is also $\mu^+(E) \geq \mu(E) \geq \mu^-(E)$ with equality if and only if E is semistable. In particular if C is an elliptic curve, an indecomposable vector bundle E on C is semistable and hence $\mu(E) = \mu^-(E) = \mu^+(E)$.

The following definitions are standard in the theory of polarized varieties. A good reference is [11]. A *polarized variety* is a pair (X, L) where X is a smooth projective n-dimensional variety and L is an ample line bundle on X. Its *sectional genus*, denoted $g(X, L)$, is defined by $2g(X, L) - 2 = (K_X + (n-1)L)L^{n-1}$. Given any n-dimensional polarized variety (X, L) its Δ-*genus* is defined by $\Delta(X, L) = \dim(X) + L^n - h^0(X, L)$. A polarized variety (X, L) has a *ladder* if there exists a sequence of reduced and irreducible subvarieties $X = X_n \supset X_{n-1} \ldots \supset X_1$ of X where $X_j \in |L_{j+1}| = |L_{|X_{j+1}}|$. Each (X_j, L_j) is called a *rung* of the ladder. If L is generated by global sections (X, L) has a ladder. A rung (X_j, L_j) is *regular* if $H^0(X_{j+1}, L_{|X_{j+1}}) \to H^0(X_j, L_{|X_j})$ is onto. The ladder is regular if all the rungs are regular. If the ladder is regular $\Delta(X_j, L_j) = \Delta(X, L)$ for all $1 \leq j \leq n$. A variety $X \subset \mathbb{P}^N$ is *k-normal* for some $k \in \mathbb{Z}$ if $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k)) \to H^0(\mathcal{O}_X(k))$ is onto. Equivalently, if $\mathcal{I}_X$ is the ideal sheaf of X, X is k-normal if $h^1(\mathcal{I}_X(k)) = 0$. X is *projectively normal* (*p. n.*) if it is k-normal for all $k \geq 1$. A polarized pair (X, L) with L very ample is called k-normal or projectively normal if X is k-normal or p.n. in the embedding given by $|L|$. A polarized variety (X, L) with L very ample is always 1-normal (linearly normal).

A line bundle L on X is *normally (or simply) generated* if the graded algebra $G(X, L) = \bigoplus_{t \geq 0} H^0(X, tL)$ is generated by $H^0(X, L)$. L is very ample and normally generated if and only if (X, L) is p.n.

A variety $X \subset \mathbb{P}^N$ is k-regular, in the sense of Castelnuovo-Mumford, if for all $i \geq 1$ it is $h^i(\mathcal{I}_X(k-i)) = 0$. A polarized pair (X, L) with L very ample is k-regular if X is k-regular in the embedding given by $|L|$. If X is k-regular then it is $(k+1)$-regular.

2.2. General Results.

Let C be a smooth projective curve of genus g, E a vector bundle of rank n, with $n \geq 2$, over C and $\pi : X = \mathbb{P}(E) \to C$ the projectivized bundle associated to E with the natural projection π. Denote with $\mathcal{T} = \mathcal{O}_{\mathbb{P}(E)}(1)$ the tautological sheaf and with $\mathfrak{F}_P = \pi^*\mathcal{O}_C(P)$ the line bundle associated with the fibre over $P \in C$. Let T and F denote the numerical classes respectively of $\mathcal{T}$ and $\mathfrak{F}_P$. In this work we refer to a polarized variety $(X, \mathcal{T})$ as a *scroll* over a curve C if there is a vector bundle E over C such that $(X, \mathcal{T}) = (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ and $\mathcal{T}$ is very ample.

Remark 2.1. Let $\pi : (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1)) \to C$ be a n-dimensional projectivized bundle over a curve C. From Leray's Spectral sequence and standard facts about higher direct image sheaves (see for example [15] pg. 253) it follows that

$$H^1\big(\mathcal{O}_{\mathbb{P}(E)}(t)\big) = H^1\big(C, S^t E\big) \text{ for } t \geq 0$$
$$H^i(\mathcal{O}_{\mathbb{P}(E)}(t)) = 0 \text{ for } i \geq 2 \text{ and } t > -n\,.$$

Let $D \sim a\mathcal{T} + \pi^* B$, with $a \in \mathbb{Z}$, $B \in Pic(C)$ and $\deg B = b$, then $D \equiv aT + bF$. Moreover $\pi_*(\mathcal{O}_{\mathbb{P}(E)}(D)) = S^a(E) \otimes \mathcal{O}_C(B)$ and hence $\mu^-(\pi_*(\mathcal{O}_{\mathbb{P}(E)}(D)) = a\mu^-(E) + b$ (see [8]).

Regarding the ampleness, the global generation, and the normal generation of D, a few known criteria useful in the sequel are listed here:

Theorem 2.2 (Miyaoka [26])

Let E be a vector bundle over a smooth projective curve C of genus g, and $X = \mathbb{P}(E)$. If $D \equiv aT + bF$ is a line bundle over X, then D is ample if and only if $a > 0$ and $b + a\mu^-(E) > 0$.

Lemma 2.3 (see e.g. [8], Lemma 1.12)

Let E be a vector bundle over C of genus g.
i) if $\mu^-(E) > 2g - 2$ then $h^1(C, E) = 0$
ii) if $\mu^-(E) > 2g - 1$ then E is generated by global sections.

Lemma 2.4 (Butler [8], Theorem 5.1A)

Let E be a vector bundle on a smooth projective curve of genus g and let $D \equiv aT + bF$ be a divisor on $X = \mathbb{P}(E)$. If

$$b + a\mu^-(E) > 2g\,. \tag{1}$$

then D is normally generated.

A few basic facts on the *Clifford Index* of a curve are recalled. Good references are [25] and [12]. Let C be a projective curve and L be any line bundle on C. The Clifford index of L is defined as follows:

$$cl(L) = \deg(L) - 2(h^0(L) - 1).$$

The Clifford index of the curve is $cl(C) = \min\{cl(L) | h^0(L) \geq 2 \text{ and } h^1(L) \geq 2\}$. For a general curve C it is $cl(C) = \left[\frac{g-1}{2}\right]$ and in any case $cl(C) \leq \left[\frac{g-1}{2}\right]$. By Clifford's theorem a special line bundle L on C has $cl(L) \geq 0$ and the equality holds if and only if C is hyperelliptic and L is a multiple of the unique g^1_2.

If $cl(C) = 1$ then C is either a plane quintic curve or a trigonal curve.

Theorem 2.5 ([12])

Let L be a very ample line bundle on a smooth irreducible complex projective curve C. If

$$\deg(L) \geq 2g + 1 - 2h^1(L) - cl(C)$$

then (C, L) is projectively normal.

3. The Eisenbud Goto conjecture for low values of Δ.

Let $X \subset \mathbb{P}^N$ be an n dimensional projective variety of degree d. A long standing conjecture, known to us as the *Eisenbud Goto* conjecture, states that X should be $(d - (N - n) + 1)$-regular, i.e. $(degree - codimension + 1)$-regular.

Many authors worked on the conjecture for low values of the dimension and codimension of X. A nice historic account is found in [22]. Some of their results are collected in the following Theorem.

Theorem 3.1 ([14], [23])

If $X \subset \mathbb{P}^N$ is any smooth curve or any smooth surface then X is $(d - c + 1)$-regular where $d = \deg(X)$ and $c = codimension(X)$.

