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K-metric and K-normed linear spaces: survey

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Abstract

We give a short survey on some fixed point theorems which are generalizations of the classical Banach-Caccioppoli principle of contractive mappings. All these results are gathered in three theorems about existence and uniqueness of fixed points for operators which act in K-metric or K-normed linear spaces and, in particular, in local convex spaces and scales of Banach spaces. Three fixed point theorems presented in this article cover numerous applications in numerical methods, theory of integral equations, some results on iterative methods for construction of periodic solution to ordinary differential equations, existence and uniqueness results on solvability for Cauchy and Goursat problems of Ovsjannikov - Treves - Nirenberg type and so on.

The main aim of this article is to suggest new revised version of the fixed points theory in K-metric and K-normed linear spaces and show some new its applications. In other words, the article is devoted to the rehabilitation of the theory of K-metric and K-normed linear spaces in applicable analysis; more concretely, it deals with some rather subtle fixed point principles in K-metric and K-normed linear spaces, special (and unusual) K-estimates for corresponding fixed points (and which can be served as a source of usual numerical error estimates) and some nontrivial applications.

To my mind some sceptic and ironic attitude of mathematicians to the theory of K-metric and K-normed linear spaces is most likely based on the popular opinion that numerous generalizations of the classical Banach-Caccioppoli fixed point principle to the case of K-metric and K-normed linear spaces did not lead to rather serious or new important applications. The second reason is that in many cases it is possible to do a passage from a problem in a K-metric or K-normed linear space to an equivalent problem in a usual metric or normed linear space. However, the situation is likely different; in particular, the natural K-variants of the Banach-Caccioppoli fixed point principle purposed in this article allow to generalize and formulate new variants of some Bohl's and Bogoljubov's theorems on bounded solutions to differential and functional-differential equations, convergence results on Samojlenko's successive approximations and their analogs of finding periodic solutions of quasilinear differential equations, and solvability and unique solvability theorems on Cauchy and Goursat problems for partial differential equations with deteriorating operators and their abstract analogs (including the classical Cauchy-Kowalewska theorem and its abstract variants as theorems by L. Ovsjannikov, F. Treves) and so on. At last, precisely the fixed point theory in K-metric and K-normed linear spaces is a convenient tool, which allows to determine the possibility of passage from a problem in a K-metric or K-normed linear space to a problem in a usual metric or normed linear space.

In this connection there appears a need for analysis in details of general fixed points theory of operators in K-metric and K-normed linear spaces. Moreover, it is natural to try to present such variants of the fixed point principle for contractive operators in K-metric and K-normed linear spaces that allows to comprehend all usual and new nontrivial applications and, of course, to catch directions in which the further development of the theory turns out to be useful for investigations of the concrete linear and nonlinear problems as mentioned above as new ones. This article is one of the first attempts to make progress in this field.

1. Ordered linear spaces

Let $\mathbb{B}$ be a ordered linear space (over $\mathbb{R}$). In this article it means that $\mathbb{B}$ is a triple $(\mathbf{B}, \mathbf{K}, \gamma)$ of a linear space $\mathbf{B}$, a cone $\mathbf{K}$ in this space, and a class γ of convergent sequences in $\mathbf{B}$ with the following properties:

a) If $\xi, \eta \in \mathbf{K}$, and $\eta - n\xi \in \mathbf{K}$ $(n = 1, 2, \ldots)$ then $\xi = 0$.

b) Each convergent sequence (ξ_n), $\xi_n \in \mathbb{B}$, has a unique limit $\lim_{n\to\infty} \xi_n$.

c) Each stationary sequence $(\xi, \xi, \ldots)$, $\xi \in \mathbb{B}$, is convergent and $\lim_{n\to\infty} \xi = \xi$.

d) Convergence of a sequence is not broken and its limit does not change under truncation or addition a finite number of its members.

e) If a sequence (ξ_n), $\xi_n \in \mathbb{B}$, converges to $\xi_* \in \mathbb{B}$, then any its subsequence (ξ_{n_k}) converges to ξ_*.

f) The sum $(\xi_n + \eta_n)$ of two convergent sequences $(\xi_n), (\eta_n)$, $\xi_n, \eta_n \in \mathbb{B}$, is a convergent sequence and

$$\lim_{n\to\infty} (\xi_n + \eta_n) = \lim_{n\to\infty} \xi_n + \lim_{n\to\infty} \eta_n.$$

g) The product $\lambda_n \, \xi_n$ of a convergent scalar sequence $\lambda_n \in \mathbb{R}$ and a convergent sequence (ξ_n), $\xi_n \in \mathbb{B}$, is a convergent sequence and

$$\lim_{n\to\infty} (\lambda_n \cdot \xi_n) = \lim_{n\to\infty} \lambda_n \cdot \lim_{n\to\infty} \xi_n.$$

h) The limit of a convergent sequence of elements from $\mathbf{K}$ is an element from $\mathbf{K}$.

In order linear spaces one can use standard definitions and notions such as upper and lower bounds, the least upper and greatest lower bounds (supremum and infimum), minimal and maximal elements and so on. Further, usually the order and the convergence in concrete ordered linear spaces possess some special properties which are similar to corresponding properties of reals. Here are the important ones of them.

The Cantor property. *If sequences (u_n), (v_n), $u_n, v_n \in \mathbb{B}$, satisfy the conditions*

$$u_n \le u_{n+1} \le v_{n+1} \le v_n \quad (n = 1, 2, \ldots),$$

converge to a common limit:

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = z,$$

and a sequence (ξ_n), $\xi_n \in \mathbb{B}$, satisfies the conditions

$$u_n \le \xi_n \le v_n \quad (n = 1, 2, \ldots)$$

then the sequence (ξ_n) is also convergent and

$$\lim_{n\to\infty} \xi_n = z.$$

The Weierstrass property: *If (ξ_n), $\xi_n \in \mathbb{B}$, is an upper bounded and decreasing sequence (a lower bounded and increasing sequence):*

$$\xi_1 \le \xi_2 \le \ldots \le \xi_n \le \ldots \le z \quad (\xi_1 \ge \xi_2 \ge \ldots \ge \xi_n \ge \ldots \ge z),$$

then there exists $\sup_n \xi_n$ $(\inf_n \xi_n)$ and the following equality

$$\lim_{n\to\infty} \xi_n = \sup_n \xi_n \quad (\lim_{n\to\infty} \xi_n = \inf_n \xi_n)$$

holds.

The interpolation property: *If elements* $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{B}$ *satisfy the inequalities* $\xi_1, \xi_2 \le \eta_1, \eta_2$, *then there exists an element* $\zeta \in \mathbb{B}$ *such that* $\xi_1, \xi_2 \le \zeta \le \eta_1, \eta_2$.

The Riesz property: *If elements* ξ_1, ξ_2, *and* ξ *from* $\mathbb{B}$ *satisfy the inequalities* $\xi_1, \xi_2 \le \xi \quad (\xi_1, \xi_2 \ge \xi)$, *then there exists an element* $u \in \mathbb{B} \quad (v \in \mathbb{B})$ *such that* $\xi_1, \xi_2 \le u \quad (\xi_1, \xi_2 \ge v)$ *and the inequalities* $\xi_1, \xi_2 \le \xi \quad (\xi_1, \xi_2 \ge \xi)$ *imply the inequality* $u \le \xi \quad (v \ge \xi)$. The element u (v) is defined uniquely; it is *the supremum (the infimum)* of elements ξ_1 and ξ_2 and denoted by $\sup\{\xi_1, \xi_2\}$ $(\inf\{\xi_1, \xi_2\})$ or $\xi_1 \vee \xi_2\,(\xi_1 \wedge \xi_2)$. In ordered linear spaces with the Riesz property the function $\xi \to |\xi|$, $|\xi| = \xi \vee (-\xi)$, is defined; the element $|\xi|$ is called *modul* or *absolute value* of ξ in $\mathbb{B}$.

The countable Dedekind property: *Each upper or lower bounded countable set* $\mathcal{M}$ *has the least upper bound* $\sup \mathcal{M}$ *or, correspondingly, the greatest lower bound* $\inf \mathcal{M}$.

The Dedekind property: *Each upper or lower bounded set* $\mathcal{M}$ *has the least upper bound* $\sup \mathcal{M}$ *or, correspondingly, the greatest lower bound* $\inf \mathcal{M}$.

Sometimes a (partial) multiplication is defined in ordered linear spaces. In other words, for some pairs $\xi_1, \xi_2 \in \mathbb{B}$ its product $\xi_1 \cdot \xi_2$ is defined and the function $\xi_1, \xi_2 \to \xi_1 \cdot \xi_2$ has usual algebraic and order properties of multiplication. In what follows an ordered linear space $\mathbb{B}$ is called *an ordered semi-algebra*, if the partial commutative multiplication with the unit $\mathbf{1}$ is defined in $\mathbb{B}$. Furthermore, an ordered semi-algebra $\mathbb{B}$ is T-ordered semi-algebra if there exists a power function ξ^t $(\xi \in \mathbf{K},\ 0 < t < \tau(\xi))$ in $\mathbb{B}$ ($\tau(\xi)$ is a functional with values in $(0, \infty]$) with usual monotonicity properties, and the multiplication and this power function are agreed by usual way:

$$\xi^{a+b} = \xi^a \cdot \xi^b \quad \big(\xi \in \mathbf{K},\ 0 < a, b, a + b < \tau(\xi)\big),$$

$$(\xi_1 \cdot \xi_2)^t = \xi_1^t \cdot \xi_2^t \quad \big(\xi_1, \xi_2 \in \mathbf{K},\ 0 < t < \tau(\xi_1), \tau(\xi_2)\big).$$

Below a special class of T-ordered semi-algebras is rather useful. T-ordered semi-algebra is called *C-ordered semi-algebra*, if it possesses the countable Dedekind property and, moreover, the following Cauchy property holds true.

