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Generalized Jacobi forms and abelian schemes over arithmetic varieties

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Abstract

We generalize Jacobi forms of an arbitrary degree and construct torus bundles over abelian schemes whose sections can be identified with such generalized Jacobi forms.

1. Introduction

Jacobi forms of degree one were first systematically introduced by Eichler and Zagier in [2], and they are essentially automorphic forms for the Jacobi group which is the semidirect product of $SL(2,\mathbb{R})$ and a Heisenberg group. Since then various aspects of such Jacobi forms have been investigated in numerous papers. The theory of Jacobi forms of higher degree has also been developed and studied by several authors (see e.g. [6], [16], [17], [21]). Such Jacobi forms are closely connected with Siegel modular forms of half-integral weight as well as with certain theta series. The purpose of this paper is to discuss a geometric interpretation of Jacobi forms of degree > 1.

Mixed Shimura varieties generalize Shimura varieties (cf. [15], [18]), and they play an essential role in the study of boundaries and toroidal compactifications of Shimura varieties (see e.g. [1], [3], [4], [5]). Examples of mixed Shimura varieties include abelian schemes over Shimura varieties, torus bundles over such abelian schemes, and of course (pure) Shimura varieties.

In this paper we introduce generalized Jacobi forms associated to a symplectic representation of a semisimple Lie group, construct an example of a mixed Shimura variety that is a one-dimensional torus bundle over an abelian scheme, and identify the sections of such a bundle with generalized Jacobi forms.

2. Jacobi forms

In this section we review the definition of Jacobi forms of an arbitrary degree. Let H be the Heisenberg group which consists of triples $[u, v, \lambda]$ with $u, v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and whose multiplication operation is given by

$$[u, v, \lambda] \cdot [u', v', \lambda'] = [u + u', v + v', \lambda + \lambda' + uv'^t - vu'^t]$$

for $\lambda \in \mathbb{R}$ and the elements $u, v, u', v' \in \mathbb{R}^n$ considered as row vectors. Then the symplectic group $Sp(n, \mathbb{R})$ operates on H by

$$[u, v, \lambda] \cdot M = [(u, v) \cdot M, \lambda]$$

for $M \in Sp(n, \mathbb{R})$ and $[u, v, \lambda] \in H$. The Jacobi group G_0^J is the semidirect product $Sp(n, \mathbb{R}) \ltimes H$ with its multiplication operation given by

$$\begin{aligned}(M, [u, v, \lambda]) \cdot (M', [u', v', \lambda']) &= (MM', ([u, v, \lambda] \cdot M') \cdot [u', v', \lambda']) \\ &= (MM', [\tilde{u} + u', \tilde{v} + v', \lambda + \lambda' + \tilde{u}v'^t - \tilde{v}u'^t]),\end{aligned}$$

where $\tilde{u}, \tilde{v} \in \mathbb{R}^n$ with $(\tilde{u}, \tilde{v}) = (u, v) \cdot M'$. As usual the symplectic group $Sp(n, \mathbb{R})$ operates on the Siegel upper half space

$$\mathcal{H}^n = \{z \in M_{n \times n}(\mathbb{C}) \mid z = z^t, \ \operatorname{Im} z \gg 0\}$$

by

$$M\langle z \rangle = (Az + B)(Cz + D)^{-1} \quad \text{for} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}) \quad \text{and} \quad z \in \mathcal{H}^n,$$

and the Jacobi group G_0^J operates on $\mathcal{H}^n \times \mathbb{C}^n$ by

$$(M, [u, v, \lambda]) \cdot (z, w) = (M\langle z \rangle, (w + uz + v)(Cz + D)^{-1}).$$

Let $\Gamma_0 \subset Sp(n, \mathbb{Z})$ be a torsion-free discrete subgroup of $Sp(n, \mathbb{R})$, and consider the associated discrete subgroup

$$\Gamma_0^J = \{(M, [u, v, \lambda]) \in G_0^J \mid M \in \Gamma_0; \ u, v \in \mathbb{Z}^n; \ \lambda \in \mathbb{Z}\}$$

of G_0^J. Given nonnegative integers l and m we define the map $J_{l,m} : G_0^J \times (\mathcal{H}^n \times \mathbb{C}^n) \to \mathbb{C}^\times$ by