In this section we would like to offer a proof of the conjecture for linearly normal smooth varieties with low Δ-genus. Let (X, L) be a polarized variety with L very ample. The above conjecture can be restated for the embedding given by $|L|$ in terms of Δ-genus as follows:

Conjecture. Let (X, L) be a polarized variety with L very ample. Then (X, L) is $(\Delta + 2)$-regular.

Remark 3.2. It is straightforward to check that hypersurfaces of degree d are always d-regular and not $(d - 1)$-regular. This shows that the conjecture is indeed sharp. On the other hand there are varieties $X \subset \mathbb{P}^N$ which are k-regular for $k < d - c + 1$. This motivates Definition 3.4.

Remark 3.3. It is a classical adjunction theoretic results that given (X, L) with L very ample, $K_X + tL$ is globally generated, and in particular $h^0(K_X + tL) \neq 0$, for $t \geq n$ unless $t = n$ and $(X, L) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. This fact, Remark 3.2 and the sequence $0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \to 0$ suitably twisted show that no linearly normal non degenerate n-dimensional variety $X \subset \mathbb{P}^N$ can be k-regular for $k \leq 1$. Therefore in what follows we will always assume $k \geq 2$ when dealing with k-regularity.

Definition 3.4. Let $X \subset \mathbb{P}^N$ be a n-dimensional variety of degree d. Let

$$r(X) = \text{Min}\left\{k \in \mathbb{Z} | X \text{ is } k-\text{regular}\right\}.$$

A variety X is **extremal** if $r(X) = d - (N - n) - 1$. A polarized variety (X, L) with L very ample is **extremal** if it is extremal in the embedding given by $|L|$, i.e. if $r(X, L) = \text{Min}\{k \in \mathbb{Z} | (X, L) \text{ is } k-\text{regular}\} = \Delta + 2$.

In what follows we will prove the above conjecture for all linearly normal manifolds with $\Delta \leq 5$ obtaining along the way the value of $r(X, L)$ for most of the same manifolds.

Lemma 3.5

Let $X \subset \mathbb{P}^N$ be a smooth n-dimensional variety and let $Y \subset \mathbb{P}^{N-1}$ be a generic hyperplane section.
i) If X is k-regular then Y is k-regular.
ii) If Y is k-regular and X is $(k-1)$-normal then X is k-regular.
iii) If X is $(r(Y)-1)$-normal then $r(X) = r(Y)$.

Proof. The exact sequence

$$0 \to \mathcal{I}_X(k-i) \to \mathcal{I}_X(k-i+1) \to \mathcal{I}_Y(k-i+1) \to 0\,. \tag{2}$$

immediately gives i). To see ii) consider again sequence (2). The k-regularity of Y gives $h^{i-1}(\mathcal{I}_Y(k-i+1)) = 0$ for all $i \geq 2$. Since k regularity implies $k+1$-regularity it is $h^i(\mathcal{I}_Y(k-i+1)) = 0$ for all $i \geq 1$. Therefore $h^i(\mathcal{I}_X(k-i)) = h^i(\mathcal{I}_X(k-i+1))$ for all $i \geq 2$ from (2) and iteratively $h^i(\mathcal{I}_X(k-i)) = h^i(\mathcal{I}_X(k-i+t)$ for all $i \geq 2$ and for all $t \geq 1$. Letting t grow, Serre's vanishing theorem gives $h^i(\mathcal{I}_X(k-i+t) = 0$ for all $i \geq 2$ and all $t \geq 1$ and thus $h^i(\mathcal{I}_X(k-i)) = 0$ for all $i \geq 2$. Because X is assumed $(k-1)$-normal it is $h^1(\mathcal{I}_X(k-1)) = 0$ which concludes the proof of ii). Now iii) follows immediately from i) and ii). $\square$

Lemma 3.6

Let (X, L) be a polarized variety with L very ample. Let $Y \in |L|$ be a generic element and assume $H^0(X, L) \to H^0(Y, L_{|_Y})$ is onto. Then $i), ii), iii)$ as in Lemma 3.5 hold if we replace X by (X, L) and Y by $(Y, L_{|_Y})$.

Proof. Let $h^0(L) = N+1$. The surjectivity condition on the restriction map between global sections of L and $L_{|_Y}$ guarantees that $|L_{|_Y}|$ embeds Y as a linearly normal manifold in $\mathbb{P}^{N-1}$, therefore the same proof as in Lemma 3.5 applies. □

Remark 3.7. Let (X, L) be a polarized variety with L very ample. Let $Y \in |L|$ be a generic element and assume $H^0(X, L) \to H^0(Y, L_{|_Y})$ is onto. Then [11] Corollary 2.5 shows that if $(Y, L_{|_Y})$ is projectively normal, so is (X, L). Therefore when the ladder is regular and Y is p.n. Lemma 3.6 gives $r(X, L) = r(Y, L_{|_Y})$.

Lemma 3.8

Let (C, L) be a projectively normal curve with $g \geq 1$.
Then $r(C, L) = \mathrm{Min}\,\{t \geq 3 | h^1((t-2)L) = 0\}$.

Proof. Let $h^0(L) = N+1$ so that $C \subset \mathbb{P}^N$. It is $h^1(\mathcal{I}_C(k-1)) = 0$ for all $k \geq 2$ because of the projective normality assumption. The sequence

$$0 \to \mathcal{I}_C(k-i) \to \mathcal{O}_{\mathbb{P}^N}(k-i) \to (k-i)L \to 0$$

easily gives $h^i(\mathcal{I}_C(k-i)) = 0$ for all $i \geq 3$ and $k \geq 2$.

The same sequence gives $h^2(\mathcal{I}_C(k-2)) = h^1((k-2)L)$ and since $h^1(\mathcal{O}_C) = g \geq 1$ it is $r(C, L) = \mathrm{Min}\,\{t \geq 3 | h^1((t-2)L) = 0\}$. □

In order to apply the above lemmata in one occasion the projective normality of a particular class of surfaces of degree nine needs to be established. The following Lemma also improves [7]. Here $\mathbb{F}_1$ denotes the Hirzebruch rational ruled surface of invariant $e = 1$, $\pi : Bl_t S \to S$ denotes the blow up of a surface S at t points, E_i are the exceptional divisors of the blow up, $\mathfrak{C}_0 = \pi^*(C_0)$ denotes the pull back of the line bundle associated with the fundamental section of $\mathbb{F}_1$ and $\mathfrak{f} = \pi^*(f)$ the pull back of the one associated with any fibre f of the natural projection $p : \mathbb{F}_1 \to \mathbb{P}^1$.

Lemma 3.9

Let $(S, L) = (Bl_{12}\mathbb{F}_1, 3\mathfrak{C}_0 + 5\mathfrak{f} - \sum_i E_i)$. Then (S, L) is projectively normal.