The Cauchy property: *If a series*

$$\mathrm{s}(\xi_n) = \sum_{n=0}^{\infty} \xi_n$$

with $\xi_n \in \mathbf{K}$ is convergent then $\mathbf{r}(\xi_n) \leq \mathbf{1}$; conversely, if $\mathbf{r}(\xi_n) \leq \mathbf{1}$ and $\mathbf{1} - \mathbf{r}(\xi_n)$ has a positive inverse element then the series above is convergent; here

$$\mathbf{r}(\xi_n) = \limsup_{n\to\infty} (\xi_n)^{1/n}.$$

In the last formula and below we used a standard definition of the upper limit

$$\limsup_{n\to\infty} \xi_n = \inf_n \sup_{k\geq n} \xi_k.$$

The conditions '$\mathbf{r}(\xi_n) \leq \mathbf{1}$' and '$\mathbf{r}(\xi_n) \leq \mathbf{1}$, $\mathbf{1} - \mathbf{r}(\xi_n)$ is an invertible element' are substitutions of the usual inequalities $\mathbf{r}(\xi_n) \leq \mathbf{1}$ and $\mathbf{r}(\xi_n) < \mathbf{1}$.

Now we are in position to present the list of main types of convergence considered in ordered linear spaces.

Kantorovich o-convergence. A sequence (ξ_n), $\xi_n \in \mathbb{B}$, *o-converges* to an element $\xi_* \in \mathbb{B}$ if there exist two sequences (u_n), $u_n \in \mathbb{B}$, and (v_n), $v_n \in \mathbb{B}$, for which

$$u_n \leq \xi_n \leq v_n,\ u_n \leq u_{n+1},\, v_n \geq v_{n+1} \quad (n = 1, 2, \dots),$$

and

$$\xi_* = \sup_n u_n = \inf_n v_n.$$

Kantorovich t-convergence. A sequence (ξ_n), $\xi_n \in \mathbb{B}$, *t-converges* to an element $\xi_* \in \mathbb{B}$ if for each sequence (n_k) satisfying the condition $n_k \to \infty$ there exists a subsequence (n_{k_j}) such that the sequence $(\xi_{n_{k_j}})$ o-converges to ξ_*.

Kantorovich r-convergence. A sequence (ξ_n), $\xi_n \in \mathbb{B}$, *r-converges* to an element $\xi_* \in \mathbb{B}$ if there exists an element $r \in \mathbf{K}$ and a sequence (ε_n) of positive reals satisfying the condition $\varepsilon_n \to 0$ for which the inequalities $-\varepsilon_n\, r \leq \xi_n - \xi_* \leq \varepsilon_n\, r$ $(n = 1, 2, \dots)$ hold.

$\mathcal{E}$-convergence. Let $\mathcal{E}$ be a subset of nonzero elements of the cone $\mathbf{K}$, and the two following properties hold:

(a) $\bigcap_{\varepsilon\in\mathcal{E}} < 0, \varepsilon >= \{0\}$,

(b) for each $\varepsilon \in \mathcal{E}$ there exists $\tilde{\varepsilon} \in \mathcal{E}$ such that $\tilde{\varepsilon} + \tilde{\varepsilon} \leq \varepsilon$.

A sequence (ξ_n), $\xi_n \in \mathbb{B}$, *$\mathcal{E}$-converges* to an element $\xi_* \in \mathbb{B}$ if for any $\varepsilon \in \mathcal{E}$ there exists a number N_ε such that the inequality $n > N_\varepsilon$ implies the inequalities $-\varepsilon \leq \xi_n - \xi_* \leq \varepsilon$.

Normed convergence. Let $||\cdot||$ be a norm in the space $\mathbb{B}$ and the cone $\mathbf{K}$ is closed in the topology generated with this norm. As usual, a sequence (ξ_n), $\xi_n \in \mathbb{B}$, *converges in the norm* $||\cdot||$ to an element $\xi_* \in \mathbb{B}$ if $\lim_{n\to\infty} ||\xi_n - \xi_*|| = 0$.

Γ-weak convergence. Let Γ be a total subspace of linear functionals on the space $\mathbb{B}$. A sequence (ξ_n), $\xi_n \in \mathbb{B}$, Γ-*weak converges* to an element $\xi_* \in \mathbb{B}$ if for each $\ell \in \Gamma$

$$\lim_{n\to\infty} \ell(\xi_n) = \ell(\xi_*).$$

Notice that the property **h)** for Γ-weak convergence may not hold; however, this property holds if

$$\mathbf{K} = \big\{\xi \in \mathbb{B} : \ell(\xi) \geq 0 \quad (\ell \in \mathbf{K}^*)\big\}$$

where

$$\mathbf{K}^* = \big\{\ell \in \Gamma : \ell(\xi) \geq 0 \quad (\xi \in \mathbf{K})\big\}.$$

One of the most important properties of a convergence is its completeness property. As usual, an ordered linear space is called *sequentially complete* if its classes of convergent and fundamental sequences coincide each other. However, there appear some difficulties concerned with definition of the class of fundamental sequences for a class considered of convergent sequences (of course, these difficulties do not exist in the case when the convergence is generated with a metric or a topology).

The most natural definition of fundamental sequences seems to be the following: a sequence (ξ_n), $\xi_n \in \mathbb{B}$, is called *fundamental* (with respect to a convergence γ considered) if for any subsequences (l_k), (m_k) satisfying the condition $l_k, m_k \to \infty$ the sequence $(\xi_{l_k} - \xi_{m_k})$ converges to zero. However under this definition the completeness property for most of concrete ordered linear spaces turns unknown (in particular, in the case of Kantorovich o-convergence). L.V. Kantorovich offered a different definition of the completeness property for ordered linear spaces.

A sequence (ξ_n), $\xi_n \in \mathbb{B}$, is called *fundamental* (in the Kantorovich sense), if there exists a sequence (z_n), $z_n \in \mathbb{B}$, for which $z_n \to 0$ and such that the inequalities

$$-z_n \leq \xi_l - \xi_k \leq z_n \quad (l, k \geq n)$$

hold. A space $\mathbb{B}$ is called *sequentially complete* (in the Kantorovich sense), if the classes of convergent and fundamental (in the Kantorovich sense) sequences coincide.

In conclusion of this section we present main examples of ordered linear spaces which can be utilized in applications.

1. The simplest and most important example of order linear spaces with convergence is the usual set of reals with natural structures; in this case

$$\mathbb{B} = \mathbb{R}, \quad \mathbf{K} = [0, \infty), \quad \gamma = \{\text{the usual convergence}\}.$$

2. The following example is the number finite-dimensional space $\mathbb{R}^m$ with natural structures; in this case

$$\mathbb{B} = \mathbb{R}^m, \quad \mathbf{K} = \big\{\xi = (\xi_1, \cdots, \xi_m) : \xi_1, \cdots, \xi_m \geq 0\big\},$$

$$\gamma = \{\text{the coordinate-wise convergence}\}.$$

3. The space s with usual structures is the simplest example of an infinite-dimensional ordered linear space with convergence:

$$\mathbb{B} = s, \quad \mathbf{K} = \big\{\xi = (\xi_1, \cdots, \xi_m, \cdots) : \ \xi_1, \cdots, \xi_m, \cdots \geq 0\big\},$$

$$\gamma = \{\text{the coordinate-wise convergence}\}.$$

4. Let Ω be an arbitrary set and $\mathcal{F}(\Omega, \mathbb{R})$ is a linear space of functions defined on Ω and taking their values in $\mathbb{R}$. Then

$$\mathbb{B} = \mathcal{F}(\Omega, \mathbb{R}), \quad \mathbf{K} = \big\{\xi(\omega) : \xi(\omega) \geq 0 \ (\omega \in \Omega)\big\},$$

$$\gamma = \{\text{the pointwise convergence}\}$$

is a new example of an ordered linear space. This space covers the foregoing as special cases.

5. Let Ω be an arbitrary set, $\mathcal{A}$ be a σ-algebra of subsets in Ω, and λ be a countably additive and σ-finite measure on $\mathcal{A}$; further, let $\mathbb{S}$ be the linear space of real measurable functions on Ω (in deed, of equivalence classes of such functions) with usual structures. Then

$$\mathbb{B} = \mathbb{S}(\Omega, \mathbb{R}), \quad \mathbf{K} = \big\{\xi(\omega) : \xi(\omega) \geq 0 \ (\omega \in \Omega)\big\},$$

$$\gamma = \{\text{the pointwise almost everywhere or metric convergence}\}$$

is one of the most important examples of ordered linear spaces with convergence; the o-convergence coincides with the pointwise almost everywhere convergence and the t-convergence coincides with the convergence with respect to measure on subsets of $\mathcal{A}$ with finite measure or the metric convergence. As in the previous case the space $\mathbb{S}$ is a generalization of three first foregoing examples.

6. Let again Ω be an arbitrary set, $\mathcal{A}$ be a σ-algebra of subsets in Ω, and λ be a countably additive and σ-finite measure on $\mathcal{A}$. A Banach space of scalar functions from $\mathbb{S}$ is called *ideal* if the relations $|z_1| \le |z_2|$, $z_1 \in \mathcal{S}$ and $z_2 \in \mathbb{Z}$ imply the relations $z_1 \in \mathbb{Z}$ and $||z_1|| \le ||z_2||$ (all inequalities are considered as ones everywhere). The following ordered linear space with convergence

$$\mathbb{B} = \mathbb{Z}, \quad \mathbf{K} = \big\{ z(\omega) : \, z(\omega) \ge 0 \ (\omega \in \Omega) \big\},$$

$$\gamma = \{\text{the pointwise, normed, weak or other convergence}\}$$

is one of the most extensively studied and greatest applicable utility.

