Collect. Math. **48**, 4-6 (1997), 743–771

Jung constants of Orlicz function spaces

Zhongdao Ren*

Department of Mathematics, Suzhou University, Suzhou 215006, P.R. China
Current address: *Department of Mathematics, University of California, Riverside, CA 92521*

Shutao Chen

Department of Mathematics, University of Iowa, Iowa City, IA 52242

Abstract

Estimation of the Jung constants of Orlicz function spaces equipped with either Luxemburg norm or Orlicz norm is given. The exact values of the Jung constants of a class of reflexive Orlicz function spaces have been found by using a new quantitative index of N-functions.

§ 1. Preliminaries

Let X be a normed linear space and $A \subset X$ be a bounded set. The diameter of A is $d(A) = \sup\{\|x - y\| : x, y \in A\}$. If $z \in X$, we set $r(A, z) = \sup\{\|x - z\| : x \in A\}$. For $A, B \subset X$, $r(A, B) = \inf\{r(A, z) : z \in B\}$ is the relative Chebyshev radius of A with respect to B and $r(A, X)$ is the absolute Chebyshev radius of A. Clearly, $r(A, z) = r(\overline{co}(A), z)$, $r(A, B) = r(\overline{co}(A), B)$ and $r(A, X) = r(\overline{co}(A), X)$.

Definition 1.1. (Jung[8]) The Jung constant $JC(X)$ of a normed linear space X is defined to be

$$JC(X) = \sup\left\{\frac{r(A, X)}{d(A)} : A \subset X \text{ bounded}, d(A) > 0\right\}. \tag{1}$$

* Partially supported by NSF of Jiangsu Province, P. R. China.

Clearly, $1/2 \leq JC(X) \leq 1$ always holds. Pichugov [12] computed $JC(L^p)$(see also Corollary 4.5 in Section 4). Amir [1] proved that if X is a dual space, then

$$JC(X) = \sup\left\{\frac{r(A,X)}{d(A)} : A \subset X \text{ finite}\,,\ d(A) > 0\right\}. \tag{2}$$

By using (2), Amir obtained the following.

Lemma 1.2 (see [1, Proposition 2.5 (b)])

Let $(X_\alpha)_{\alpha \in D}$ be a net of linear subspaces of the Banach space X, directed by inclusion, such that $\overline{\cup_{\alpha\in D} X_\alpha} = X$. If X is a dual space and each X_α admits a norm-1 linear projection P_α, then $JC(X) = \sup_{\alpha\in D} JC(X_\alpha) = \lim_{\alpha\in D} JC(X_\alpha)$.

Lemma 1.3 (Pichugov [12])

Let X_n be a real n-dimensional normed space and let A be a bounded closed convex set in X with $r(A, X_n)$ being its Chebyshev radius. Then the point x is its Chebyshev center if and only if there exists an integer $N \leq n+1$ for which

(a) there are $x_i \in A$, $i \leq N$ such that $\|x_i - x\| = r(A, X_n)$ for all $i \leq N$;
(b) there are $f_i \in X_n^*$, the dual space of X_n, $i \leq N$ such that $\|f_i\| = 1$ and $\langle x_i - x, f_i\rangle = \|x_i - x\|$ for all $i \leq N$;
(c) there are $c_i \geq 0$, $i \leq N$ such that $\sum_{i=1}^N c_i = 1$ and $\sum_{i=1}^N c_i f_i = 0$.