Proof. The projective normality of linearly normal degree nine surfaces was studied in [7]. Let (S, L) be a surface of degree 9 and sectional genus 5, embedded in $\mathbb{P}^5$. The surface under consideration was established to be projectively normal unless its generic curve section C is trigonal and $L_{|_C} = K_C - M + D$ where M is a divisor in the g^1_3 and D is a divisor of degree 4 giving a foursecant line for C. Therefore if S were not p.n. it would admit an infinite number of $k \geq 4$-secant lines. On the other hand a careful study of the embedding shows that S contains only a finite number of lines and that the only lines with self intersection ≥ -1 are the 12 exceptional divisors E_i. Thus the formulas contained in [24] can be used. A straightforward calculation using [24] shows that S cannot have a infinite number of $k \geq 4$-secants, contradiction. □

Theorem 3.10

Let (X, L) be a n dimensional polarized pair, $n \geq 2$, with a ladder. Assume $g = g(X, L) \geq \Delta(X, L) = \Delta$ and $d = L^n \geq 2\Delta + 1$. Then:
i) The curve section $(C, L_{|C})$ is k-regular if and only if (X, L) is k-regular and $r(X, L) = r(C, L_{|C})$.
ii) Either $\Delta = 0, 1$ and (X, L) is extremal or $\Delta \geq 2$ and $r(X, L) = 3$.

Proof. From [11] Theorem (3.5) and from the fact that a normally generated ample line bundle is automatically very ample it follows that L is very ample, $g = \Delta$, the ladder is regular and every rung of the ladder is projectively normal. Therefore Lemma 3.6 immediately gives i).

Then (X, L) is extremal if and only if the curve section $(C, L_{|C})$ is such. Extremal linearly normal curves were classified in [14] and they are either rational or elliptic normal curves. Therefore (X, L) is extremal if and only if $\Delta = g = 0, 1$. Now assume $\Delta \geq 2$ and thus (X, L) not extremal. The curve section $(C, L_{|C})$ is embedded in $\mathbb{P}^M$ where $M = d - \Delta$. Since $h^M(\mathcal{O}_{\mathbb{P}^M}(k - M)) = h^0(\mathcal{O}_{\mathbb{P}^M}(-1 - k)) = 0$ for all $k \geq 0$ the sequence $0 \to \mathcal{I}_C(k - i) \to \mathcal{O}_{\mathbb{P}^M}(k - i) \to \mathcal{O}_C(k - i) \to 0$ shows that $h^i(\mathcal{I}_C(k - i)) = h^{i-1}(\mathcal{O}_C(k - i))$ for all $i \geq 2$ and all $k \geq 0$. Therefore, because $h^1(\mathcal{O}_C) = g \geq 2$, it must be $r(X, L) \geq 3$. If $i \geq 3$ then clearly $h^{i-1}(\mathcal{O}_C(3 - i)) = 0$ and thus $h^i(\mathcal{I}_C(3 - i)) = 0$. It is also $h^2(\mathcal{I}_C(1)) = h^1(\mathcal{O}_C(1)) = h^1(L_{|C}) = 0$ because $g = \Delta$ and $d \geq 2\Delta + 1 > 2g - 2$. Since every rung of the ladder is projectively normal, in particular $h^1(\mathcal{I}_C(2)) = 0$ and thus $(C, L_{|C})$ is 3-regular. We can conclude that $r(X, L) = r(C, L_{|C}) = 3$. □

Proposition 3.11

Let $(X, \mathcal{T}) = (\mathbb{P}(\mathcal{E}),\ \mathcal{O}_{\mathbb{P}(E)}(1))$ be a scroll over an elliptic curve. Then $r(X, \mathcal{T}) = 3$.

Proof. Because $h^2(\mathcal{I}_X) = h^1(\mathcal{O}_X) = 1$ it is $r(X, L) \geq 3$. We need to show that $h^i(\mathcal{I}_X(3 - i)) = 0$ for all $i \geq 1$. Notice that $|\mathcal{T}|$ embeds X into $\mathbb{P}^N$ as a variety of degree d where $N = d - 1$. Let $i = 1$. It is known, cf. [8] and [2], that elliptic scrolls are projectively normal, so $h^1(\mathcal{I}_X(2)) = 0$. Let $i = 2$. From Remark 2.1 it is $h^2(\mathcal{I}_X(1)) = h^1(\mathcal{O}_X(1)) = h^1(C, E)$. Because E is very ample it is $\mu^-(E) > 0$ which, by Lemma 2.3 implies $h^1(C, E) = 0$.

For $i = N$ it is $h^N(\mathcal{O}_{\mathbb{P}^N}(3-N)) = h^0(\mathcal{O}_{\mathbb{P}^N}(-4)) = 0$. Therefore it follows that $h^i(\mathcal{I}_X(3-i)) = h^{i-1}(\mathcal{O}_X(3-i))$ for all $i \geq 3$. Remark 2.1 gives $h^{i-1}(\mathcal{O}_X(3-i)) = 0$ for $3 \leq i \leq n+1$ while clearly $h^{i-1}(\mathcal{O}_X(3-i)) = 0$ for $i > n+1$ since $n = \dim X$.

Therefore $h^i(\mathcal{I}_X(3-i)) = 0$ for all $i \geq 3$. □

Lemma 3.12

Let $(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ be a n-dimensional scroll over a curve of genus $g \geq 2$. If $\deg(E) > 2g-2$ then $\Delta \geq 2n+g-3$.

Proof. Because $d = \deg(\det E) = \deg(E) > 2g-2$, it is $h^0(\det E) = 1+d-g$ by Riemann Roch. Combining this with the inequality $h^0(\det E) \geq h^0(E)+r-2$ found in [21], it follows that $h^0(E) \leq d-n+3-g$ and therefore $\Delta \geq 2n+g-3$. □

Remark 3.13. Notice that the above Lemma 3.12 rules out the existence of scrolls of degree 10 over a curve of genus $g = 3$ left as an open possibility in [10].

We can now prove the main theorems of this section. For $\Delta \leq 3$ we establish the conjecture and give the value of $r(X, L)$ for all pairs. For $\Delta = 4, 5$ we establish the conjecture and collect in a remark the known values of $r(X, L)$.

Theorem 3.14

Let (X, L) be a n-dimensional polarized pair with X smooth, L very ample and $\Delta \leq 5$. Then (X, L) is $\Delta+2$-regular.

Proof. Because of Theorem 3.1 and Remark 3.2, the blanket hypothesis $n \geq 3$ and *codimX* ≥ 2 will be in place throughout this proof.

Case 1. $\Delta \leq 1$

If $\Delta = 0$ then (X, L) is extremal by Theorem 3.10. Assume $\Delta = 1$, because $g = 0$ implies $\Delta = 0$, see [11] Proposition (3.4), it is $g \geq 1$. Because (X, L) is not a hypersurface it is $d \geq 3$ and again Theorem 3.10 gives (X, L) extremal.

Case 2. $\Delta = 2$

If $g \leq 1$ then (X, L) must be a two dimensional elliptic scroll, see [11] Theorem (10.2). Proposition 3.11 gives $r = 3$. Let $g \geq 2$. Because X is not a hypersurface it is $h^0(L) \geq n+3$. This implies $\Delta \leq d-3$ i.e. $d \geq 5$ and then Theorem 3.10 gives $r = 3$.

Case 3. $\Delta = 3$

From [6] it follows that complete intersections of type $(2,3)$ have $r = 4$. Following [18] Theorem 4.8 and section 7, it follows from Theorem 3.10 and Proposition 3.11 that the only varieties left to investigate are Bordiga threefolds scrolls in $\mathbb{P}^5$. They have the following resolution with $N = 5$.