In all these examples we deals with a space of scalar functions (or equivalence classes of scalar functions), a cone of nonnegative functions (classes of functions) and either usual point-wise convergence or a natural convergence generated by some natural norm or some total family of linear functionals. In the following we do not consider other examples (in particular, linear spaces with cones of monotone or convex functions, linear spaces of vector-functions, or linear spaces of set functions). It is not difficult to noticed which properties presented above hold for the foregoing ordered linear spaces.

Theory of ordered linear spaces goes back to investigations of F. Riesz [75] and H. Freudental [41]; in the following it was developed by a few independent schools in different directions. Maybe this circumstance is a main reasoning of absence of unify terminology, unify approaches, and unify object of investigation. In the former Soviet Union there were at least three independent science schools. The first of them was begun from fundamental investigations of L.V. Kantorovich and his pupils [49-54,92]; they studied ordered linear spaces with strong additional properties such as the Dedekind property and others. The second school goes back to investigations of M.G. Krein and his pupils [59] and then of M.A. Krasnosel'skii and his pupils [55-57]; these authors studied Banach spaces ordered with a closed cone and linear and nonlinear operators acting in such spaces. The third school is related with investigations of M.J. Antonovskii, V.G. Boltjanskii, and T.A. Sarymsakov [9], which studied so called topological semi-fields or, in other words, ordered semi-algebras. Some original approach to ordered linear spaces in framework of modular spaces was suggested by H. Nakano [67]. A major contribution to the ordered linear space theory was done by G. Birkhoff [15], I. Namioka [68], H. Schaefer [79-80] and others.

2. K-metric and K-normed linear spaces

Let $\mathbb{B}$ be an ordered linear space with a cone $\mathbf{K}$ and a convergence γ and $\mathbf{X}$ be an arbitrary set. A function $\rho(\cdot,\cdot)$ defined on the set $\mathbf{X}\times\mathbf{X}$ with values in $\mathbb{B}$ is called a *K-metric* on $\mathbf{X}$ if the following properties hold:

a) $\rho(x_1,x_2)\geq 0 \quad (x_1,x_2\in\mathbf{X})$.
b) *The equality* $\rho(x_1,x_2)=0$ $(x_1,x_2\in\mathbf{X})$ *is equivalent to the equality* $x_1=x_2$.
c) $\rho(x_1,x_2)=\rho(x_2,x_1) \quad (x_1,x_2\in\mathbf{X})$.
d) $\rho(x_1,x_3)\leq\rho(x_1,x_2)+\rho(x_2,x_3) \quad (x_1,x_2,x_3\in\mathbf{X})$.

The pair $\mathbb{X}=(\mathbf{X},\rho)$ where $\mathbf{X}$ is a set and ρ is a K-metric on $\mathbf{X}$ is called *K-metric space.*

The most important class of K-metric spaces is a class of K-normed linear spaces.

Let $\mathbf{X}$ be a linear space. A function $|||\cdot|||$ defined on $\mathbf{X}$ and taking values in $\mathbb{B}$ is called a *K-norm* if it has the following properties:
a) $|||x|||\geq 0 \quad (x\in\mathbf{X})$.
b) *The equality* $|||x|||=0$ $(x\in\mathbf{X})$ *is equivalent to the equality* $x=0$.
c) $|||\lambda x|||=|\lambda|\,|||x||| \quad (\lambda\in\mathbb{R},x\in\mathbf{X})$.
d) $|||x_1+x_2|||\leq|||x_1|||+|||x_2||| \quad (x_1,x_2\in\mathbf{X})$.

The pair $\mathbb{X}=(\mathbf{X},|||\cdot|||)$ where $\mathbf{X}$ is a linear space and $|||\cdot|||$ is a K-norm on $\mathbf{X}$ is called *K-normed linear space.*

Each K-normed linear space $\mathbb{X}$ is a K-metric space with K-metric in $\mathbb{X}$ defined by means of the formula

$$\rho(x_1,x_2)=|||x_1-x_2||| \quad (x_1,x_2\in\mathbb{X}).$$

K-metrics in K-normed linear spaces have two additional important properties:

$$\rho(x_1+z,x_2+z)=\rho(x_1,x_2) \quad (x_1,x_2,z\in\mathbb{X})$$

(*the invariance with respect to shifts property*) and

$$\rho(\lambda x_1,\lambda x_2)=|\lambda|\,\rho(x_1,x_2) \quad (\lambda\in\mathbb{R},\,x_1,x_2\in\mathbb{X})$$

(*the homogeneity with respect to homotheties property*).

In standard manner one can define a notion of a convergence (the K-convergence) in K-metric spaces. A sequence (x_n), $x_n\in\mathbb{X}$, is called *convergent*, if there exists an element $x_* \in \mathbb{X}$ such that the sequence $(\rho(x_n, x_*))$ is convergent to zero in the space $\mathbb{B}$. This convergence has usual properties.

In what follows some sequential completeness property of the K-metric space $\mathbb{X}$ is basic. There are some variants of corresponding definitions of fundamental sequences, however, the standard definition is usually not convenient.

It is natural to give the following natural modification of the definition of fundamental sequences: a sequence (x_n), $x_n \in \mathbb{X}$, is called *fundamental* (with respect to a convergence γ considered in $\mathbb{B}$), if for any subsequences (l_k), (m_k) satisfying the condition l_k, $m_k \to \infty$ the sequence $(\rho(x_{l_k}, x_{m_k}))$ converges to zero. Under this definition the completeness property for most of concrete K-metric spaces turns out to be unknown. As a result, usually, due to L.V. Kantorovich another modification of the standard definition for fundamental sequences is used.

A sequence (x_n), $x_n \in \mathbb{X}$, is called *fundamental (in the Kantorovich sense)*, if there exists a sequence (z_n), $z_n \in \mathbb{B}$, for which $z_n \to 0$ and such that the inequalities

$$\rho(x_l, x_k) \leq z_n \quad (l, k \geq n,\, n = 1, 2, \ldots)$$

hold. A space $\mathbb{X}$ is called *sequentially complete (in the Kantorovich sense)* if the classes of convergent and fundamental sequences coincide.

A closed notion of the completeness property is connected with the classical Weierstrass test of absolute convergence of series. A K-metric space $\mathbb{X}$ is called *sequentially complete (in the Weierstrass sense)*, if each sequence $x_n \in \mathbb{X}$ such that

$$\sum_{n=1}^{\infty} \rho(x_n, x_{n+1}) < \infty$$

is convergent in the space $\mathbb{X}$ (the inequality above as usual means, that the corresponding series is convergent in the space $\mathbb{B}$).

In the case of K-normed linear spaces this definition can be formulated in traditional form. Recall that the series

$$\mathbf{s}(x_n) = \sum_{n=1}^{\infty} x_n.$$

is called *convergent* if the sequence of its partial sums is convergent in the space $\mathbb{X}$, and *absolutely convergent* if the series

$$\mathbf{s}(|||x_n|||) = \sum_{n=1}^{\infty} |||x_n|||$$

is convergent in the space $\mathbb{B}$. *K-normed linear space $\mathbb{X}$ is sequentially complete in the Weierstrass sense if and only if each absolutely convergent series in it is convergent.*

Now we present the list of basic types of mathematical structures which can be considered as K-metric or K-normed linear spaces.

1. Metric spaces and normed linear spaces. Of course, usual metric and normed linear spaces are K-metric and K-normed linear spaces; in this case $\mathbb{B} = \mathbb{R}$.

2. Bi-metric spaces and bi-normed linear spaces. It is an important (for numerous applications) class of K-metric and K-normed linear spaces for which $\mathbb{B} = \mathbb{R}^2$. Obviously, it is easy to define a usual metric or norm in order to the K-convergence in the original space coincides with the convergence generated with the metric or norm; unfortunately, after the corresponding passing we usually lose important geometrical properties of these spaces. Typical examples of bi-metric and bi-normed linear spaces are *equipped Hilbert spaces, Hölder spaces, spaces with two norms, spaces of smooth functions* and some others.

3. m-metric spaces and m-normed linear spaces. This class of K-metric and K-normed linear spaces is formal generalization of the foregoing one; in this case $\mathbb{B} = \mathbb{R}^m$. Typical examples of m-metric and m-normed linear spaces are *spaces of vector functions* (with values in $\mathbb{R}^m$), *finite direct sum of metric and normed linear spaces, spaces of smooth functions* and others.

4. Countably metric spaces and countably normed spaces. It is K-metric and K-normed linear spaces in which K-metric or K-norm takes its values in the space $\mathbb{B} = s$. The passing to usual metric (sometimes to normed linear) spaces is possible but important geometrical properties of the original space lose as a rule completely. But this class is rather interesting; it covers classical *metrizable local convex spaces, spaces of infinitely differentiable and analytical functions, countably normed linear spaces, discrete and continuous scales of Banach spaces* and lot of others. In particular, if $\mathbb{X}$ is a local convex space and $\mathcal{B} = \{\mathcal{O}_1, \ldots, \mathcal{O}_n, \ldots\}$ is its countable base of balanced convex neighborhoods of 0 in $\mathbb{X}$ then

$$|||x||| = \big(\mu_{\mathcal{O}_1}(x), \ldots, \mu_{\mathcal{O}_n}(x), \ldots\big),$$

where $\mu_{\mathcal{O}}$ is the Minkowski functional of a set $\mathcal{O}$, is a K-norm in $\mathbb{X}$ with values in s.