In this case, $\sum_{i=1}^N \sum_{j=1}^N c_i c_j \langle x_i - x_j, f_i - f_j\rangle = 2r(A, X_n)$. If $1 \leq \lambda \leq 2$ and

$$\Lambda = \sum_{i=1}^N \sum_{j=1}^N c_i c_j \{\langle x_i - x_j, f_i - f_j\rangle\}^\lambda,$$

then

$$\frac{2^\lambda [r(A,X_n)]^\lambda}{\left(\frac{n}{n+1}\right)^{\lambda-1}} \leq \Lambda \leq [d(A)]^\lambda \sum_{i=1}^N \sum_{j=1}^N c_i c_j \|f_i - f_j\|^\lambda \,. \tag{3}$$

Lemma 1.4 (Pichugov[12])

Let X be a separable and dual space. If $\{x_1, x_2, \cdots\}$ is a dense set in X and $X_n = \operatorname{span}\{x_i : 1 \leq i \leq n\}$, then

$$JC(X) \leq \liminf_{n\to\infty} JC(X_n)\,. \tag{4}$$

Recall that Bynum [2] defined the normal structure coefficient $N(X)$ of a Banach space X by

$$N(X) = \inf\left\{\frac{d(A)}{r(A,A)} : A \subset X \text{ closed bounded convex}\,,\ d(A) > 0\right\}.$$

Maluta [11] denoted $[N(X)]^{-1}$ by $\tilde{N}(X)$ and proved that $2^{-1/2} \le \tilde{N}(X)$ for every infinite-dimensional Banach space X. Amir [1] pointed out that for every Banach space X,

$$\frac{1}{2} \le JC(X) \le \tilde{N}(X) \le 1\,. \tag{5}$$

Next we introduce some basic facts on Orlicz space. Let

$$\Phi(u) = \int_0^{|u|} \phi(t)dt \qquad \text{and} \qquad \Psi(v) = \int_0^{|v|} \psi(s)ds$$

be a pair of complementary N-functions. The Orlicz function space $L^\Phi(\Omega)$ on $\Omega = [0,1]$ or $[0,\infty)$ is defined to be the set $\{x : x \text{ is Lebesgue measurable on } \Omega \text{ and } \rho_\Phi(\lambda x) = \int_\Omega \Phi[\lambda x(t)]dt < \infty \text{ for some } \lambda > 0\}$. The Luxemburg norm and the Orlicz norm are defined respectively by

$$\|x\|_{(\Phi)} = \inf\left\{c > 0 : \rho_\Phi\left(\frac{x}{c}\right) \le 1\right\}$$

and

$$\|x\|_\Phi = \sup\left\{\int_\Omega |x(t)y(t)|dt : \rho_\Psi(y) \le 1\right\}.$$

The norms are equivalent: $\|x\|_{(\Phi)} \le \|x\|_\Phi \le 2\|x\|_{(\Phi)}$. The closed separable subspace $E^\Phi(\Omega)$ of $L^\Phi(\Omega)$ is defined to be the set $\{x \in L^\Phi(\Omega) : \rho_\Phi(\lambda x) < \infty \text{ for all } \lambda > 0\}$. By the same way we define the Orlicz sequence space ℓ^Φ and its closed separable subspace h^Φ. An important parameter for analysis in an Orlicz space is the rate of growth of the underling N-function. An N-function $\Phi(u)$ is said to satisfy the Δ_2-condition for large u (for small u or for all $u \ge 0$), in symbol $\Phi \in \Delta_2(\infty)(\Phi \in \Delta_2(0)$ or $\Phi \in \Delta_2)$, if there exist $u_0 > 0$ and $K > 2$ such that $\Phi(2u) \le K\Phi(u)$ for $u \ge u_0$ (for $0 \le u \le u_0$ or for $u \ge 0$). An N-function $\Phi(u)$ is said to satisfy the ∇_2-condition for large u, in symbol $\Phi \in \nabla_2(\infty)$, if there exist $u_0 > 0$ and $a > 1$ such that $\Phi(u) \le \frac{1}{2a}\Phi(au)$ for $u \ge u_0$. Similarly we define $\Phi \in \nabla_2(0)$ and $\Phi \in \nabla_2$. The basic facts on Orlicz spaces can be found in [9], [10] and [14]. For instance, $L^\Phi[0,1]$ ($L^\Phi[0,\infty)$ or ℓ^Φ) is separable if and only if $\Phi \in \Delta_2(\infty)$ ($\Phi \in \Delta_2$ or

$\Phi \in \Delta_2(0)$); $L^\Phi[0,1]$ ($L^\Phi[0,\infty)$ or ℓ^Φ) is reflexive if and only if $\Phi \in \triangle_2(\infty) \bigcap \nabla_2(\infty)$ ($\Phi \in \triangle_2 \bigcap \nabla_2$ or $\Phi \in \triangle_2(0) \bigcap \nabla_2(0)$).