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-4)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^N}(-3)^{\oplus 4} \to \mathcal{I}_X \to 0\,. \tag{3}$$

Equalities $h^{N-1}(\mathcal{I}_X(3-N)) = h^0(\mathcal{O}_{\mathbb{P}^N}) = 1$ and $h^i(\mathcal{I}_X(3-i)) = 0$ for all $i \geq 1$ are straightforward to see, therefore $r = 3$.

Case 4. $\Delta = 4$

Varieties with $\Delta = 4$ are classified. Let us follow the list of varieties given in [19] Theorem 3. Threefolds in $\mathbb{P}^5$ with $d = 7$, $g = 5$ or $g = 6$ have respective resolutions as in (4) and (5) with $N = 5$.

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-5) \oplus \mathcal{O}_{\mathbb{P}^N}(-4) \to \mathcal{O}_{\mathbb{P}^N}(-3)^{\oplus 3} \to \mathcal{I}_X \to 0\,. \tag{4}$$

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-5)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^N}(-2) \oplus \mathcal{O}_{\mathbb{P}^N}(-4)^{\oplus 2} \to \mathcal{I}_X \to 0\,. \tag{5}$$

The resolutions (4) and (5) quickly show that $r = 4$. Complete intersections of type $(2,2,2)$ have $r = 4$ by [6]. Scrolls over a genus 2 curve must be two-dimensional while elliptic scrolls are taken care of by Proposition 3.11.

Let now $q = 0$ and $g = 4$. If $d \geq 9$ Theorem 3.10 gives $r = 3$. On the other hand since $\Delta = 4$ and the codimension must be at least two, it follows that $d \geq 7$. Let us now compare the varieties under consideration with the lists of manifolds of degree 7 and 8 given in [18] and [20].

If $d = 8$ then $X \subset \mathbb{P}^6$ is a threefold scroll over the quadric surface. Since $q = 0$ the ladder is regular. Consider the curve section $(C, L_{|C})$. Such a $(C, L_{|C})$ is known to be non hyperelliptic (see [20]) and thus Theorem 2.5 gives $(C, L_{|C})$ p.n. Since $d = 8 > 2g - 2 = 6$ it is $h^1(L_{|C}) = 0$ and thus $r(C, L_{|C}) = 3$ by Lemma 3.8. Because the ladder is regular $r(X, L) = 3$ by Remark 3.7.

If $d = 7$ then $X \subset \mathbb{P}^5$ is Palatini's scroll over the cubic surface. A resolution for $\mathcal{I}_X$ is found in [4]:

$$0 \to \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 4} \to \Omega^1(-2) \to \mathcal{I}_X \to 0\,.$$

A simple cohomological calculation gives $r = 4$.

Case 5. $\Delta = 5$

Theorem 3.10 takes care of cases with $g \geq 5$ and $d \geq 11$. Manifolds with degree $d \leq 10$ were classified by various authors and we will examine them later in the proof. Let us now assume $d \geq 11$ and $g \leq 4$. Because $\Delta = 5$ and elliptic scrolls are dealt with in Proposition 3.11, it must be $g \geq 2$. Varieties of low sectional genus were classified in [18]. Let us follow the lists given there. If $g = 2$ scrolls over a curve are the only manifolds to be considered. On the other hand such scrolls of genus 2 have $\Delta = 2n$ (cf. [7]) so there are no manifolds to examine. If $g = 3$ scrolls over curves are ruled out by Lemma 3.12 and scroll over $\mathbb{P}^2$, having $q = 0$, are ruled out by [18] Theorem 4.8 iv). If $g = 4$ scrolls over curves are again ruled out by Lemma 3.12. Using standard numerical relations (see for example [9] (0.14)) one sees that there are no hyperquadric fibrations of dimension $n \geq 3$, $g = 4$, $\Delta = 5$ over $\mathbb{P}^1$ or over an elliptic curve. Let now (X, L) be a threefold which is a scroll over a surface $(Y, \mathcal{L})$ with $q(Y) = 0$, $g(X, L) = 4$. Because $h^1(\mathcal{O}_X) = q(Y) = 0$, recalling that a general hyperplane section of X is birational to Y and thus regular, the ladder is regular and then $\Delta(X, L) = \Delta(C, L_{|C}) = 4$ by Riemann Roch.

Let us now consider the cases with $d \leq 10$ by looking at the classification found in [20], [9], [10]. The first non trivial case occurs with $d = 8$. (X, L) is a threefold in $\mathbb{P}^5$, admitting a fibration over $\mathbb{P}^1$ with generic fibres complete intersections of type $(2, 2)$ in $\mathbb{P}^4$. A resolution of the ideal of this variety can be found in [4]. A standard cohomological calculation shows that $r(X, L) = 4$.

Let now $d = 9$. From [9] all varieties to be considered are threefolds in $\mathbb{P}^6$ with $g = 5, 6, 7 \geq \Delta$ and $d = 9 \geq 2\Delta - 1$. Thus the ladder is regular, see [11] Theorem 3.5. Let $(S, L_{|S})$ be the surface section and let $(C, L_{|C})$ be the curve section. The projective normality of linearly normal surfaces of degree nine was studied in [7]. Comparing the list given there with [9] and using Lemma 3.9 $(S, L_{|S})$ is seen to be projectively normal. Remark 3.7 then gives $r(X, L) = r(S, L_{|S}) = r(C, L_{|C})$. Let now $g = 5$. Then $h^1(tL_C) = 0$ for all $t \geq 1$ and from the structural sequence of C in $\mathbb{P}^4$ it is easy to see that $r(C, L_{|C}) = 3$ if and only if $(C, L_{|C})$ is 2-normal. On the other hand [9] shows that in this case $h^1(\mathcal{O}_S) = 0$ and since $h^1(L_C) = 0$ it must be $h^1(L_S) = 0$ and thus $h^2(\mathcal{I}_S(1)) = 0$. Now the 2-normality of $(S, L_{|S})$ implies the 2-normality of C as can be seen from $0 \to \mathcal{I}_S(1) \to \mathcal{I}_S(2) \to \mathcal{I}_C(2) \to 0$ and therefore $r(X, L) = 3$.

Let now $g = 6$. First notice that since $h^0(L_C) = 5$ it is $h^1(L_C) = 1$ and thus $0 \to \mathcal{I}_C(1) \to \mathcal{O}_{\mathbb{P}^4}(1) \to L_C \to 0$ shows that $h^2(\mathcal{I}_C(1)) = h^1(L_C) = 1$ i.e. $(C, L_{|C})$ cannot be 3-regular. Consider the sequence

$$0 \to tL_S \to (t+1)L_{|S} \to (t+1)L_{|C} \to 0 \tag{6}$$

for all $t \geq 1$. Because $\deg(t+1)L_{|C} = 9(t+1) > 2g-2$ it is $h^1((t+1)L_{|C}) = 0$ for all $t \geq 1$. Therefore the above sequence gives $h^2(tL_{|S}) = h^2((t+1)L_{|S}) = 0$ for all $t \geq 1$ and thus $h^2(tL_{|S}) = 0$ for all $t \geq 1$ by Serre's Theorem. From [9] we know that $q(S) = 0$ and $p_g(S) = 1$. Thus the sequence $0 \to \mathcal{O}_S \to L_{|S} \to L_{|C} \to 0$ gives $h^1(L_{|S}) = h^2(L_{|S}) = 0$. Then the sequence (6) for $t = 1$ gives $h^1(2L_{|S}) = 0$. The sequence $0 \to \mathcal{I}_S(2) \to \mathcal{O}_{\mathbb{P}^5}(2) \to 2L_{|S} \to 0$ gives $h^2(\mathcal{I}_s(2)) = h^1(2L_S) = 0$. Then the sequence $0 \to \mathcal{I}_S(2) \to \mathcal{I}_S(3) \to \mathcal{I}_C(3) \to 0$, recalling that $(S, L_{|S})$ is projectively normal, gives $h^1(\mathcal{I}_C(3)) = h^2(\mathcal{I}_S(2)) = 0$, i.e. $(C, L_{|C})$ is 3-normal. The structure sequence for C in $\mathbb{P}^4$ then easily shows that $(C, L_{|C})$ is 4-regular and thus $r(X, L) = 4$.