5. $\mathcal{F}$-metric spaces and $\mathcal{F}$-normed linear spaces. In these K-metric and K-normed linear spaces K-metric and K-norm take their values in the space $\mathbb{B} = \mathcal{F}(\Omega, \mathbb{R})$. This type of spaces covers arbitrary local convex spaces, Ovsjannikov's scales of Banach spaces and others. Really all uniform spaces can be considered as $\mathcal{F}$-metric ones and, similarly, all topological linear (not only local convex) spaces as $\mathcal{F}$-normed linear spaces. However as of now there do not exist serious examples of utilization of these spaces in the fixed point theory.

6. $\mathbb{S}$-metric spaces and $\mathbb{S}$-normed linear spaces. In these K-metric and K-normed linear spaces K-metric and K-norm take their values in the space $\mathbb{B} = \mathcal{S}(\Omega, \mathbb{R})$.

7. Riesz and Kantorovich spaces. Each Riesz space (ordered linear space with the Riesz property) and, in particular, each Kantorovich space (ordered linear space with Dedekind property) is a K-normed linear space; in this case

$$\mathbb{B} = \mathbb{X}, \quad |||x||| = |x|,$$

where $|x| = \sup\{x, -x\}$. Partial cases of these spaces are ideal spaces and Riesz spaces.

8. Bochner function spaces. Each Bochner function space can be considered as K-normed linear space with

$$\mathbb{B} = \mathbb{Z}, \quad |||x||| = ||x(t)||,$$

where $\mathbb{Z} \subseteq \mathcal{S}(\Omega, \mathbb{R})$ is an ideal Banach space; usual ideal spaces are a special case of these spaces yet.

9. Ideal-metric and ideal-normed linear spaces. In this class of K-metric and K-normed linear spaces K-metric or K-norm takes its values from an ideal Banach space $\mathbb{Z}$ of sequences or functions; Bochner function spaces lie in this class. This class of K-metric and K-normed linear spaces plays an important role in the operator equations theory since allows to use all enormous arsenal of analysis in Banach spaces.

10. Direct sums of metric or Banach spaces. Each direct sum $\sum_{\omega \in \Omega} E(\omega)$ of metric or Banach spaces $E(\omega)$ from some family $\mathcal{E} = \{E(\omega) : \omega \in \Omega\}$ is really K-metric or K-normed linear space for which

$$\mathbb{B} = \mathcal{F}(\Omega, \mathbb{R}), \quad |||x(\cdot)|||(\omega) = ||x(\omega)||_\omega \ (\omega \in \Omega).$$

Theory of K-metric and K-normed linear spaces were developed in different directions; here we have special interest in the fixed point theory for operators in K-metric and K-normed linear spaces. This aspect was intensively studied by L.V. Kantorovich and then B.Z. Vulikh [50-53,92]; they considered the case when $\mathbb{B}$ was a K-space (or, in other words, an abstract linear space with the order generated with the Dedekind property and the Kantorovich o-convergence generated with this cone). The second direction deals with situation when $\mathbb{B}$ is a Banach space ordered with a closed cone $\mathbf{K}$, it is developed by A.I. Perov and his pupils [62,70-71], E.M. Mukhamadijev - V.J. Stetsenko [66] and others. The third direction is related to the case when $\mathbb{B}$ is a semi-field and systematically studied by M.J. Antonovskii, V.G. Boltjanskii and T.A Sarymsakov [9], and then A.V. Mironov and T.A. Sarymsakov [65]. At least it is necessary to cite articles by J. Schröder [81-84], L. Collatz [25], E. Bohl [16-17], J. Vandergraft [90], in which were made numerous attempts to construct general theory. Remark here articles by N.V. Azbelev and Z.B. Tsaljuk [13], J.W. Daniel [26], G. Kurepa [60], W.J. Kammerer - R.H. Kasriel [48], S.N. Slugin [85-86], J. Reinermann [74], N.S. Kurpel' [61], and articles of mathematicians from Yugoslavia and Rumania.

3. Banach-Caccioppoli's principle with linear Lipschitz condition

In this section K-analogs of the classical Banach-Caccioppoli fixed point principle of contracting mappings are presented. More exactly, here is formulated the general fixed point principle for operators in K-metric spaces $\mathbb{X}$ with K-metric $\rho(\cdot,\cdot)$ satisfying the Lipschitz condition of type

$$\rho(Ax_1, Ax_2) \leq Q\,\rho(x_1, x_2) \quad (x_1, x_2 \in \mathbb{X}) \tag{1}$$

where Q is a linear and nonnegative ($Q\mathbf{K} \subseteq \mathbf{K}$) operator acting in $\mathbb{B}$.

The main assumption in the classical Banach-Caccioppoli fixed point principle is the inequality $Q < 1$; the latter has a sense because Q is a usual number. However, in general case this inequality loses sense and, thus, must be modified. The most natural and simple modification is based on the usual school theorem: *the inequality $Q < 1$ is equivalent to existence of a sum for an infinite geometrical progression with ratio Q, or, in other words, to the convergence of the Neuman series*

$$\nu(Q) = \sum_{n=0}^{\infty} Q^n.$$

In what follows we use the following definitions:

$$\mathcal{C}(Q) = \Big\{\xi \in \mathbf{K} : \lim_{n\to\infty} Q^n\xi = 0\Big\}, \tag{2}$$

$$\mathcal{D}(Q) = \Big\{\xi \in \mathbf{K} : \sum_{n=0}^{\infty} Q^n\xi < \infty\Big\} \tag{3}$$

(the inequality here, as usual, means the convergence of the corresponding series). The equality

$$H(Q)\xi = \sum_{n=0}^{\infty} Q^n\xi, \tag{4}$$

obviously, is defined a linear nonnegative operator $H(Q)$ on the set $\mathcal{D}(Q)$; in the case when 1 is not a positive eigenvalue of the operator Q (i.e. the equality $Q\xi = \xi,\ \xi \in \mathbf{K}$ implies $\xi = 0$) the operator $H(Q)$ is usually denoted with $(I - Q)^{-1}$. It is evident that the sets $\mathcal{C}(Q)$ and $\mathcal{D}$ are normal (a set $\mathcal{M}$ is normal if $\eta \leq \xi,\ \eta \in \mathbf{K},\ \xi \in \mathcal{M}$ implies $\eta \in \mathcal{M}$).

Of course, analysis of the sets $\mathcal{C}(Q)$ and $\mathcal{D}(Q)$ in general case demands getting over serious difficulties. However as of now a lot of concrete facts in this field are well known; the main of them are gathered in the following statement.

Proposition 1

Let Q be a linear nonnegative operator in the space $\mathbb{B}$. Then the following assertions hold:
a) The equality $\mathcal{D}(Q) = \mathbf{K}$ is true if and only if $\mathcal{C}(Q) = \mathbf{K}$ and $(I - Q)\ \ \mathbf{K} = \mathbf{K}$.
b) In general case the following relations

$$\mathcal{D}(Q) = (I - Q)\mathcal{C}(Q) \subseteq \mathcal{C}(Q)$$

hold.
c) In the case when $\mathbb{B}$ is a Banach space with a generating and normal cone $\mathbf{K}$, the equality $\mathcal{D}(Q) = \mathbf{K}$ is hold if

$$\rho(Q) < 1;$$

this equality is necessary and sufficient if the operator Q is compact or, at least, its peripherical spectrum is Fredholm.
d) In the case when $\mathbb{B}$ is an C-ordered semi-algebra the inequality $r(Q)\xi \leq \mathbf{1}$ and the positive invertibility of the element $1 - r(Q)\xi$ imply the inclusion $< 0, \xi > \subseteq \mathcal{D}(Q)$; conversely, the inequality $r(Q)\xi \not\leq \mathbf{1}$ implies the relation $< \xi, \infty > \cap \mathcal{D}(Q) = \emptyset$; here

$$r(Q)\xi = \limsup_{n\to\infty} (Q^n\xi)^{1/n}.$$

Notice that the operator $H(Q)$ satisfies two functional identities:

$$H(Q) = QH(Q) + I \tag{5}$$

and

$$H(Q) = H(Q)Q + I. \tag{6}$$

The general fixed point principle can be formulated in the following manner:

Theorem 1

Let $\mathbb{X}$ be a sequentially complete (in the Weierstrass sense) K-metric space and A be an operator which acts in $\mathbb{X}$ and satisfies the Lipschitz condition (1) in which the Lipschitz coefficient Q is a linear and nonnegative operator acting in $\mathbb{B}$. Assume that

$$\rho(Ax_0, x_0) \in \mathcal{D}(Q).$$

Then A has a fixed point $x_* \in \mathbb{X}$ which lies in the ball $B(x_0, H(Q)(\rho(Ax_0, x_0)))$; further this fixed point is the limit of successive approximations $x_{n+1} = Ax_n \quad (n = 0, 1, \ldots)$, and satisfies the inequalities

$$\rho(x_n, x_*) \leq H(Q)Q^n \rho(Ax_0, x_0) \quad (n = 0, 1, \ldots);$$

at last, this fixed point is unique on the set

$$\mathcal{U}(x_0, Q) = \big\{x \in \mathbb{X} : \rho(x, x_0) \in \mathcal{C}(Q)\big\}.$$

Error estimates in Theorem 1 are completely similar to classical estimates for operators in metric spaces, however, they are really more complicated. So, even in simple cases analysis of asymptotic behavior of the sequence $(H(Q)Q^n\xi)$ for different initial values $\xi \in \mathbf{K}$ is not sufficient.