$$\begin{aligned}J_{l,m}((M, [u, v, \lambda]), (z, w)) &= \det(Cz + D)^l \\ &\quad \cdot \mathbf{e}^m(-\lambda + w(Cz + D)^{-1}Cw^t - 2u(Cz + D)^{-1}w^t - uM\langle z \rangle u^t - vu^t),\end{aligned}$$

where $\mathbf{e}^m(*) = e^{2\pi m i(*)}$. Then $J_{l,m}$ is an automorphy factor of G_0^J, i.e., it satisfies the relation

$$J_{l,m}(\gamma\gamma', (z,w)) = J_{l,m}(\gamma, \gamma'(z,w)) \cdot J_{l,m}(\gamma', (z,w))$$

for $\gamma, \gamma' \in G_0^J$ and $(z,w) \in \mathcal{H}^n \times \mathbb{C}^n$. Thus there is an operation of G_0^J on the space of functions on $\mathcal{H}^n \times \mathbb{C}^n$ by

$$\begin{aligned}
&(F|_{l,m}(M,[u,v,\lambda]))(z,w)\\
&\quad = J_{l,m}((M,[u,v,\lambda]),(z,w))^{-1} \cdot F((M,[u,v,\lambda]) \cdot (z,w))\\
&\quad = J_{l,m}((M,[u,v,\lambda]),(z,w))^{-1} \cdot F(M\langle z\rangle, (w+uz+v)(Cz+D)^{-1})\,.
\end{aligned}$$

DEFINITION 2.1. A *Jacobi form of weight l and index m for* Γ_0^J is a holomorphic map $F : \mathcal{H}^n \times \mathbb{C}^n \to \mathbb{C}^\times$ satisfying the following conditions:

(i) $F|_{l,m}(M,[u,v,\lambda]) = F$ for all $(M,[u,v,\lambda]) \in \Gamma_0^J$.

(ii) For each $M \in Sp(n,\mathbb{Z})$ the function $F|_{l,m}(M,[0,0,0])$ has a Fourier expansion of the form

$$(F|_{l,m}(M,[0,0,0]))(z,w) = \sum_{T,r} a(T,r) \cdot \mathbf{e}(\mathrm{Tr}(Tz)/\nu) \cdot \mathbf{e}(r^t w)$$

for some $\nu \in \mathbb{Z}$, where T runs over $n \times n$ symmetric half integral matrices and r over the elements of $\mathbb{Z}^n$, and $a(T,r) \neq 0$ only if the matrix $\left(\begin{smallmatrix} T/\nu & r/2 \\ r^t/2 & m \end{smallmatrix}\right)$ is positive semi-definite.

Remark 2.2. More general definition of Jacobi forms of degree > 1 can also be considered (see e.g. [15], [16], [21]). For example, if ρ is the determinant map and if $j = 1$, then Definition 1.1 in [21] reduces to Definition 2.1 above.

If $n \geq 2$, then for certain types of groups Γ_0^J the condition (i) in Definition 2.1 implies the condition (ii), and so (ii) is not necessary. In fact we have the following Koecher's principle for Jacobi forms:

Proposition 2.3

Let $n \geq 2$ and let Γ_0 be a subgroup of $Sp(n,\mathbb{Z})$ of finite index. Then any holomorphic function F satisfying the condition (i) in Definition 2.1 also satisfies the condition (ii).

Proof. See [21, Lemma 1.6]. □

3. Torus bundles over abelian schemes

In this section we construct abelian schemes over arithmetic varieties as well as torus bundles over such abelian schemes. Let the discrete groups $\Gamma_0 \subset Sp(n, \mathbb{Z})$ and $\Gamma_0^J \subset G_0^J$ be as in §2. We set $Y_0 = \Gamma_0^J \backslash \mathcal{H}^n \times \mathbb{C}^n$, where the quotient is taken using the operation of G_0^J on $\mathcal{H}^n \times \mathbb{C}^n$ described in §2. Then we have

$$Y_0 \cong \Gamma_0 \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n) \backslash \mathcal{H}^n \times \mathbb{C}^n,$$

where the operation of the semidirect product $\Gamma_0 \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n)$ on $\mathcal{H}^n \times \mathbb{C}^n$ is given by

$$(\gamma, u, v) \cdot (z, w) = (\gamma\langle z\rangle, (w + uz + v)(Cz + D)^{-1})$$