Let now $g = 7$. Noticing that $h^1(L_{|C}) = 2$ and recalling from [9] that in this case $q(S) = 0$ and $p_g(S) = 2$, the same argument as above shows that $(C, L_{|C})$ is not 3-regular but it is 4-regular thus $r(X, L) = 4$.

Let now $d = 10$. From [10] we see that $h^1(\mathcal{O}_X) = 0$ and therefore the ladder of these manifolds is regular. Following the list given in [10] let X be a sectional genus 6, codimension 4 Mukai manifold of dimension 3 or 4. The curve section $(C, L_{|C})$ is then a canonical curve in $\mathbb{P}^5$ and as such it is projectively normal. Because $h^1(K_C) = 1$, $h^1(2K_C) = 0$ and the ladder is regular, it follows from Lemma 3.8 and Remark 3.7 that $r(X, L) = r(C, L_{|C}) = 4$.

Let now X be any of the remaining threefolds of degree 10 in $\mathbb{P}^7$, all of which have $g = 5$, according to [10]. Let $(C, L_{|C})$ be a generic curve section. From the classification of manifolds with hyperelliptic section (see [5]) it follows that either X is a hyperquadric fibration over $\mathbb{P}^1$ or C is not hyperelliptic. In the latter case it is $cl(C) \geq 1$ and therefore Theorem 2.5 gives the projective normality of C. Because $g = 5$ it is $h^1(L_{|C}) = 0$ and then Lemma 3.8, the regularity of the ladder and Remark 3.7 give $r(X, L) = r(C, L_{|C}) = 3$.

Let $(X, L) \xrightarrow{\pi} \mathbb{P}^1$ now be a hyperquadric fibration. Consider $W = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1,1,1,1))$ and let $\mathcal{T} = \mathcal{O}_W(1)$. From [10] it follows that $X \in |2\mathcal{T} + \pi^*(\mathcal{O}_{\mathbb{P}^1}(2))|$ and $L = \mathcal{T}_{|X}$. The higher vanishing $h^i(\mathcal{I}_X(k-i)) = 0$ for $i \geq 2$ required for the k-regularity of X are easily obtained for all $k \geq 3$ from the sequences

$$0 \to \mathcal{I}_X(k-i) \to \mathcal{O}_{\mathbb{P}^7}(k-i) \to \mathcal{O}_X(k-i) \to 0$$

$$0 \to (k-2-i)\mathcal{T} + \pi^*(\mathcal{O}_{\mathbb{P}^1}(-2)) \to (k-i)\mathcal{T} \to \mathcal{O}_X(k-i) \to 0$$

recalling Remark 2.1.

Notice that $|\mathcal{T}|$ embeds W in $\mathbb{P}^7$ and the embedding is projectively normal, i.e. $H^0(\mathcal{O}_{\mathbb{P}^7}(k)) \to H^0(W, \mathcal{O}_W(k))$ is onto for all $k \geq 1$. Therefore X is k-normal in the embedding given by $\mathcal{T}_{|X}$, for some k, if and only if $H^0(W, \mathcal{O}_W(k)) \to H^0(X, \mathcal{O}_X(k))$ is surjective and this happens if and only if $H^1(W, (k-2)\mathcal{T} + \pi^*(\mathcal{O}_{\mathbb{P}^1}(-2)) = 0$.

It is $H^1(W, (k-2)\mathcal{T} + \pi^*(\mathcal{O}_{\mathbb{P}^1}(-2))) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \otimes S^{k-2}\mathcal{O}_{\mathbb{P}^1}(1,1,1,1))$.

Combining Lemma 2.3 and the fact that $\mu^-(\mathcal{O}_{\mathbb{P}^1}(-2) \otimes S^{k-2}\mathcal{O}_{\mathbb{P}^1}(1,1,1,1)) = k-4$ it is $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \otimes S^{k-2}\mathcal{O}_{\mathbb{P}^1}(1,1,1,1)) = 0$ for all $k \geq 3$. On the other hand $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 1$ so $r(X, L) = 3$. □

Corollary 3.15

Let (X, L) be a n-dimensional polarized pair with X smooth, L very ample and $\Delta \leq 3$. Then
i) (X, L) is extremal if and only if it is either a hypersurface or $\Delta = 0, 1$.
ii) If $\Delta = 2$ then $r(X, L) = 3$.
iii) If $\Delta = 3$ then $r(X, L) = 3$ unless (X, L) is a complete intersection of type $(2, 3)$ or a curve of genus 3 embedded in $\mathbb{P}^3$ as a curve of type $(2, 4)$ on a smooth quadric hypersurface. In both these cases $r(X, L) = 4$.

Proof. From the proof of Theorem 3.14 there are only curves and surfaces with $\Delta = 3$ to consider. If X is a curve, since $c \geq 2$, it must be $g \geq 3$ and $d \geq 6$. If $d \geq 7$ Theorem 3.10 gives $r = 3$. If $d = 6$ then $X \subset \mathbb{P}^3$ and [18] section 7 gives three possible types for X. X is linked to a twisted cubic by two cubic hypersurfaces, X is of type $(2, 4)$ on a smooth quadric or X is a complete intersection of type $(2, 3)$. In the first case $\mathcal{I}_X$ has a resolution as in (3) for $N = 3$ and therefore $r = 3$. In the second case X is not 2-normal and therefore $r \geq 4$. By Theorem 3.1 $r \leq 5$. By [14] X cannot be extremal, therefore $r = 4$. From [6] it follows that complete intersections of type $(2, 3)$ have $r = 4$.

Assume $n = 2$. As above complete intersections of type $(2, 3)$ have $r = 4$. Following [18] Theorem 4.8 and section 7, it follows from Theorem 3.10 and Proposition 3.11 that the only varieties left to investigate are Bordiga surfaces in $\mathbb{P}^4$. They have resolutions as in (3) with $N = 4$. It is straightforward to check $r = 3$. □

Corollary 3.16

Let (X, L) be a n-dimensional polarized pair with $n \geq 3$, X smooth, L very ample, $\Delta = 4$ and (X, L) not a hypersurface. Then $r(X, L) = 3$ unless (X, L) is a complete intersection of type $(2, 2, 2)$ or any threefold in $\mathbb{P}^5$ of degree 7 in which cases $r(X, L) = 4$.