The simplest situation is when ξ is a nonnegative eigenvector e_0 of the linear nonnegative operator Q (in the case of Banach spaces classical theorems on linear nonnegative operators describe natural conditions of existence of nonnegative eigenvectors with eigenvalue which coincides with the spectral radius $\rho\,(Q)$ of the operator Q). In this case the equalities

$$H(Q)Q^n e_0 = \frac{\rho^n(Q)}{1 - \rho(Q)}\, e_0 \quad (n = 1, 2, \ldots)$$

holds and analysis of the sequence $(H(Q)Q^n e_0)$ is reduced to analysis of the usual number sequence. A weaker assertion holds for elements $\xi \in \mathbf{K}$ from the order component $\mathbf{K}(e_0)$ generated with the eigenvector e_0 which by definition is a set of $\xi \in \mathbf{K}$ for which the inequalities $\alpha e_0 < \xi < \beta e_0$ are true for suitable $\alpha, \beta,\ 0 < \alpha < \beta < \infty$. The inequalities

$$\alpha_{e_0}(\xi)\, \frac{\rho^n(Q)}{1 - \rho(Q)}\, e_0 \leq H(Q)Q^n\xi \leq \beta_{e_0}(\xi)\, \frac{\rho^n(Q)}{1 - \rho(Q)}\, e_0 \quad (n = 1, 2, \ldots),$$

where

$$\alpha_u(\xi) = \sup\{\alpha : \xi \geq \alpha u\},\ \ \beta_u(\xi) = \inf\{\beta : \xi \leq \beta u\} \quad (u \in \mathbf{K})$$

characterize rather well asymptotic behavior of the sequence $(H(Q)Q^n\xi)$ with $\xi \in \mathbf{K}(e_0)$. The similar inequalities

$$\alpha_{e_0}(Q^s\xi)\,\frac{\rho^{n-s}(Q)}{1-\rho(Q)}\,e_0 \le W(Q)Q^n\xi \le \beta_{e_0}(Q^s\xi)\,\frac{\rho^{n-s}(Q)}{1-\rho(Q)}\,e_0 \quad (n=1,2,\ldots)$$

are true for ξ from the set $\mathbf{K}_s(Q) = Q^{(-s)}(\mathbf{K}(e_0)),\ s = 1,2,\ldots$

It is obvious that the nonnegative eigenvector e_0 of the operator Q can be found in the explicit form only in exceptional cases. However, in numerous cases it can be replaced with a vector $u_0 = Q\mathbf{1}$ (or $u_0 = Q^s\mathbf{1}$ for a suitable $s > 1$); here $\mathbf{1}$ is a weak order unit in $\mathbf{K}$. In these cases the nonnegative eigenvector e_0 is usually lies in the component $\mathbf{K}(u_0)$ generated with u_0, or, in other words, the inequality

$$\delta = \inf\{\beta\alpha^{-1}:\ \alpha\,u_0 < e_0 < \beta\,u_0\} < \infty$$

holds. Unfortunately, simple and general conditions, under which these inequalities hold true are unknown; besides in many cases the operator Q is e_0-bounded (i.e. $\mathbf{K}_s(Q) = \mathbb{B}$ for some s); in this situation the conditions of e_0-boundedness and u_0-boundedness of the operator Q are equivalent each other.

In many important cases the operator Q has no nonnegative eigenvectors; in particular, the latter is true if Q is a compact Volterra integral operator. In this case the asymptotic behavior of sequences $(H(Q)Q^nz)$ $(z \in \mathbf{K})$ can be described if simple formulas for iterations $Q^n \quad (n = 1,2,\ldots)$ are known; the latter is true, for example, for Liouville's integral operators.

In some applications it were useful to make analysis in details of the set $T_n(Q)$ of elements $z \in \mathbf{K}$ satisfying the inequalities $Q^nz \le cz$ for suitable $c \in (0,\infty)$ and subadditive functionals

$$\gamma_n(z) = \inf\{c:\ Q^nz \le cz\} \quad (n = 0,1,...)$$

defined on these sets. In particular, it is of interest to find conditions, under which the formula

$$\rho(Q) = \inf\{\gamma_1(z):\ z \in \mathbf{K}\}$$

holds (the answer is, of course, evident if Q possesses an nonnegative eigenvector).

The estimates considered above are estimates in ordered linear space $\mathbb{B}$. If $\mathbb{B}$ is a Banach space it is natural to pass from such order estimates to usual estimates with norms. Usually the corresponding passing is trivial in the case the cone $\mathbf{K}$ is normal; the latter means the existence of a constant L such that the order inequalities $\xi \le \eta$, $(\xi,\eta \in \mathbf{K})$ imply the scalar inequality $||\xi|| \le L\,||\eta||$. However one can consider usual scalar estimates for the sequence $(H(Q)Q^n\rho(Ax_0,x_0))$ without analysis corresponding order inequalities. This problem is studied insufficiently; some new results and unsolved problems in this field is presented in [97,98].

It is not possible to say who discovered and proved Theorem 1. Below we present only a list of mathematicians who dealt with special cases of Theorem 1 or its analogs: M.J. Antonovskii, V.G. Boltjanskii, T.A. Sarymsakov [9], N.V. Azbelev - Z.B. Tsaljuk [13], E. Bohl [16-17], L. Collatz [25], J.W. Daniel [26], N.A. Evkhuta - P.P. Zabrejko [39], F. Gandac [42-44], W.J. Kammerer - R.H. Kasriel [48], L.V. Kantorovich [49-52,54], M.A. Krasnosel'skii - G.M. Vainikko - P.P. Zabrejko - J.B. Rutitskii - V.J. Stetsenko [58], G. Kurepa [60], N.S. Kurpel' [61], A.V. Mironov - T.A. Sarymsakov [65], E.M. Mukhamadiev - V.J. Stetsenko [66], J.M. Ortega - W.C. Rheingoldt [69], A.I. Perov [70], A.I. Perov, A.V. Kibenko [71], J.V. Radyno [72-73],J. Reinermann [74], T. Sabirov - F. Nazarov - E.M. Mukhamadiev [77], J. Schröder [81-84], S.N. Slugin [85- 86], E. Tarafdar [87], B.Z. Vulikh [92], P.P. Zabrejko [94-101], P.P. Zabrejko - T.A. Makarevich [103-104], P.P. Zabrejko - J.V. Radyno [105].

4. Banach-Caccioppoli's principle with nonlinear Lipschitz condition

This section is devoted to K-analogs of the classical Banach-Caccioppoli principle of contractive mappings A which act in K-metric spaces $\mathbb{X}$ with K-metric $\rho(\cdot,\cdot)$ and satisfy the Lipschitz condition of type

$$\rho(Ax_1,Ax_2)\le Q\,\rho(x_1,x_2)\quad(x_1,x_2\in\mathbb{X})\tag{7}$$

where Q is a nonlinear and nonnegative operator acting in $\mathbb{B}$. The condition (7) is called *the nonlinear Lipschitz condition.*

Fixed points of operators satisfying the nonlinear Lipschitz conditions in metric spaces were begun to study intensively since sixties (see s.g. [47]); however, analogous operators acting in K-metric spaces were almost not of interest.

There are two approaches to investigation of operators satisfying the nonlinear Lipschitz conditions. The first of them is based on analysis of Neuman series for the operator coefficient Q and the second is based on analysis of successive approximations with small shift-perturbations of the operator coefficient Q. By contrast to linear case both methods lead to different statements and results obtained with these methods are not comparable in general.

In what follows we need some definitions. Let the nonlinear nonnegative operator $Q : \mathbf{K} \to \mathbf{K}$ satisfy the condition $Q0 = 0$ and be monotone (the inequality $\xi \le \eta$ ($\xi, \eta \in \mathbf{K}$) implies the inequality $Q\xi \le Q\eta$). Set

$$\mathcal{C}(Q) = \Big\{\xi \in \mathbf{K} : \lim_{n\to\infty} Q^n \xi = 0\Big\}, \tag{8}$$

$$\mathcal{L}(Q) = \Big\{\xi \in \mathbf{K} : \sum_{n=0}^{\infty} Q^n \xi < \infty\Big\} \tag{9}$$

(as usual, an inequality here means convergence of a corresponding series with nonnegative members), and

$$\mathcal{M}(Q) = \Big\{\xi \in \mathbf{K} : Q\xi \le \xi, \lim_{n\to\infty} Q^n\xi = 0, \lim_{n\to\infty} S_n(\xi) < \infty\Big\} \tag{10}$$

where

$$S_0(\xi) = 0, \quad S_{n+1}(\xi) = QS_n(\xi) + \xi \quad (n = 1, 2, \dots)$$

(an inequality in (10) means convergence of a corresponding monotonically increasing sequence).

The equality

$$L(Q)\xi = \sum_{n=0}^{\infty} Q^n \xi, \tag{11}$$

obviously, defines a nonlinear nonnegative and monotone operator $L(Q)$ on the set $\mathcal{L}(Q)$; this operator satisfies the equation

$$L(Q) = L(Q)Q + I. \tag{12}$$

Similarly, the equality

$$M(Q)\xi = \lim_{n\to\infty} S_n(\xi), \tag{13}$$

obviously, defines a nonlinear nonnegative and monotone operator $M(Q)$ on the set $\mathcal{M}(Q)$; this operator satisfies the equation

$$M(Q) = QM(Q) + I. \tag{14}$$

In the linear case both operators $L(Q)$ and $M(Q)$ coincides with the operator $H(Q) = (I - Q)^{-1}$ which is defined with natural way; in the nonlinear case operators $L(Q)$ and $M(Q)$ are different.