for $u, v \in \mathbb{Z}^n$, $(z, w) \in \mathbb{C}^n$ and $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_0$. Let $X_0 = \Gamma_0 \backslash \mathcal{H}^n$ be the Siegel modular variety determined by Γ_0. Then the natural projection $\mathcal{H}^n \times \mathbb{C}^n \to \mathcal{H}^n$ induces the map $Y_0 \to X_0$ which is a fiber bundle with its fiber isomorphic to the quotient $\mathbb{C}^n/(\mathbb{Z}^n \times \mathbb{Z}^n)$. In fact, the fiber has a natural structure of a polarized abelian variety, and Y_0 is a universal family of abelian varieties (see [19, p. 200]). The discrete subgroup Γ_0^J of G_0^J operates on $(\mathcal{H}^n \times \mathbb{C}^n) \times \mathbb{C}^\times$ by

$$\gamma \cdot ((z, w), \zeta) = (\gamma \cdot (z, w), J_{l,m}(\gamma, (z, w))\zeta)$$

for $\gamma \in \Gamma_0^J$, $(z, w) \in \mathcal{H}^n \times \mathbb{C}^n$, $\zeta \in \mathbb{C}^\times$. (Note that $J_{l,m}(\gamma, (z, w))\zeta \neq 0$ for all $\gamma \in \Gamma_0^J$ and $(z, w) \in \mathcal{H}^n \times \mathbb{C}^n$.) We set

$$\mathcal{T}_0 = \Gamma_0^J \backslash (\mathcal{H}^n \times \mathbb{C}^n) \times \mathbb{C}^\times,$$

where the quotient is taken with respect to the operation described above. Then the natural projection map $(\mathcal{H}^n \times \mathbb{C}^n) \times \mathbb{C}^\times \to \mathcal{H}^n \times \mathbb{C}^n$ induces the map $\mathcal{T}_0 \to Y_0$ which has the structure of a principal fiber bundle with fiber the one-dimensional complex torus $\mathbb{C}^\times$.

Let G be a semisimple Lie group, and let K be a maximal compact subgroup of G. We assume that the associated symmetric space $\mathcal{D} = G/K$ has a G-invariant complex structure. Let $\rho : G \to Sp(n, \mathbb{R})$ be a homomorphism of Lie groups and let $\tau : \mathcal{D} \to \mathcal{H}^n$ be a holomorphic map such that

$$\tau(gz) = \rho(g)\tau(z)$$

for all $z \in \mathcal{D}$. Let Γ be a torsion free arithmetic subgroup of G with respect to some $\mathbb{Q}$-structure of G such that $\rho(\Gamma) \subset \Gamma_0$, and let $X = \Gamma \backslash \mathcal{D}$ be the corresponding arithmetic variety. Then the map τ induces a morphism $\tau_X : X \to X_0 = \Gamma_0\backslash\mathcal{H}^n$ of arithmetic varieties. Let $Y = \tau_X^* Y_0$ be the fiber bundle over X obtained by pulling back Y_0 via τ_X so that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tau_Y} & Y_0 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\tau_X} & X_0 \end{array}$$

is commutative. Then Y is an abelian scheme over X, usually known as a *Kuga fiber variety*, and its fiber is a polarized abelian variety (cf. [7], [9], [11], [13], [14], [19]). Now we obtain the fiber bundle $\mathcal{T} = \tau_Y^* \mathcal{T}_0$ over Y by pulling back $\mathcal{T}_0$ via τ_Y so that the diagram

$$\begin{array}{ccc} \mathcal{T} & \longrightarrow & \mathcal{T}_0 \\ \downarrow & & \downarrow \\ Y & \xrightarrow{\tau_Y} & Y_0 \end{array}$$

commutes. Then $\mathcal{T}$ is a principal bundle over Y whose fiber is isomorphic to the one-dimensional complex torus $\mathbb{C}^\times$.

Let H be the Heisenberg group described in §2. Then G operates on H by

$$[u, v, \lambda] \cdot g = [(u, v) \cdot \rho(g), \lambda]$$

for $g \in G$ and $[u, v, \lambda] \in H$. We define the *generalized Jacobi group* G^J to be the semidirect product $G \ltimes H$ with its multiplication operation given by

$$\begin{aligned}(g, [u, v, \lambda]) \cdot (g', [u', v', \lambda']) &= (gg', ([u, v, \lambda] \cdot g) \cdot [u', v', \lambda']) \\ &= (gg', [\tilde{u} + u', \tilde{v} + v', \lambda + \lambda' + \tilde{u}v'^t - \tilde{v}u'^t]),\end{aligned}$$