Proof. Immediate from the proof of Theorem 3.14. □

Corollary 3.17

Let (X, L) be a n-dimensional polarized pair with $n \geq 3$, X smooth, L very ample, $\Delta = 5$ and (X, L) not a hypersurface. Then $r(X, L) = 3$ unless (X, L) is in the following list, in which cases $r(X, L) = 4$.
i) $(X, L) \subset \mathbb{P}^5$ is a threefold of degree 8 fibered over $\mathbb{P}^1$ with generic fibres complete intersections of type $(2, 2)$ (see [4]).
ii) $(X, L) \subset \mathbb{P}^6$ is a threefold of degree 9, $g = 6$, obtained by blowing up a point on a Fano manifold in $\mathbb{P}^7$, (see [9]).
iii) $(X, L) \subset \mathbb{P}^6$ is a threefold of degree 9, $g = 7$, obtained by a cubic section of a cone over the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$, (see [9]).
iv) $(X, L) \subset \mathbb{P}^{n+4}$ is a Mukai manifold of degree 10, $n = 3, 4$, $g = 6$, (see [10]).

Proof. Immediate from the proof of Theorem 3.14. □

4. Scrolls over curves of genus two

Let $(X, \mathcal{T}) = (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ be an n-dimensional scroll over a curve C of genus 2. From [7] (Lemma 5.2) it follows that $\Delta(X, \mathcal{T}) = 2n$ and $h^1(\mathcal{T}) = 0$. These facts will be used without further mention throughout this section. The same conclusions can also be drawn from Lemma 2.3 and the following lemma:

Lemma 4.1

Let E be a rank r very ample vector bundle over a genus 2 curve. Then $\mu^-(E) > 3$ and $h^1(C, S^t(E)) = 0$ for $t \geq 1$.

Proof. By induction on r. If $r = 1$ then E is semistable and very ample, therefore $\mu^-(E) = \mu(E) \geq 5$. Let now $r \geq 2$ and assume $\mu^-(E) > 3$ for every very ample vector bundle of rank up to $r - 1$. From [21] it is $c_1(E) \geq 3r + 1$. If E is semistable then $\mu^-(E) = \mu(E) = \frac{d}{r} \geq 3 + \frac{1}{r} > 3$.

Let now E be non semistable. Then there is a quotient bundle $E \to Q \to 0$ such that $rk(Q) < rk(E)$ and $\mu(Q) = \mu^-(E)$. Being a quotient of a very ample bundle on a curve, Q is also very ample and by induction $\mu^-(E) = \mu(Q) \geq \mu^-(Q) > 3$.

Because $\mu^-(S^t(E)) = t\mu^-(E) > 3t \geq 3$ it is $h^1(S^t(E)) = 0$ from Lemma 2.3. □

Proposition 4.2

Let $(X, \mathcal{T})$ be a surface scroll over a curve of genus $g = 2$ with degree $T^2 = d$. Then $(X, \mathcal{T})$ is projectively normal unless $d = 8$. In this case X is as in [20] (4.2).

Proof. The projective normality of such scrolls up to degree 8 was studied in [2] where the non projectively normal surfaces in the statement can be found. Let us assume $d \geq 9$. If E is semistable then $\mu^-(E) = \mu(E) = d/2 > 4$ and therefore $(X, \mathcal{T})$ is p.n. by Lemma 2.4. Let now E be non semistable. Then E admits a Harder Narasimhan filtration of the form $0 \to D \to E$ where D is a line bundle. Let now Q be the quotient line bundle $0 \to D \to E \to Q \to 0$. From the definition of μ^- it is $\mu^-(E) = \mu(Q) = \deg Q$. Because E is very ample so must be Q. A line bundle Q on a curve of genus 2 is very ample if and only if $\deg Q \geq 5$. Thus $\mu^-(E) > 4$ and $(X, \mathcal{T})$ is p.n. by Lemma 2.4. $\square$

The above non projectively normal surface scrolls are such because they are not 2-normal, (see [2]). Indeed the next Proposition and Lemma 4.1 show that 2-normality is equivalent to projective normality for scrolls of genus 2, extending a result found in [28].

Proposition 4.3

Let $(X, \mathcal{T}) = (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ be a scroll over a smooth curve C of genus g such that $\mu^-(E) > 2g - 2$. Then $(X, \mathcal{T})$ is projectively normal if and only if it is 2-normal.

Proof. If $(X, \mathcal{T})$ is p.n. it is obviously 2-normal.

To see the converse let $n = \dim X = rkE$ and let $\pi : X \to C$ be the natural projection. Reasoning as in [28] Lemma 1.4, projective normality of $(X, \mathcal{T})$ follows from the surjectivity of the maps

$$H^0\big((j-1)\mathcal{T}\big) \otimes H^0(\mathcal{T}) \to H^0(j\mathcal{T}) \quad \text{for all} \quad j \geq 2\,. \tag{7}$$

This in turns follows, according to [27] Theorem 2, from the vanishing of

$$H^i\big(X, (j-1-i)\mathcal{T}\big) = 0 \quad \text{for all } n \geq i \geq 1 \quad \text{and for all } j \geq 2\,.$$

Because $i \leq n$ and $j \geq 2$ it is $j - 1 - i > -n$ and therefore Remark 2.1 shows that $(X, \mathcal{T})$ is p.n. if $H^1(X, (j-2)\mathcal{T}) = H^1(C, S^{j-2}E) = 0$ for all $j \geq 2$. The hypothesis $\mu^-(E) > 2g - 2$ implies $\mu^-(S^{j-2}E) = (j-2)\mu^-(E) > 2g - 2$ for all $j \geq 3$. From Lemma 2.3 it follows that $H^1(X, (j-2)\mathcal{T}) = 0$ for all $j \geq 3$. This gives all necessary surjectivity in (7) but for $j = 2$. Thus $(X, \mathcal{T})$ is p.n. if $H^0(\mathcal{T}) \otimes H^0(\mathcal{T}) \to H^0(2\mathcal{T})$ is onto, i.e. if $(X, \mathcal{T})$ is 2-normal. $\square$

Corollary 4.4

A scroll $(X, \mathcal{T})$ over a curve of genus 2 is p.n. if and only if it is 2 normal.

Proof. Immediate from Proposition 4.3 and Lemma 4.1. $\square$

Results on threefold scrolls are collected in the following proposition.

Proposition 4.5

Let $(X,\mathcal{T}) = (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ be a threefold scroll over a curve of genus 2. Let $d \neq 12$. Then $(X,\mathcal{T})$ fails to be projectively normal if and only if one of the following cases occur
i) $d = 11$.
ii) $d \geq 13$, E is not semistable and it admits a quotient $E \to \mathcal{E} \to 0$ of rank two and degree eight.

Proof. It is known, see [21] or [20] and [9], [10], that there do not exist threefold scrolls of genus two and $d \leq 10$. So assume $d \geq 11$. Because $h^1(\mathcal{T}) = 0$ it is $h^0(\mathcal{T}) = d-3$. A simple computation shows that $h^0(X, \mathcal{O}_X(2)) = 4d - 6 > h^0(\mathcal{O}_{\mathbb{P}^{d-4}}(2)) = \frac{(d-2)(d-3)}{2}$ if $d \leq 11$, so that degree 11 scrolls cannot be 2-normal. Assume $d \geq 13$. If E is semistable then $\mu^-(E) = \mu(E) = d/3 > 4$ and thus $(X,\mathcal{T})$ is p.n. by Lemma 2.4. Let now E be not semistable. Assume E does not admit a degree 8 and rank 2 quotient. All quotients $E \to Q \to 0$ must be very ample and thus it must be $\operatorname{rank} Q = 1$ and $\deg Q \geq 5$ or $\operatorname{rank} Q = 2$ and $\deg Q \geq 9$. Therefore for all Q it is $\mu(Q) > 4$ and thus $\mu^-(E) > 4$ and then $(X,\mathcal{T})$ is p.n. by Lemma 2.4.