Of course, analysis of the sets $\mathcal{L}(Q)$ and $\mathcal{M}(Q)$ in the nonlinear case and calculation of the operators $L(Q)$ and $M(Q)$ turn out to be rather difficult problem in the comparison with its analog in the linear case. However some statements keep true and in the nonlinear case.

First remark that $\mathcal{L}(Q), \mathcal{M}(Q) \subseteq \mathcal{C}(Q)$ and all three sets $\mathcal{C}(Q)$, $\mathcal{L}(Q)$ and $\mathcal{M}(Q)$ are normal as in the linear case.

An operator Q satisfies *the upper Fatou property* if the relations $z_0 \leq z_1 \leq \ldots \leq z_n \leq \ldots$, $z_* = \sup\{z_n\}$ imply the inequality $Qz_* \leq \sup\{Qz_n\}$, and, similarly, Q satisfies *the lower Fatou property* if the relations $z_0 \geq z_1 \geq \ldots \geq z_n \geq \ldots$, $z_* = \inf\{z_n\}$ imply the inequality $Qz_* \geq \sup\{Qz_n\}$.

Proposition 2

Let Q be a nonlinear nonnegative and monotone operator in the space $\mathbb{B}$. Then the following statements are true:

a) In the case when $\mathbb{B}$ is an C-ordered semi-algebra the inequality $r(Q)\xi \leq \mathbf{1}$ and the positive invertibility of the element $\mathbf{1} - r(Q)\xi$ imply the relation $< 0, \xi > \subseteq \mathcal{L}(Q)$; conversely, the inequality $r(Q)\xi \not\leq \mathbf{1}$ implies the relation $< \xi, \infty > \cap \mathcal{L}(Q) = \emptyset$; here

$$r(Q)\xi = \limsup_{n \to \infty} (Q^n \xi)^{1/n}.$$

b) If an element $\xi \in \mathbf{K}$ satisfies the inequality $Q\xi \leq \xi$, the operator Q has no nonzero fixed points on the segment $< 0, \xi >$ and has the lower Fatou property, then $< 0, \xi > \subseteq \mathcal{C}(Q)$. Conversely, if an element $\xi \in \mathbf{K}$ satisfies the inequality $Q\xi \geq \xi$, then $< \xi, \infty > \cap \mathcal{C}(Q) = \emptyset$.
c) If for an element $z \in \mathbf{K}$ there exists an element $\xi \in \mathbf{K}$ such that the inequality $Q\xi + z \leq \xi$ is true and let the operator Q have the upper Fatou property, then $< 0, z > \subseteq \mathcal{M}(Q)$. Conversely, if for an element $z \in \mathbf{K}$ there exists an element $\xi \in \mathbf{K}$ such that the inequality $Q\xi + z \geq \xi$ is true, then $< z, \infty > \cap \mathcal{M}(Q) = \emptyset$.
d) If the space $\mathbb{B}$ satisfies the Weierstrass property and the operator Q satisfies the upper Fatou property then the operator $M(Q)$ is continuous at zero.

The general fixed point principle for operators satisfying the nonlinear Lipschitz condition can be formulated in the following manner.

Theorem 2

Let $\mathbb{X}$ be a K-metric space and A be an operator A which acts in $\mathbb{X}$ and satisfies the Lipschitz condition (7) in which the Lipschitz coefficient Q is a nonlinear and nonnegative operator acting in $\mathbb{B}$. Assume that either

$$\rho(Ax_0, x_0) \in \mathcal{L}(Q)$$

and the space $\mathbb{X}$ is sequentially complete in the Weierstrass sense or

$$\rho(Ax_0, x_0) \in \mathcal{M}(Q),$$

the operator $M(Q)$ be continuous at zero, and the space $\mathbb{X}$ is sequentially complete in the Kantorovoch sense. Then A has a fixed point $x_* \in \mathbb{X}$ which lies in the ball $B(x_0, L(Q)\,\rho(Ax_0, x_0))$ or in the ball $B(x_0, M(Q)\,\rho(Ax_0, x_0))$; further, this fixed point is the limit of successive approximations $x_{n+1} = Ax_n \quad (n = 0, 1, \ldots)$, and satisfies the inequalities

$$\rho(x_n, x_*) \le L(Q)Q^n \rho(Ax_0, x_0) \quad (n = 0, 1, \ldots)$$

or, correspondingly, the inequalities

$$\big(\rho(x_n, x_*) \le M(Q)Q^n \rho(Ax_0, x_0) \quad (n = 0, 1, \ldots)\big);$$

at last, this fixed point is unique at the set

$$\mathcal{U}(x_0, Q) = \{x \in \mathbb{X} : \, \rho(x, x_0) \in \mathcal{C}(Q)\}.$$

There appears a natural question which of the operators $L(Q)$ and $M(Q)$ allows to catch the strongest statement on existence of a fixed point for the original operator A. The answer is not unique and depends on special properties of the operator Q. In particular, if the operator Q is *subadditive* (i.e. it satisfies the inequality

$$Q(\xi_1 + \xi_2) \le Q\xi_1 + Q\xi_2 \quad (\xi_1, \xi_2 \in \mathbf{K})$$

Theorem 2 gives a stronger existence result with the operator $M(Q)$; however, if the operator Q is *superadditive* (i.e., it satisfies the inequality

$$Q(\xi_1 + \xi_2) \ge Q\xi_1 + Q\xi_2 \quad (\xi_1, \xi_2 \in \mathbf{K})$$

Theorem 2 gives stronger existence result with the operator $M(Q)$.

$L(Q)$-variant of Theorem 2 seems to go back to E. Hille; some its modifications were offered by F. Gandac [42-44], S. Vakhidov [88-89], P.P. Zabrejko - T.A. Makarevich [103-104], P.P. Zabrejko [96,99-100,102]. $M(Q)$-variant of Theorem 2 seems to go back to the article by T. Wažewski [93]; the original variant was improved by M. Kwapisch [63], N.S. Kurpel' [62], J. Eisenfeld - V. Lakshmikantham [33-35], K.-J. Chung [19-24], S.T. Dzhabarov [30-32] and others. The variant presented is new.

5. Kantorovoch's majorant principle

Let A be an operator acting in K-metric space $\mathbb{X}$. By definition a nonnegative and monotone operator Φ ($\Phi(0) = 0$) is called *Kantorovich majorant* (at the point x_0) of the operator A if the inequalities

$$\rho(A\tilde{x}, Ax) \le \Phi(r+h) - \Phi(r) \quad \big(\rho(x, x_0) \le r,\ \rho(\tilde{x}, x) \le h\ (r, h \in \mathbf{K})\big) \tag{15}$$

hold. There is the deepest connection (discovered by L.V. Kantorovich) between fixed points of the operator A or, in other words, solutions to the equation

$$x = Ax$$

and solutions to the equation

$$\xi = \Phi(\xi) + a \quad \big(a = \rho(x_0, Ax_0)\big). \tag{16}$$

This fact allows to catch fixed point of the operator A if we have sufficient information about solutions of the essentially simpler equation (16). As a result there appears a problem of constructing Kantorovich majorants to the operator A; moreover, there appears a problem of constructing of optimal Kantorovich majorants. Both problems lead to definite difficulties even for concrete nonlinear operators (some results in this direction were obtained in [108] for superposition and Uryson integral operators; further statements one can find in [12]).

In Kantorovich spaces one can write abstract formulas for Kantorovich majorants. In particular the formula

$$\Phi(\xi) = \sup\left\{\sum_{\sigma=1}^{s} \Delta(\xi_{\sigma-1}, \xi_\sigma - \xi_{\sigma-1}) : 0 = \xi_0 \le \xi_1 \le \ldots \le \xi_s = \xi,\ s = 1, 2, \ldots\right\}$$

holds; here

$$\Delta(a, b) = \sup\,\{\rho(Ax, A\tilde{x}) :\ \rho(x, x_0) \le a, \rho(\tilde{x}, x) \le b\}.$$

There are formulas of other type; the following section is devoted to some of them.

Put

$$S_{n+1}(z, \xi) = \Phi(z)\,\big(S_n(z, \xi)\big) \quad \big(S_0(z, \xi) = \xi,\ n = 0, 1, \ldots\big)$$

where

$$\Phi(z)\,(\xi) = \Phi(\xi) + z$$

and, further,

$$\mathcal{S}(\Phi) = \left\{z \in \mathbf{K} : \lim_{n\to\infty} S_n(z, 0) < \infty\right\}, \tag{17}$$

$$S(\Phi)(z) = \lim_{n\to\infty} S_n(z, 0) \tag{18}$$

and, at last,

$$\mathcal{T}(\Phi, z) = \left\{\xi \in \mathbf{K} : \lim_{n\to\infty} S_n(z, \xi) = S(\Phi)(z)\right\}. \tag{19}$$

Theorem 3

Let $\mathbb{X}$ be a sequentially complete (in the Weierstrass sense) K-metric space, A be an operator acting in the space $\mathbb{X}$, and Φ be its Kantorovich majorant. Assume that the condition

$$\rho(Ax_0, x_0) \in \mathcal{S}(\Phi)$$

holds. Then A has a fixed point $x_* \in \mathbb{X}$ which lies in the ball $B(x_0, S(\Phi)\rho(Ax_0, x_0))$; further, this fixed point is the limit of successive approximations $x_{n+1} = Ax_n \quad (n = 0, 1, \ldots)$ and satisfies the inequalities

$$\rho(x_n, x_*) \leq S(\Phi)\big(\rho(Ax_0, x_0)\big) - S_n\big(\rho(Ax_0, x_0)\big) \quad (n = 0, 1, \ldots),$$

at last, this fixed point is unique in the set

$$\mathcal{U}(x_0, \Phi) = \big\{x \in \mathbb{X} : \ \rho(x, x_0) \in \mathcal{T}\big(\Phi, \rho(Ax_0, x_0)\big)\big\}.$$