A new quantitative index of $\Phi(u)$ is provided by the following six constants:

$$\alpha_\Phi = \liminf_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}\,, \qquad \beta_\Phi = \limsup_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}\,, \tag{6}$$

$$\alpha_\Phi^0 = \liminf_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \qquad \beta_\Phi^0 = \limsup_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} \tag{7}$$

and

$$\bar{\alpha}_\Phi = \inf\left\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u < \infty\right\}, \qquad \bar{\beta}_\Phi = \sup\left\{\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u < \infty\right\}. \tag{8}$$

The following result will play the leading role in this paper.

Theorem 1.5

(i) $\Phi \notin \triangle_2(\infty) \Leftrightarrow \beta_\Phi = 1, \Phi \notin \nabla_2(\infty) \Leftrightarrow \alpha_\Phi = 1/2;$
(ii) $\Phi \notin \triangle_2(0) \Leftrightarrow \beta_\Phi^0 = 1, \Phi \notin \nabla_2(0) \Leftrightarrow \alpha_\Phi^0 = 1/2;$
(iii) $\Phi \notin \triangle_2 \Leftrightarrow \bar{\beta}_\Phi = 1, \Phi \notin \nabla_2 \Leftrightarrow \bar{\alpha}_\Phi = 1/2.$

The proof of Theorem 1.5 can be found in [14, p. 23] and [15].

Another quantitative index of Φ is well known and is provided by the following six constants:

$$A_\Phi = \liminf_{t\to\infty} \frac{t\phi(t)}{\Phi(t)}\,, \qquad B_\Phi = \limsup_{t\to\infty} \frac{t\phi(t)}{\Phi(t)}\,, \tag{9}$$

$$A_\Phi^0 = \liminf_{t\to 0} \frac{t\phi(t)}{\Phi(t)}\,, \qquad B_\Phi^0 = \limsup_{t\to 0} \frac{t\phi(t)}{\Phi(t)} \tag{10}$$

and

$$\bar{A}_\Phi = \inf\left\{\frac{t\phi(t)}{\Phi(t)} : 0 < t < \infty\right\}, \qquad \bar{B}_\Phi = \sup\left\{\frac{t\phi(t)}{\Phi(t)} : 0 < t < \infty\right\}. \tag{11}$$

It is also known that $\Phi \notin \triangle_2(\infty) \Leftrightarrow B_\Phi = \infty$, $\Phi \notin \nabla_2(\infty) \Leftrightarrow A_\Phi = 1$, $\Phi \notin \triangle_2(0) \Leftrightarrow B_\Phi^0 = \infty$, $\Phi \notin \nabla_2(0) \Leftrightarrow A_\Phi^0 = 1$, $\Phi \notin \triangle_2 \Leftrightarrow \bar{B}_\Phi = \infty$ and $\Phi \notin \nabla_2 \Leftrightarrow \bar{A}_\Phi = 1$. Furthermore, we have the following.