Let now E be not semistable with a quotient $E \to \mathcal{E} \to 0$ of degree 8 and rank 2. Notice that $(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$ is one of the non 2-normal surfaces of degree eight embedded in $\mathbb{P}^5$ studied in [2]. Let D be the line bundle of degree $d-8$ such that

$$0 \to D \to E \to \mathcal{E} \to 0\,. \tag{8}$$

Since $d - 8 > 2$ it is $h^1(D) = 0$ and thus $H^0(E) = H^0(\mathcal{E}) \oplus H^0(D)$. Therefore

$$S^2\big(H^0(E)\big) = S^2\big(H^0(\mathcal{E})\big) \bigoplus H^0(\mathcal{E}) \otimes H^0(D) \bigoplus S^2\big(H^0(D)\big)\,. \tag{9}$$

Consider the sequence obtained by tensoring (8) with D:

$$0 \to D \otimes D \to E \otimes D \to \mathcal{E} \otimes D \to 0\,. \tag{10}$$

Because $\deg(D \otimes D) = 2(d-8) > 2$ and $\mu^-(\mathcal{E} \otimes D) = \mu^-(\mathcal{E}) + \mu^-(D) = d - 4 > 2$ it follows that $h^1(D \otimes D) = h^1(\mathcal{E} \otimes D) = 0$ and thus $H^0(E \otimes D) = H^0(D \otimes D) \oplus H^0(\mathcal{E} \otimes D)$ and $h^1(E \otimes D) = 0$. Considering now the exact sequence

$$0 \to D \otimes E \to S^2(E) \to S^2(\mathcal{E}) \to 0 \tag{11}$$

it follows that

$$H^0\big(S^2(E)\big) = H^0\big(S^2(\mathcal{E})\big) \bigoplus H^0(\mathcal{E} \otimes D) \bigoplus H^0(D \otimes D)\,. \tag{12}$$

Putting together (12) and (9) it follows that the map $\phi : S^2(H^0(E)) \to H^0(S^2(E))$ decomposes as

$$\begin{gathered}\big[S^2(H^0(\mathcal{E})) \xrightarrow{\alpha} H^0(S^2(\mathcal{E}))\big] \\ \bigoplus \\ [H^0(\mathcal{E}) \otimes H^0(D) \xrightarrow{\beta} H^0(\mathcal{E} \otimes D)] \bigoplus [S^2(H^0(D)) \xrightarrow{\gamma} H^0(D \otimes D)]\,.\end{gathered}$$

It was proven in [2] that α is not surjective, therefore ϕ cannot be surjective, i.e. $(X, \mathcal{T})$ cannot be 2-normal. $\square$

Remark 4.6. The existence of degree 11 and 12 threefold scrolls over curves of genus 2 is an open problem. If a degree 12 such scroll $(X, \mathcal{T}) = (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ exists then it is not difficult to see that E must be semistable. If it were not semistable then there would be a destabilizing subbundle $\mathcal{F}$ either of rank 1 such that $\deg \mathcal{F} > 4$ or of rank 2 such that $\deg \mathcal{F} > 8$.

In both cases the resulting quotient $0 \to \mathcal{F} \to E \to Q \to 0$ could not be very ample for degree reasons, which is a contradiction.

Remark 4.7. Let $(X, \mathcal{T}) = (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ again be a 3 dimensional scroll over a curve of genus 2. If (X, L) is projectively normal, recalling that $h^1(E) = 0$, the same argument used in Proposition 3.11 gives $r(X, L) = 3$. If (X, L) is not p.n., notice that if $d \geq 13$ it follows that $h^0(\mathcal{T}) \geq 10$ and thus $(X, \mathcal{T})$ is $(\Delta + 2)$-regular, i.e. 8-regular from [29]. When $d = 11, 12$ it is easy to check that $h^i(\mathcal{I}_X(8 - i)) = 0$ for all $i \geq 2$ while we were not able to establish the 7-normality of these manifolds.

5. $\mathbb{P}^{r-1}$ bundles over an elliptic curve

Throughout this section let E be a vector bundle of rank r and degree d over an elliptic curve C. Let $(X, \mathcal{T}) = (\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ and let D be a divisor on X numerically equivalent to $aT + bF$. Assume D is very ample. The projective normality of the embedding given by D was studied by Homma [16], [17] when $r = 2$, and in a more general setting by Butler [8] (see also [2]). In this section the case of $a = 2$ and $r = 3$ is addressed and Butler's results are improved in some cases.

Lemma 5.1

Let $E = \bigoplus_i E_i$ be a decomposable vector bundle over an elliptic curve. Then

$$\mu^+(E) = \max_i \left\{\mu(E_i)\right\}.$$

Proof. This is essentially [1] Lemma 2.8, reinterpreted from the point of view of μ^+ instead of μ^-. $\square$

Lemma 5.2

Let $(X, \mathcal{T})$ be as above and let M_s be a divisor on X whose numerical class is $M_s \equiv T + sF$. Let $m = \min\left\{t \in \mathbb{Z} | h^0(M_t) > 0\right\}$. Then $m = -[\mu^+(E)]$ and there exists a smooth $S \equiv T + mF$.

Proof. From [8] it follows that for any vector bundle $\mathcal{G}$ over an elliptic curve $\mu^+(\mathcal{G}) < 0$ implies $h^0(\mathcal{G}) = 0$. For simplicity of notation let $m^* = -[\mu^+(E)]$. We need to show that $m = m^*$. Let $\mathcal{L}_t$ be a line bundle on C with degree t. If $t < m^*$, then $t = m^* - x$ for some integer $x \geq 1$. Then $\mu^+(E \otimes \mathcal{L}_t) = \mu^+(E) + t = \mu^+(E) + m^* - x < 1 - x \leq 0$. Therefore $h^0(M_t) = 0$ if $t < m^*$ and thus

$$m^* \leq m\,. \tag{13}$$

Let now E be indecomposable and thus semistable. Because $\mu(E) = \mu^+(E)$ and $-m^* \leq \mu^+(E)$ it is $d + rm^* \geq 0$. If $d + rm^* > 0$ then $h^0(M_{m^*}) > 0$. If $d + rm^* = 0$ then, as in [3], a line bundle $\mathcal{L}_{m^*}$ of degree m^* can be found by a suitable twist of degree zero, such that $h^0(E \otimes \mathcal{L}_{m^*}) = 1$.

Let now $E = \bigoplus_i E_i$ be decomposable. Then $h^0(E) = \bigoplus_i h^0(E_i)$. Let $E_{\hat{i}}$ be one of the components such that $\mu^+(E) = \mu(E_{\hat{i}})$. As $E_{\hat{i}}$ is indecomposable it follows from above that there exists a line bundle $\mathcal{L}_{m^*}$ of degree m^* such that $h^0(E_{\hat{i}} \otimes \mathcal{L}_{m^*}) > 0$.