It is natural way there appears a problem on relations between Theorems 1-2 and Theorem 3. One can see that (1) is a special case of (7) and (15); thus, Theorem 1 is a consequence of both Theorems 2 and 3. Further, (7) implies (15) if

$$Qh \leq \Phi(r + h) - \Phi(r);$$

it holds for $\Phi = Q$ provided that Q is a superadditive operator. Therefore, in this special case Theorem 2 is a consequence of Theorem 3; unfortunately, this case is unnatural and seems not to be interesting. In Kantorovich spaces under conditions of Theorem 2 one can try to define a Kantorovich majorant Φ by the formula

$$\Phi(r) = \lim_{n\to\infty} \Phi_n(r)$$

where

$$\Phi_0(r) = Q(r), \ \Phi_{n+1}(r) = \sup\{\Phi_n(\xi) + Q(r - \xi) : \quad (0 \leq \xi \leq r)\} \quad (n = 0, 1, \ldots);$$

if this process is convergent the corresponding Kantorovich majorant is the best one. But one can notice that usually direct application of Theorem 2 in this and other similar cases is more effective than application of Theorem 3 with the corresponding Kantorovich majorant. At last, it is necessary to note that (15) implies (7) with $Qh = \Phi(r + h) - \Phi(r)$ on the ball $B(x_0, r) \subseteq \mathbb{X}$. This passing usually lead to serious loss of information about the operator A.

Thus, Theorems 2 and 3 are independent from each other. There are other interesting relations between Theorems 1-3; some of them are discussed in the following section.

Theorem 3 is really stated by L.V. Kantorovich [51,54].

6. The Lipschitz condition and Kantorovich majorants

In many applications we deals with operators satisfying the local Lipschitz condition of type

$$\rho(Ax_1, Ax_2) \le Q(r)\,\rho(x_1, x_2) \quad \big(x_1, x_2 \in B(x_0, r), r \in \mathbf{K}\big) \tag{20}$$

where $Q(r)$ is a monotone function defined on the cone $\mathbf{K}$ and taking values in the cone $\mathcal{K}(\mathbb{B})$ of linear nonnegative operators acting in the space $\mathbb{B}$. It is easy to formulate some fixed point principles for such operators using Theorem 1. Really, if for some $r \in \mathbf{K}$ the operator $Q(r)$ satisfies the conditions

$$\rho(x_0, Ax_0) \in \mathcal{S}\big(Q(r)\big), \quad \rho(x_0, Ax_0) \le \big(I - Q(r)\big)r$$

then the operator A satisfies the condition of Theorem 1 in which the space $\mathbb{X}$ is changed with its subspace $B(x_0, r)$. However there exists an essentially more effective approach. This approach is based on the passing from the local Lipschitz condition (20) to the condition (15) with a suitable Kantorovich majorant $\Phi(r)$. Evident reasoning shows that such approach will be more effective than smaller a Kantorovich majorant $\Phi(r)$. However, a problem of construction of minimal Kantorovich majorant is open in general case.

Remark that the simplest Kantorovich majorant for the operator A satisfying the local Lipschitz condition (20) can be defined by the formula

$$\Phi(r) = Q(r)r$$

(this fact is an evident consequence of the inequalities $\rho(A\tilde{x}, Ax) \le Q(r+h)\,h = Q(r+h)(r+h) - Q(r+h)r \le Q(r+h)(r+h) - Q(r)r = \Phi(r+h) - \Phi(r) \quad (\rho(x, x_0) \le r,\ \rho(x, \tilde{x}) \le h)$).

Assume that $\mathbb{X}$ is a K-normed linear space (although many constructions and speculations can be enlarged on more general types of K-metric spaces). In this case the passing from the local Lipschitz condition to the Kantorovich majorant condition is possible in the case when the operator-function $Q(r)$ is potential or, in other words, has a primitive $\Phi(r)$. As it is well known the potentiality conditions for an operator-function $Q(r)$ in the case when $\mathbb{B} \neq \mathbb{R}$ are rigid and not trivial; they are equivalent to independence on a way of a curve integral

$$I(\mathcal{L}) = \int_{\mathcal{L}} Q(t)\, dt;$$

if these conditions hold then the corresponding potential is defined by the curve integral

$$\Phi(r) = \int_0^r Q(t)\, dt.$$

It is evident to see that this Kantorovich majorant is minimal. The passing described for differentiable operators A in normed linear spaces was discovered by L.V. Kantorovich [52] and for differentiable operators A in K-metric spaces was done by B.Z. Vulikh [92]; in general case it was considered in [12,101].

In other cases it is necessary to pass from the original Lipschitz coefficient $Q(r)$ to the larger Lipschitz coefficients $\tilde{Q}(r)$ which has a primitive; in this case the operator

$$\Phi(r) = \int_0^r \tilde{Q}(t)\, dt$$

is a Kantorovich majorant for the operator A at the point x_0. It is unknown, whether it is minimal; moreover, the passing $Q(r) \to \tilde{Q}(r)$ was almost not studied.

Consider now the cases $\mathbb{B} = \mathbb{R}^m$ and $\mathbb{B} = s$. Then Kantorovich majorant can be defined by the simple formula

$$\Phi(r) = \int_{\Pi(r)} Q(t)\, dt,$$

where $\Pi(r)$ is the multi-path consisting the closed intervals

$$\Pi_j(r) = \{(r_1, \ldots, r_{j-1}, t, r_{j+1}, \ldots, r_m) :\ 0 \leq t \leq\ r_m\}$$

$$(j \in \{1, \ldots, m\},\ r = (r_1, \ldots, r_m) \in \mathbf{K})$$

in the case $\mathbb{B} = \mathbb{R}^m$ and

$$\Pi_j(r) = \{(r_1, \ldots, r_{j-1}, t, r_{j+1}, \ldots) :\ 0 \leq t \leq\ r_m\}$$

$$(j \in \{1, 2, \ldots\},\ r = (r_1, r_2, \ldots) \in \mathbf{K})$$

in the case $\mathbb{B} = s$. This passing was suggested by T.V. Savchenko [78] and then studied in [40,106].

In conclusion of this section we consider some special conditions of Lipschitz-like type (these conditions are systematically studied by L.V. Ovsjannikov in connection with Cauchy problem for singular differential equations in Banach spaces). Let $\mathbb{B}$ is a space of functions $z(\omega)$ which are defined on a set Ω and take their values from a cone $\mathbf{k}$ in a Banach space $\mathbb{Z}$, $\mathbf{K} = \{z(\omega) \in \mathbb{Z} : z(\omega) \in \mathbf{k} \ (\omega \in \Omega)\}$. The operator A satisfies *the Ovsjannikov condition* if

$$\rho(Ax_1, Ax_2)(\omega'') \leq C(\omega', \omega'')\, \rho(x_1, x_2)(\omega') \quad \big((\omega', \omega'') \in \mathcal{W},\ x_1, x_2 \in \mathbb{X}\big), \tag{21}$$

where $C(\omega', \omega'') \quad ((\omega', \omega'')) \in \mathcal{W})$ is a family of linear nonnegative operators acting in the space $\mathbb{Z}$, $\mathcal{W}$ is a sufficiently 'rich' subset of $\Omega \times \Omega$ or, in other words, a subset for which the following property holds

$$H(\mathcal{W}) \cap (\omega \times \Omega) \neq \emptyset,\ H(\mathcal{W}) \cap (\omega \times \Omega) \neq \emptyset \quad (\omega \in \Omega);$$

here

$$H(\mathcal{W}) = \bigcap_{n=1}^{\infty} H_n(\mathcal{W}), \quad H_n(\mathcal{W}) = \big\{(\omega', \omega'') : \Delta_n(\omega', \omega'') \neq \emptyset\big\},$$

$$\Delta_n(\omega', \omega'') = \{(\omega_0, \omega_1, \ldots, \omega_n) : \omega_0 = \omega',\ \omega_n = \omega'',\ (\omega_{j-1}, \omega_j) \in \mathcal{W} \ \ (j = 1, \ldots, n)\}.$$

It is easy to see that Ovsjannikov condition (21) for the operator A is equivalent the usual Lipschitz condition (7) in which the operator Q is defined by the formula

$$Qz(\omega) = \inf_{\tilde{\omega} \in \Omega} C(\omega, \tilde{\omega})\, z(\tilde{\omega}) \tag{22}$$

provided that the right hand side of (22) has a sense. This operator is homogeneous but nonlinear (in deed it is a suplinear operator); the latter leads to natural difficulties. However in many cases one can pass to a rougher Lipschitz condition (1) with a linear operator coefficient. Such pass can de realized with many ways; we describe two of them.