Proposition 1.6

Let Φ and Ψ be a pair of complementary N-functions. Then

$$\frac{1}{A_\Phi} + \frac{1}{B_\Psi} = \frac{1}{A^0_\Phi} + \frac{1}{B^0_\Psi} = \frac{1}{\bar{A}_\Phi} + \frac{1}{\bar{B}_\Psi} = 1\,. \tag{12}$$

Proposition 1.7

Let $\Phi(u)$ be an N-function. Then

$$2^{-1/A_\Phi} \le \alpha_\Phi \le \beta_\Phi \le 2^{-1/B_\Phi}, \tag{13}$$

$$2^{-1/A^0_\Phi} \le \alpha^0_\Phi \le \beta^0_\Phi \le 2^{-1/B^0_\Phi} \tag{14}$$

and

$$2^{-1/\bar{A}_\Phi} \le \bar{\alpha}_\Phi \le \bar{\beta}_\Phi \le 2^{-1/\bar{B}_\Phi}. \tag{15}$$

The proofs of Propositions 1.6 and 1.7 can be found in [14, p. 27], [10] and [15]. In this paper, we only deal with Orlicz function spaces. The Jung constants of Orlicz sequence spaces will be discussed in another paper.

Finally, we need some properties of Hadamard matrix, which can be found in [12], [7] and [6]. The Hadamard matrix $H_{(n+1)\times(n+1)}$ of order $(n+1)$ is defined to be a square matrix with entries ± 1 and with pairwise orthogonal rows. $H_{(n+1)\times(n+1)}$ is said to be in normalized form, if its first column and row consist only of one. Removing the first column of $H_{(n+1)\times(n+1)}$, we obtain matrix $H_{n\times(n+1)}$, which is used in [12] and [7, Lemma 2].

Example 1.8: If $n + 1 = 4$, one has

$$H_{4\times 4} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

and

$$H_{3\times 4} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Let Φ, Ψ be a pair of complementary N-functions and $\Omega = [0,1]$ with the usual Lebesgue measure μ. For any given $u \ge 1$, we divide the interval $[0, 1/u]$ into four parts: $G_1 = [0, 1/4u), G_2 = [1/4u, 2/4u), G_3 = [2/4u, 3/4u)$ and $G_4 = [3/4u, 1/u]$. Let χ_{G_i} be the characteristic function of G_i and $a = \Phi^{-1}(4u/3)$. By

$$(x_1, x_2, x_3, x_4) = a(\chi_{G_2}, \chi_{G_3}, \chi_{G_4})H_{3\times 4}$$

we denote

$$\begin{aligned}
x_1(t) &= a[\chi_{G_2}(t) + \chi_{G_3}(t) + \chi_{G_4}(t)]\,,\\
x_2(t) &= a[\chi_{G_2}(t) - \chi_{G_3}(t) - \chi_{G_4}(t)]\,,\\
x_3(t) &= a[-\chi_{G_2}(t) + \chi_{G_3}(t) - \chi_{G_4}(t)]\,,\\
x_4(t) &= a[-\chi_{G_2}(t) - \chi_{G_3}(t) + \chi_{G_4}(t)]\,.
\end{aligned}$$

Since $\mu(\bigcup_{i=2}^4 G_i) = 3/4u$ and $1/2\,\mu(\bigcup_{i=1}^4 G_i) = 1/2u$, we have $\|x_i\|_{(\Phi)} = 1, 1 \le i \le 4$ and for $i \ne j$

$$\|x_i - x_j\|_{(\Phi)} = \frac{2a}{\Phi^{-1}(2u)}\,.$$

Put $b = \frac{3}{4u}\Phi^{-1}(\frac{4u}{3})$, $y_i(t) = \frac{1}{ab}x_i(t)$ and $c_i = 1/4$ for $1 \le i \le 4$. Then $\sum_{i=1}^4 c_i = 1$, $\|y_i\|_\Psi = 1$, $\sum_{i=1}^4 c_i y_i = 0$ and $\langle x_i - 0, y_i\rangle = \int_0^1 x_i(t)y_i(t)dt = 1 = \|x_i - 0\|_{(\Phi)}$. Therefore, by Lemma 1.3, the set $A_4 = co\{x_i : 1 \le i \le 4\}$ has zero as its Chebyshev center in $X_4[0, \frac{1}{u}] = \text{span}\,\{\chi_{G_i} : 1 \le i \le 4\} \subset L^{(\Phi)}[0, \frac{1}{u}]$ (see also Lemma 2 in [7]). It follows from (1) that

$$JC\left(X_4\Big[0, \frac{1}{u}\Big]\right) \ge \frac{r\big(A_4, X_4[0, \frac{1}{u}]\big)}{d(A_4)} \ge \frac{\Phi^{-1}(2u)}{2\Phi^{-1}(\frac{4u}{3})}\,.$$