where $\tilde{u}, \tilde{v} \in \mathbb{R}^n$ with $(\tilde{u}, \tilde{v}) = (u, v) \cdot \rho(g')$. Then G^J operates on $\mathcal{D} \times \mathbb{C}^n$ by

$$(g, [u, v, \lambda]) \cdot (z, w) = (gz, (w + u\tau(z) + v)(C_\rho \tau(z) + D_\rho)^{-1})$$

for $g \in G$, $[u, v, \lambda] \in H$, $(z, w) \in \mathcal{D} \times \mathbb{C}^n$, and

$$\rho(g) = \begin{pmatrix} A_\rho & B_\rho \\ C_\rho & D_\rho \end{pmatrix} \in Sp(n, \mathbb{R}).$$

Lemma 3.1

Let Y be the Kuga fiber variety constructed above, and let

$$\Gamma^J = \left\{(g, [u, v, \lambda]) \in G^J \mid g \in \Gamma;\ u, v \in \mathbb{Z}^n;\ \lambda \in \mathbb{Z}\right\}.$$

Then there is a canonical isomorphism

$$Y \cong \Gamma^J \backslash \mathcal{D} \times \mathbb{C}^n .$$

Proof. From the construction of $Y = \tau_X^* Y_0$ it follows that

$$Y \cong \Gamma \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n) \backslash \mathcal{D} \times \mathbb{C}^n ,$$

where the quotient is taken with respect to the operation of $\Gamma \ltimes (\mathbb{Z}^n \times \mathbb{Z}^n)$ on $\mathcal{D} \times \mathbb{C}^n$ given by

$$(g, u, v) \cdot (z, w) = (gz, (w + u\tau(z) + v)(C_\rho \tau(z) + D_\rho)^{-1})$$

for $g \in G$, $u, v \in \mathbb{Z}^n$ and $(z, w) \in \mathcal{D} \times \mathbb{C}^n$. However, this operation coincides with the one of Γ^J on $\mathcal{D} \times \mathbb{C}^n$; hence the lemma follows. □

4. Generalized Jacobi forms

Let H be the Heisenberg group described in §2, and let the homomorphism $\rho : G \to Sp(n, \mathbb{R})$ and the holomorphic map $\tau : \mathcal{D} \to \mathcal{H}^n$ be as in § 3. In this section we define generalized Jacobi forms associated to Γ, ρ and τ, and describe a geometric interpretation of such Jacobi forms.

Let $G^J = G \ltimes H$ be the generalized Jacobi group described in §3. Given integers l and m we define the map $J^{\rho,\tau}_{l,m} : G^J \times (\mathcal{D} \times \mathbb{C}^n) \to \mathbb{C}^\times$ by

$$\begin{aligned} J^{\rho,\tau}_{l,m}((g, [u, v, \lambda]), (z, w)) &= J_{l,m}((\rho(g), [u, v, \lambda]), (\tau(z), w)) \\ &= \det(C_\rho \tau(z) + D_\rho)^l \cdot \mathbf{e}^m(-\lambda + w(C_\rho \tau(z) + D_\rho)^{-1} C_\rho w^t \\ &\qquad - 2u(C_\rho \tau(z) + D_\rho)^{-1} w^t - u(\tau(gz))u^t - vu^t) , \end{aligned}$$

where $\mathbf{e}^m(*) = e^{2\pi m i (*)}$ as in §2 and $\rho(g) = \begin{pmatrix} A_\rho & B_\rho \\ C_\rho & D_\rho \end{pmatrix} \subset Sp(n, \mathbb{R})$.

Lemma 4.1

The map $J^{\rho,\tau}_{l,m}$ is an automorphy factor of G^J, i.e., it satisfies

$$J^{\rho,\tau}_{l,m}(\gamma\gamma', (z, w)) = J^{\rho,\tau}_{l,m}(\gamma, \gamma' \cdot (z, w)) \cdot J^{\rho,\tau}_{l,m}(\gamma', (z, w))$$

for $\gamma, \gamma' \in G^J$ and $(z, w) \in \mathcal{D} \times \mathbb{C}^n$.