Let λ be a probabilistic measure on the set Ω and

$$\tilde{C}(\omega', \omega'') = \begin{cases} C(\omega', \omega'') & \text{if } (\omega', \omega'') \in \mathcal{W}, \\ 0 & \text{if } (\omega', \omega'') \notin \mathcal{W}. \end{cases}$$

Then the Ovsjannikov (21) condition implies the Lipschitz condition (1), in which the operator coefficient Q is defined with the formula

$$Qz(\omega) = \int_{\Omega} \tilde{C}(\omega, \tilde{\omega})\, z(\tilde{\omega})\, d\lambda(\omega). \tag{23}$$

Let $\theta:\ \Omega \to \Omega$ be a mapping of the set Ω into itself for which $(\omega, \theta(\omega)) \in \mathcal{W}\ (\omega \in \Omega)$; in this case the Ovsjannikov condition implies the Lipschitz condition (1), in which the operator coefficient Q is defined by the formula

$$Qz(\omega) = C\big(\omega, \theta(\omega)\big)\, z\big(\theta(\omega)\big). \tag{24}$$

Analysis of different properties of the nonlinear operator (22) and linear operators (23) and (24) can usually be realized on standard schemes; remark that analysis of the operator (24) is often reduced to applying Antonevich-Kitover-Lebedev theorem on the spectral radius of generalized shift operators (see [6,8]).

In conclusion of this section we remark that the foregoing fixed point principles (Theorem 1-3) are formulated as statements in terms of K-metric and K-normed linear spaces. However, as was noticed above local convex spaces, general topological linear spaces and even arbitrary uniform spaces are special types of K-metric (or K-normed linear) spaces; thus one can formulate specializations of Theorem 1-3 for these types of spaces. We omit exact assertions, however, note that articles [3-5, 13,18,27-29,46,64,76,87,91,94] are devoted to fixed point principles for operators acting in uniform, local convex spaces and so on.

7. Applications

The fixed point theory for operators in K-metric and K-normed linear spaces has numerous applications in analysis, theory of differential and integral equations, numerous methods and so on. However, we omit classical and standard applications (one can find the description of numerous applications in details, s.g. [10-12,13,25, 53-54,59,61,69-71,76,81-84,94,106]). In this section we restrict ourselves only some new and unexpected applications; moreover, for reasons of room, we give only general moments concerning these applications.

7.1. Bogoljubov's and Bohl's theorems about bounded solutions of differential equations. Let $\mathbb{X}$ be a Banach space. As is well known the problem on bounded (on the whole axe $\mathbb{R}$) solutions for differential or functional-differential equations of type

$$\frac{dx}{dt} = Ax + F(x), \tag{25}$$

where A is linear and $F(\cdot)$ is nonlinear operators in the space of $\mathcal{C}(\mathbb{R}, \mathbb{X})$ of bounded and continuous functions $x(t) : \mathbb{R} \to \mathbb{X}$. In special case when F is a superposition operator $Fx(s) = f(s, x(s))$ generated with a function $f(s, u) : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ these assumptions cover different types of ordinary and partial differential equations; in general case these assumptions as well cover differential equations with retarded arguments, some types of stochastic differential equations and so on.

The problem on bounded solution to (25) is equivalent to Hammerstein's integral equation

$$x(t) = \int_{-\infty}^{+\infty} g(t,s)\, F\big(x(s)\big)\, ds, \tag{26}$$

where $g(t,s)$ is the Green function of the problem on bounded solutions for the linear differential equation $x' = Ax$. Classical Bohl's and Bogoljubov's theorems on existence and continuous continuation of bounded solutions to Equation (25) are based on applying Banach-Caccioppoli fixed point principle to the operator A defined by the right hand side of (26) in the space $\mathcal{C}(\mathbb{R}, \mathbb{X})$. It turns out that essential sharpening of Bohl and Bogoljubov theorems can be obtained if the operator A is considered in the bi-normed linear space $\mathcal{C}^1(\mathbb{R}, \mathbb{X})$ of bounded and differentiable functions with bounded and continuous derivatives. In this case the operator A satisfies the local Lipschitz condition with the operator-function coefficient

$$Q(r) = \begin{pmatrix} a(r) & b(r) \\ c(r) & d(r) \end{pmatrix}$$

where $a(r), b(r), c(r), d(r)$ are defined by the nonlinear operator $F(\cdot)$. The application of Theorem 1 (or Theorem 3) leads to generalizations of Bohl's and Bogoljubov's theorems. Notice that the inequality $\rho(Q(r)) < 1$ is equivalent to the scalar inequalities

$$a(r), d(r) < 1, \quad b(r)c(r) < \big(1 - a(r)\big)\big(1 - d(r)\big)$$

which allows $b(r)$ to be large! Exact results and their numerous modifications are described in details in [1,2,105].

7.2. Samojlenko's successive approximations method in the oscillations theory and theory of boundary value problems. Let $\mathbb{X}$ be a Banach space. The problem on ω-periodic solutions for differential equation

$$\frac{dx}{dt} = f(t,x), \tag{27}$$

where $f(\cdot,\cdot)$ is a ω-periodic with respect to t and continuous with respect to both variables is equivalent to the following system of Hammerstein's integral equation

$$x(t) = \xi + \int_0^{\omega} g(t,s)\, f(s, x(s))\, ds, \tag{28}$$

where

$$g(t,s)=\begin{cases}1-\frac{t}{\omega} & \text{if } 0\le s\le t\le\omega\\ \frac{t}{\omega} & \text{if } 0\le t<s\le\omega\end{cases}$$

and the operator equation

$$\int_0^\omega f\big(s,x(s)\big)\,ds=0.$$

Samojlenko's successive approximations

$$x_{n+1}(t,\xi)=A(\xi)x_n(t,\xi)\quad\big(x_0(t,\xi)=\xi,\ n=0,1,\dots\big)$$

with parameter $\xi\in\mathbb{X}$ for the operator $A(\xi)$ which is defined by the right hand side of (28); the parameter ξ is defined by the equation

$$\int_0^\omega f\big(s,w(s,\xi)\big)\,ds=0\quad\Big(w(t,\xi)=\lim_{n\to\infty}x_n(t,\xi)\Big).$$

Analysis of convergence of Samojlenko's approximations is reduced to applying Theorem 1 for the operator $A(\xi)$ in the space $\mathcal{C}([0,\omega],\mathbb{X})$; the operator A satisfies the Lipschitz condition with the operator coefficient

$$Qz(s)=c\int_0^\omega g(t,s)\,z(s)\,ds$$

where c is a Lipschitz coefficient of the function $f(t,x)$ with respect to x. Simple calculations show that $\rho(Q)=\kappa\,c$, where $\kappa=(4\theta^2)^{-1}$ and θ are positive roots of the transcendent equation

$$\int_0^\theta \exp\sigma^2\,d\sigma=\frac{1}{2\theta}\exp\theta^2$$

($\kappa\approx 0,293$). Exact results and their numerous modifications for ordinary and partial differential equations are described in details in [36-39].

7.3. The Cauchy and Goursat problems for differential equations with deteriorating operators. Let $\mathbb{Z}(\omega)$ ($\omega\in[0,1]$) be a family of Banach spaces $\mathbb{Z}(\omega)$ such that

$$\mathbb{Z}(\omega'')\subseteq\mathbb{Z}(\omega'),\quad ||z||_{\omega'}\le||z||_{\omega''}\quad(0\le\omega'\le\omega''\le1).$$

Furthermore, let $\mathbb{X}$ be a K-normed linear spaces of elements from $\mathbb{X}(0)$ with the natural K-norm $|||x|||(\omega)=||x||_\omega$ taking its values from the space $\mathbb{B}$ of real functions on $[0,1]$ with values in $\mathbb{Z}$. Consider the Cauchy problem for differential equation

$$\frac{dx}{dt}=f(t,x),\tag{29}$$

where $f(\cdot,\cdot)$ is a function continuous with respect to t, satisfying the Ovsjannikov condition

$$||f(t,x_1)-f(t,x_2)||_{\omega''} \le c(\omega',\omega'')\,||x_1-x_2||_{\omega'} \;\; (0 \le t \le T,\ x_1,x_2 \in \mathbb{X},\ \omega',\omega'' \in [0,1])$$

with respect to x with a real function $c(\omega',\omega'')$ taking finite values for $\omega' > \omega''$, and, in addition, closed in a natural sense with respect to x. This Cauchy problem is equivalent to the following Volterra's integral equation

$$x(t) = \xi + \int_0^t f\big(s,x(s)\big)\,ds. \tag{30}$$

The operator A generated by the right hand side of this integral equation satisfies the Ovsjannikov condition with the operator coefficient

$$C(\omega',\omega'')z(t,\omega'') = c(\omega',\omega'')\int_0^t z(s,\omega')\,ds \quad (0 \le \omega'' \le \omega' \le 1).$$

The application of Theorem 2 (or even Theorem 1) to the operator A allows to state existence (uniqueness) of a solution to (29) on the interval $[0,T]$ in the space $\mathbb{X}(\omega)$ if

$$\limsup_{n\to\infty}\left(\frac{c_n(0,\omega)}{n!}\right)^{1/n} < \frac{1}{T} \quad \left(\limsup_{n\to\infty}\left(\frac{c_n(\omega,1)}{n!}\right)^{1/n} < \frac{1}{T}\right);$$

here

$$c_n(\omega',\omega'') = \inf_{\omega'=\omega_0>\omega_1>\ldots>\omega_n=\omega''} c(\omega_0,\omega_1)\,c(\omega_1,\omega_2)\,\ldots\,c(\omega_{n-1},\omega_n).$$

In classical case $c(\omega',\omega'') = c(\omega''-\omega')^{-1}$ and $c_n(\omega',\omega'') = c^n\,n^n\,(\omega''-\omega')^{-n}$. This result covers the classical Cauchy-Kowalewskaja theorem and its numerous analogs by Gevreut for partial differential equations as well as their abstract variants, in particular, Ovsjannikov's theorem and their variants suggested by F. Treves and his successors. Exact results and their numerous modifications are described in details in [11,14,96,100,102-104].

Analogous speculations are applied to study of Goursat problem for differential equations, in inverse problems of the scattering theory and other.

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