In general, if $n + 1 = 2^m$ for some $m \ge 1$, we choose $a_n = \Phi^{-1}(\frac{n+1}{n}u)$, $b_n = \frac{n}{(n+1)u}\,\Phi^{-1}(\frac{n+1}{n}u)$,

$$(x_1, x_2, \cdots, x_{n+1}) = a_n(\chi_{G_2}, \chi_{G_3}, \cdots, \chi_{G_{n+1}})H_{n\times(n+1)}$$

and $y_i = \frac{1}{a_n b_n}\,x_i, c_i = \frac{1}{n+1}$. Finally, it follows from Lemma 1.3 that

$$JC\left(X_{n+1}\Big[0, \frac{1}{u}\Big]\right) \ge \frac{\Phi^{-1}(2u)}{2\Phi^{-1}(\frac{n+1}{n}u)} > \frac{\Phi^{-1}(2u)}{2(\frac{n+1}{n})\Phi^{-1}(u)}\,. \tag{16}$$

We conclude this section by the following.

Remark 1.9. Let X be a Banach space and let $A \subset S(X) = \{x \in X : \|x\| = 1\}$. If there exists a $z_0 \in X$ such that $r(A, z_0) \le 2$, then

$$\|z_0\| \le 3\,. \tag{17}$$

In fact, if $\|z_0\| > 3$, one has

$$r(A, z_0) = \sup\big[\|x - z_0\| : x \in A\big] \geq \sup\big[\|z_0\| - \|x\| : x \in A\big] > 2\,,$$

which is a contradiction.

§ 2. Lower Bounds of $JC(L^{(\Phi)}(\Omega))$

Theorem 2.1

Let Φ be an N-function. Then the Jung constant of $L^{(\Phi)}[0,1] = (L^{\Phi}[0,1], \|\cdot\|_{(\Phi)})$ satisfies

$$\beta_\Phi \leq JC\big(L^{(\Phi)}[0,1]\big)\,. \tag{18}$$

Furthermore, if $\Phi \in \triangle_2(\infty)$, we also have

$$\frac{1}{2\alpha_\Phi} \leq JC\big(L^{(\Phi)}[0,1]\big)\,. \tag{19}$$

Proof. We first show (18). By (6), there exist $1 < v_k \nearrow \infty$ such that

$$\lim_{k\to\infty} \frac{\Phi^{-1}(v_k)}{\Phi^{-1}(2v_k)} = \beta_\Phi\,. \tag{20}$$

For any given $1/2 > \epsilon > 0$, there is a $v_0 \in \{v_k : k \geq 1\}$ such that

$$\frac{\Phi^{-1}(v_0}{\Phi^{-1}(2v_0)} > \beta_\Phi - \epsilon \tag{21}$$

and

$$\Phi^{-1}(2v_0) > \frac{6\Phi^{-1}(2)}{\epsilon}\,. \tag{22}$$

An integer $n_0 > 1$ can be found such that $2v_0 - 1 < n_0 \leq 2v_0$. Thus,

$$\frac{2v_0}{n_0} < 1 + \frac{1}{n_0} < 2\,. \tag{23}$$

Put $e_i = [\frac{i-1}{2v_0}, \frac{i}{2v_0})$ for $1 \leq i \leq n_0$ and define $A = \{x_i : 1 \leq i \leq n_0\}$, where