Therefore $h^0(M_{m^*}) > 0$ and thus $m \leq m^*$ which combined with (13) gives $m = m^*$.

If $S \equiv T + m^*F$ is singular it must be reducible as $S' \cup (m^* - t)F$ where $S' \equiv T + tF$ with $t < m^*$ which is not possible because of the minimality of $m^* = m$. Therefore there must be a smooth $S \equiv T + m^*F$. $\square$

Lemma 5.3

Let $(X, \mathcal{T})$ and D be as above with $r = 3$, $a \geq 2$, and D very ample. If
i) there exists an ample smooth surface $S \equiv T + xF$ for some $x \in \mathbb{Z}$;
ii) $(a-1)\mu^-(E) + b - x > 1$
then the embedding of X given by D is projectively normal if and only if it is 2-normal.

Proof. If the embedding is p.n. it is obviously 2-normal. As in Lemma 4.3 the projective normality follows from the surjectivity of the maps

$$H^0\big((j-1)D\big) \otimes H^0(D) \to H^0(jD) \quad \text{for all } \ j \geq 2\,. \tag{14}$$

Assume $j \geq 4$. Surjectivity in (14) follows, according to [27], from the vanishing of

$$H^i\big(X, (j-1-i)D\big) = 0 \ \text{ for all } 3 \geq i \geq 1 \quad \text{and for all } j \geq 4\,.$$

Notice that $R^q\pi_*((j-1-i)D) = R^q\pi_*(a(j-1-i)\mathcal{T}) \otimes \mathcal{M}_{i,j}$ where $\mathcal{M}_{i,j}$ is a line bundle on C of degree $(j-1-i)b$. Notice also, e.g. [15], that $R^q\pi_*(a(j-1-i)\mathcal{T}) = 0$ unless $q = 0$ and $j-1-i \geq 0$, or $q = 2$ and $a(j-1-i) \leq -3$. Since $a \geq 2$, $i \leq 3$ and $j \geq 4$, the last inequality is never satisfied . For $j \geq 4$ Leray's spectral sequence shows that it is enough to show $H^1(X, (j-2)D) = H^1(C, S^{a(j-2)}E \otimes \mathcal{M}_{1,j}) = 0$ which is guaranteed by D being ample. This gives all necessary surjectivity in (14) but for $j = 2, 3$. If $j = 2$ the map $H^0(D) \otimes H^0(D) \to H^0(2D)$ is onto by assumption, being the embedding 2-normal. Assume now $j = 3$. Let $S \equiv T + xF$ be the smooth element whose existence is given by assumption *i*). Ampleness of D gives $a\mu^-(E) + b > 0$. Combining this with condition *ii*) it follows from Lemma 2.3 that $H^1(tD - S) = 0$ for $t = 1, 2, 3$. In particular, following Homma, the commutative diagram below is obtained:

$$\begin{array}{ccccccc}
0 \to H^0(D-S) \otimes H^0(2D) & \to & H^0(D) \otimes H^0(2D) & \to & H^0(S, D_{|S}) \otimes H^0(2D) \to 0 \\
\downarrow^{\alpha} & & \downarrow^{\beta} & & \downarrow^{\gamma} \\
0 \to H^0(3D-S) & \to & H^0(3D) & \to & H^0(S, 3D_{|S}) \to 0
\end{array} \tag{15}$$

The surjectivity of β will follow from the surjectivity of α and γ. From [16] and [17] it follows that $D_{|S}$ is normally generated. Because $H^0(2D) \to H^0(2D_{|S})$ is surjective from above, it follows that γ is onto.

Lemma 2.3 and condition *ii*) give $D - S \equiv (a-1)T + (b-x)F$ generated by global sections. Using this fact and noticing that $H^1(D+S) = 0$ being D very ample and S ample, it is straightforward to check that $H^i(2D - i(D-S)) = 0$ for all $i \geq 1$. Therefore by [27] α is onto. □

Proposition 5.4

Let $(X, \mathcal{T})$ and $D \equiv 2T + bF$ be as above. If
i) there exists an ample smooth divisor $Y \equiv T + xF$ for some $x \in \mathbb{Z}$;
ii) $\mu^-(E) + b - x > 1$
then $|D|$ gives a 2-normal embedding of X if $|D_{|_Y}|$ gives a 2-normal embedding of Y.

Proof. The proof proceeds along the same lines of the case $j = 3$ in the proof of Lemma 5.3. Let $Y \equiv T + xF$ be the smooth element whose existence is given by assumption *i*). Ampleness of D gives $2\mu^-(E) + b > 0$. Combining this with condition *ii*) it follows that $H^1(tD - Y) = 0$ for $t = 1, 2$. In particular the following commutative diagram is obtained:

$$\begin{array}{ccccccccc} 0 \to & H^0(D-Y)\otimes H^0(D) & \to & H^0(D)\otimes H^0(D) & \to & H^0(Y, D_{|_Y})\otimes H^0(D) & \to 0 \\ & \downarrow^{\alpha} & & \downarrow^{\beta} & & \downarrow^{\gamma} & \\ 0 \to & H^0(2D-Y) & \to & H^0(2D) & \to & H^0(Y, 2D_{|_Y}) & \to 0\,. \end{array} \tag{16}$$

The surjectivity of β will follow from the surjectivity of α and γ.

Because $H^0(D) \to H^0(D_{|_Y})$ is onto from above and

$$H^0(D_{|_Y}) \otimes H^0(D_{|_Y}) \to H^0(2D_{|_Y})$$

is onto by assumption it follows that γ is onto.

Condition *ii*) is equivalent to $D - S \equiv T + (b - x)F$ being generated by global sections (see Lemma 2.3 and [1] Lemma 2.9). Using this fact and noticing that $H^1(Y) = 0$ being Y ample, it is straightforward to check that $H^i(D - i(D - Y)) = 0$ for all $i \geq 1$. Therefore by [27] α is onto. □

Corollary 5.5

Let $(X, \mathcal{T})$ and D be as above with $r = 3$ and $a = 2$. If
i) $\mu^-(E) > [\mu^+(E)]$
ii) $\mu^-(E) + b > 1 - [\mu^+(E)]$
then $|D|$ gives a projectively normal embedding.

Proof. Let $x = -[\mu^+(E)]$, notice that condition i) and Theorem 2.2 give ampleness of $T + xF$. Now combine Lemma 5.2, Lemma 5.3, Proposition 5.4 and the fact that a very ample line bundle on a 2 dimensional scroll over an elliptic curve is always normally generated by [16] and [17]. □

Remark 5.6. Let E be an indecomposable vector bundle of rank $r = 3$ and degree $d \equiv 1(3)$. For simplicity let us assume that E has been normalized, so $d = 1$. Since E is indecomposable it is semistable and $\mu^-(E) = \mu^+(E) = \mu(E) = 1/3$. The hypothesis of Corollary 5.5 are satisfied for $D \equiv 2T + F$, and such a D is very ample from [1] Theorem 4.5. Therefore $|D|$ gives a projectively normal embedding. Notice that Butler's results [8], see Lemma 2.4, were not able to establish the normal generation of such a D.

Remark 5.7. It is straightforward to check that for a divisor D as in Corollary 5.5 it is always $h^0(D) \geq 10$ and therefore the embedding given by $|D|$ satisfies the Eisenbud Goto conjecture (see [29]).

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