Proof. If $(z,w) \in \mathcal{D} \times \mathbb{C}^n$ and $\gamma, \gamma' \in G^J$ with $\gamma = (g,[u,v,\lambda])$, $\gamma' = (g',[u',v',\lambda'])$, then

$$\begin{aligned} J^{\rho,\tau}_{l,m}(\gamma\gamma',(z,w)) &= J^{\rho,\tau}_{l,m}((gg',[\tilde{u}+u',\tilde{v}+v',\lambda+\lambda'+\tilde{u}v'^t-\tilde{v}u'^t]),(z,w)) \\ &= J_{l,m}((\rho(gg'),[\tilde{u}+u',\tilde{v}+v',\lambda+\lambda'+\tilde{u}v'^t-\tilde{v}u'^t]),(\tau(z),w))\,, \end{aligned}$$

where $(\tilde{u},\tilde{v}) = (u,v)\cdot\rho(g')$. If $\delta = (\rho(g),[u,v,\lambda])$ and $\delta' = (\rho(g'),[u',v',\lambda'])$, then we have

$$J^{\rho,\tau}_{l,m}(\gamma\gamma',(z,w)) = J_{l,m}(\delta\delta',(\tau(z),w)) = J_{l,m}(\delta,\delta'\cdot(\tau(z),w))\cdot J_{l,m}(\delta',(\tau(z),w)),$$

since $\rho(gg') = \rho(g)\rho(g')$ and $J_{l,m}$ is an automorphy factor of G^J_0. However, we have

$$J_{l,m}(\delta',(\tau(z),w)) = J^{\rho,\tau}_{l,m}(\gamma',(z,w)),$$

and, since $\rho(g')\tau(z) = \tau(g'z)$, we get

$$\begin{aligned} J_{l,m}(\delta,\delta'\cdot(\tau(z),w)) &= J_{l,m}(\delta,(\rho(g')\tau(z),(w+u'\tau(z)+v')(C\tau(z)+D)^{-1})) \\ &= J^{\rho,\tau}_{l,m}(\gamma,\gamma'\cdot(z,w))\,. \end{aligned}$$

Therefore the lemma follows. □

Since $J^{\rho,\tau}_{l,m}$ is an automorphy factor of G^J_0, we obtain an operation of G^J on the space of functions F on $\mathcal{D}\times\mathbb{C}^n$ by

$$\begin{aligned} &(F|^{\rho,\tau}_{l,m}(g,[u,v,\lambda]))(z,w) \\ &\quad = J^{\rho,\tau}_{l,m}((g,[u,v,\lambda]),(z,w))^{-1}\cdot F((g,[u,v,\lambda])\cdot(z,w)) \\ &\quad = J^{\rho,\tau}_{l,m}((g,[u,v,\lambda]),(z,w))^{-1}\cdot F(gz,(w+u\tau(z)+v)(C_\rho z+D_\rho)^{-1})\,. \end{aligned}$$

DEFINITION 4.2. Let Γ^J be as in Lemma 3.1. A holomorphic map $F:\mathcal{D}\times\mathbb{C}^n\to\mathbb{C}^\times$ is a *generalized Jacobi form of weight* l *and index* m *for* Γ^J *associated to* ρ *and* τ if there is a Jacobi form $F':\mathcal{H}^n\times\mathbb{C}^n\to\mathbb{C}$ of weight l and index m for the discrete subgroup

$$\Gamma^J_\rho = \big\{(\rho(\gamma),[u,v,\lambda])\in G^J_0 \mid g\in\Gamma,\ u,v\in\mathbb{Z}^n,\ \lambda\in\mathbb{Z}\big\}$$

of G^J_0 such that $F(z,w) = F'(\tau(z),w)$ for all $(z,w)\in\mathcal{D}\times\mathbb{C}^n$.

Proposition 4.3

Let F be a generalized Jacobi form of weight l and index m for Γ^J associated to ρ and τ. Then F satisfies the following conditions:

(i) $F|^{\rho,\tau}_{l,m}\gamma = F$ for all $\gamma \in \Gamma^J$.

(ii) For each $g \in \Gamma$ the function $F|^{\rho,\tau}_{l,m}(g, [0, 0, 0])$ has a Fourier expansion of the form

$$(F|^{\rho,\tau}_{l,m}(g, [0, 0, 0]))(z, w) = \sum_{T,r} a(T, r) \cdot \mathbf{e}(\mathrm{Tr}(\mathrm{T}\tau(\mathrm{z}))/\nu) \cdot \mathbf{e}(\mathrm{r}^{\mathrm{t}}\mathrm{w}) \,,$$

where T runs over $n \times n$ symmetric half integral matrices and r over the elements of $\mathbb{Z}^n$, with a suitable $\nu \in \mathbb{Z}$ and $a(T, r) \neq 0$ only if the matrix $\begin{pmatrix} T/\nu & r/2 \\ r^t/2 & m \end{pmatrix}$ is positive semi-definite.