$$x_i(t) = \Phi^{-1}(2v_0)\chi_{e_i}(t)\,.$$

Then $\|x_i\|_{(\Phi)} = 1$ for all $i \le n_0$. By (21) one has for $i \ne j$

$$\|x_i - x_j\|_{(\Phi)} = \Phi^{-1}(2v_0)\|\chi_{e_i \cup e_j}\|_{(\Phi)} = \frac{\Phi^{-1}(2v_0)}{\Phi^{-1}(v_0)} < \frac{1}{\beta_\Phi - \epsilon}\,,$$

i.e., $d(A) < 1/(\beta_\Phi - \epsilon)$.

Let $r_0 = r(A, L^{(\Phi)}[0,1])$. Then there exists some $z \in L^{(\Phi)}[0,1]$ such that for all $1 \le i \le n_0$

$$\|x_i - z\|_{(\Phi)} \le r(A, z) < r_0 + \frac{\epsilon}{2}\,. \tag{24}$$

Put $z_1(t) = z(t)\chi_e(t)$, where $e = \cup_{i=1}^{n_0} e_i = [0, \frac{n_0}{2v_0}) \subset [0,1]$. Then $|x_i(t) - z_1(t)| = |(x_i(t) - z(t))\chi_e(t)| \le |x_i(t) - z(t)|$ for $t \in [0,1]$ and $1 \le i \le n_0$. Therefore,

$$r(A, z_1) \le r(A, z)\,. \tag{25}$$

Further, let $F_i = \{t \in e_i : z_1(t) \le \Phi^{-1}(2v_0)\}$ and put

$$z_2(t) = \sum_{j=1}^{n_0} \left\{ z_1(t)\chi_{F_j}(t) + \left[2\Phi^{-1}(2v_0) - z_1(t)\right]\chi_{e_j - F_j}(t) \right\}.$$

Then $z_2(t) \le \Phi^{-1}(2v_0)$ and $|x_i(t) - z_2(t)| \le |x_i(t) - z_1(t)|$ for all $1 \le i \le n_0$ and all $t \in e$. Thus,

$$r(A, z_2) \le r(A, z_1)\,. \tag{26}$$

Put $F = \{t \in e : 0 \le z_2(t)\}$ and define $z_3(t) = z_2(t)\chi_F(t)$. Then $0 \le z_3(t) \le \Phi^{-1}(2v_0)$, $|x_i(t) - z_3(t)| \le |x_i(t) - z_2(t)|$ and

$$r(A, z_3) \le r(A, z_2)\,. \tag{27}$$

Now let us define $z_0(t) = \sum_{j=1}^{n_0} b_j \chi_{e_j}(t)$, where

$$b_j = \frac{1}{\mu(e_j)} \int_{e_j} z_3(t)dt\,.$$

For each $1 \le i \le n_0$ and any $R_i > \|x_i - z_3\|_{(\Phi)}$, we have from Jensen integral inequality(see [9, p. 62])

$$\begin{aligned}
1 &\ge \rho_\Phi\left(\frac{x_i - z_3}{R_i}\right) \\
&= \sum_{j \ne i} \int_{e_j} \Phi\left(\frac{0 - z_3(t)}{R_i}\right) dt + \int_{e_i} \Phi\left(\frac{\Phi^{-1}(2v_0) - z_3(t)}{R_i}\right) dt \\
&\ge \sum_{j \ne i} \mu(e_j)\Phi\left(\frac{1}{\mu(e_j)} \int_{e_j} \frac{z_3(t)}{R_i} dt\right) + \mu(e_i)\Phi\left(\frac{1}{\mu(e_i)} \int_{e_i} \frac{\Phi^{-1}(2v_0) - z_3(t)}{R_i} dt\right) \\
&= \sum_{j \ne i} \mu(e_j)\Phi\left(\frac{b_j}{R_i}\right) + \mu(e_i)\Phi\left(\frac{\Phi^{-1}(2v_0) - b_i}{R_i}\right) \\
&= \rho_\Phi\left(\frac{x_i - z_0}{R_i}\right),
\end{aligned}$$