Proof. Let F' be a Jacobi form for Γ^J_ρ such that $F'(\tau(z), w) = F(z, w)$ and let $\gamma = (g, [u, v, \lambda]) \in \Gamma^J$. Then we have

$$\begin{aligned} F|^{\rho,\tau}_{l,m}\gamma &= J^{\rho,\tau}_{l,m}((g, [u, v, \lambda]), (z, w))^{-1} \cdot F(gz, (w+u\tau(z)+v)(C_\rho z + D_\rho)^{-1}) \\ &= J_{l,m}((\rho(g), [u, v, \lambda]), (\tau(z), w))^{-1} \cdot F'(\tau(gz), (w+u\tau(z)+v)(C_\rho z+D_\rho)^{-1}) \\ &= F'(\tau(z), w) = F(z, w) \,, \end{aligned}$$

since $\tau(gz) = \rho(g)\tau(z)$ and $(\rho(g), [u, v, \lambda]) \in \Gamma^J_\rho$. Hence the condition (i) holds. As for (ii), for each $g \in \Gamma$ we have

$$\begin{aligned} F|^{\rho,\tau}_{l,m}(g, [0, 0, 0]) &= J^{\rho,\tau}_{l,m}(g, [0, 0, 0]), (z, w))^{-1} F(gz, 0) \\ &= J_{l,m}((\rho(g), [0, 0, 0]), (\tau(z), w))^{-1} F'(\rho(g)\tau(z), 0) \\ &= F'|_{l,m}(\rho(g), [0, 0, 0]) \,. \end{aligned}$$

Since $\rho(g) \in Sp(n, \mathbb{Z})$, the condition (ii) follows from the condition (ii) in Definition 2.1. □

Theorem 4.4

Assume that $\rho(\Gamma)$ is a subgroup of $Sp(n, \mathbb{Z})$ of finite index and that $n \geq 2$. Then the space of sections $H^0(Y, \widetilde{\mathcal{T}})$ of the bundle $\mathcal{T}$ over Y is canonically isomorphic to the space of generalized Jacobi forms for Γ^J of weight l and index m associated to ρ and τ, where $\widetilde{\mathcal{T}}$ denotes the sheaf of sections of $\mathcal{T}$.

Proof. The bundle $\mathcal{T}$ is the pullback $\tau_Y^*\mathcal{T}_0$ of the bundle $\mathcal{T}_0$ via the map $\tau_Y : Y \to Y_0$, and we have the commutative diagram

$$\begin{array}{ccc} (\mathcal{H}^n \times \mathbb{C}^n) \times \mathbb{C}^\times & \longrightarrow & \mathcal{T} = \Gamma_0^J \backslash (\mathcal{H}^n \times \mathbb{C}^n) \times \mathbb{C}^\times \\ \Big\downarrow & & \Big\downarrow \\ \mathcal{H}^n \times \mathbb{C}^n & \longrightarrow & Y = \Gamma_0^J \backslash (\mathcal{H}^n \times \mathbb{C}^n)\,, \end{array}$$

where the horizontal maps are natural projections. A section $f_0 : Y_0 \to \mathcal{T}_0$ can be identified with a map $f_0' : \mathcal{H}^n \times \mathbb{C}^n \to \mathbb{C}^\times$ such that the map $\tilde{f}_0 : \mathcal{H}^n \times \mathbb{C}^n \to (\mathcal{H}^n \times \mathbb{C}^n) \times \mathbb{C}^\times$ given by $\tilde{f}_0(z,w) = ((z,w), f_0'(z,w))$ is a lifting of f_0 and $\tilde{f}_0(\gamma \cdot (z,w)) = \gamma \cdot \tilde{f}_0(z,w)$ for each $\gamma \in \Gamma_0^J$ and $(z,w) \in \mathcal{H}^n \times \mathbb{C}^n$. Since $\tilde{f}_0(\gamma \cdot (z,w)) = (\gamma \cdot (z,w), f_0'(\gamma \cdot (z,w)))$ and

$$\gamma \cdot \tilde{f}_0(z,w) = \gamma \cdot ((z,w), f_0'(z,w)) = (\gamma \cdot (z,w), J_{l,m}(\gamma,(z,w)) f_0'(z,w)),$$

it follows that $f_0'(\gamma(z,w)) = J_{l,m}(\gamma,(z,w)) f_0'(z,w)$. Since $Y = \Gamma^J \backslash \mathcal{D} \times \mathbb{C}^n = \tau_X^* Y_0$ and $\tau_X : \Gamma \backslash \mathcal{D} \to \Gamma_0 \backslash \mathcal{H}^n$ is induced by $\tau : \mathcal{D} \to \mathcal{H}^n$, a section $f : Y \to \mathcal{T}$ of $\mathcal{T}$ can be identified with a map $f' : \mathcal{D} \times \mathbb{C}^n \to \mathbb{C}^\times$ such that

$$f'(z,w) = f_0'(\tau(z), w)$$

for some map $f_0' : \mathcal{H}^n \times \mathbb{C}^n \to \mathbb{C}^\times$ of the form described above. Since $n \geq 2$, such f_0' is a Jacobi forms of weight l and index m for Γ_0^J by Proposition 2.3. Now it follows from Definition 4.3 that f' is a generalized Jacobi form for Γ^J of the required type. □

Remark 4.5. The bundle $\mathcal{T}$ in Theorem 4.4 can also be regarded as the quotient $\Gamma^J \backslash (\mathcal{D} \times \mathbb{C}^n) \times \mathbb{C}^\times$ with respect to the operation of Γ^J given by

$$\gamma \cdot ((z,w), \zeta) = (\gamma \cdot (z,w), J_{l,m}^{\rho,\tau}(\gamma,(z,w))\zeta)$$

for all $\gamma \in \Gamma^J$, $(z,w) \in \mathcal{D} \times \mathbb{C}^n$ and $\zeta \in \mathbb{C}^\times$.

Remark 4.6. In Theorem 4.4 if $G = Sp(n,\mathbb{R})$, $\mathcal{D} = \mathcal{H}^n$, $\rho = \mathrm{id}$ and $\tau = \mathrm{id}$, then $\mathcal{T} = \mathcal{T}_0$ and the sections of $\mathcal{T}$ are identified with the usual Jacobi forms given in Definition 2.1.

Remark 4.7. In [20] Satake constructed principal torus bundles over abelian schemes from a different viewpoint. He then discussed the projectivity and Chern classes of the associated toric bundles.

5. Supplementary remarks

It is well-known that Jacobi forms are closely linked to certain automorphic forms such as Siegel modular forms and theta series. In this section we describe certain connections between torus bundles on abelian schemes over an arithmetic variety X and automorphic forms for the arithmetic group Γ that determines X.

Let $\mathcal{T}$, Y and X be as in §4. For each $y \in Y$ we set

$$\mathcal{T}_y^1 = \{\zeta/|\zeta|^{-1} \mid \zeta \in \mathcal{T}_y\},$$

where $\mathcal{T}_y$ is the fiber of $\mathcal{T}$ over y isomorphic to $\mathbb{C}^\times$. Then the disjoint union

$$\mathcal{T}^1 = \coprod_{y \in Y} \mathcal{T}_y^1$$

becomes a fiber bundle over Y with its fiber isomorphic to the unit circle

$$S^1 = \left\{ z \in \mathbb{C} \mid |z| = 1 \right\}.$$

Thus we obtain a circle bundle $\mathcal{T}^1$ over Y and a natural morphism $\mathcal{T} \to \mathcal{T}^1$ of bundles over the abelian scheme Y. Circle bundles over a Kuga fiber variety were also constructed in [12] in a different setting without using Jacobi forms, and in that paper the cohomology groups of such a circle bundle were expressed in terms of holomorphic automorphic forms of one variable for Γ when $X = \Gamma\backslash\mathcal{D}$ is a Riemann surface associated to a quaternion algebra.

In another direction, let $\Omega^\nu(Y)$ denote the sheaf of holomorphic ν-forms on Y with $\nu = \dim_{\mathbb{C}} Y$. Then it was shown in [12] that the space $H^0(Y, \Omega^\nu(Y))$ of sections of $\Omega^\nu(Y)$ is isomorphic to the space of mixed automorphic forms associated to Γ, ρ and τ at least when X is compact (see also [8], [10], [11]).

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