i.e., $\|x_i - z_0\|_{(\Phi)} < R_i$ and $\|x_i - z_0\|_{(\Phi)} \le \|x_i - z_3\|_{(\Phi)}$ for every i or $r(A, z_0) \le r(A, z_3)$. Note that $r_0 \le r(A, 0) = 1$. It follows from (24)-(27) that

$$r(A, z_0) < r_0 + \frac{\epsilon}{2} < 2\,. \tag{28}$$

Put $\lambda_i = b_i/\Phi^{-1}(2v_0)$ and let $\lambda_{i_0} = \min\{\lambda_i : 1 \le i \le n_0\}$. Then $0 \le \lambda_i \le 1$ and $z_0(t) = \sum_{i=1}^{n_0} \lambda_i x_i(t) \ge \lambda_{i_0}\Phi^{-1}(2v_0)\chi_e(t)$. By (17), (22) and (23) we have

$$3 \ge \|z_0\|_{(\Phi)} \ge \frac{\lambda_{i_0}\Phi^{-1}(2v_0)}{\Phi^{-1}(\frac{2v_0}{n_0})} > \frac{6\lambda_{i_0}}{\epsilon}\,,$$

i.e., $\lambda_{i_0} < \epsilon/2$. Therefore,

$$\begin{aligned} r_0 + \frac{\epsilon}{2} > r(A, z_0) &= \max_{1 \le i \le n_0} \|x_i - z_0\|_{(\Phi)} \ge \|(x_{i_0} - z_0)\chi_{e_{i_0}}\|_{(\Phi)} \\ &= \|(1 - \lambda_{i_0})x_{i_0}\|_{(\Phi)} = 1 - \lambda_{i_0} > 1 - \frac{\epsilon}{2}\,, \end{aligned}$$

i.e., $r_0 > 1 - \epsilon$. Finally,

$$JC\big(L^{(\Phi)}[0,1]\big) \ge \frac{r_0}{d(A)} > (1 - \epsilon)(\beta_\Phi - \epsilon)\,.$$

We have thus proved (18) since ϵ is arbitrary.

Next we show (19), if $\Phi \in \triangle_2(\infty)$. In this case, $L^{(\Phi)}[0,1]$ is a separable dual space. By (6), there exist $1 \le u_k \nearrow \infty$ such that $\lim_{k\to\infty} \frac{\Phi^{-1}(u_k)}{\Phi^{-1}(2u_k)} = \alpha_\Phi$. For any given $\epsilon > 0$, there is a $u_0 \in \{u_k : k \ge 1\}$ such that

$$\frac{\Phi^{-1}(u_0)}{\Phi^{-1}(2u_0)} < \alpha_\Phi + \epsilon\,. \tag{29}$$

Put $D = \{n+1 \in N :$ Hadamard matrix $H_{(n+1)\times(n+1)}$ exists$\}$. Note that D is an infinite set since $n + 1 = 2^m \in D$ for every integer m. If $n + 1 \in D$, we divide the interval $[0, \frac{1}{u_0}] \subset [0, 1]$ into $n + 1$ parts: $G_1^{(n+1)} = [0, \frac{1}{(n+1)u_0}), \cdots, G_{n+1}^{(n+1)} = [\frac{n}{(n+1)u_0}, \frac{1}{u_0}]$. Then $\mu(G_i^{(n+1)}) = \frac{1}{(n+1)u_0}$ for all $1 \le i \le n+1$. Put $X_{n+1}[0, \frac{1}{u_0}] = \mathrm{span}\,\{\chi_{G_i^{(n+1)}} : 1 \le i \le n+1\} \subset L^{(\Phi)}[0, \frac{1}{u_0}]$. One has $X_{n_1+1}[0, \frac{1}{u_0}] \subset X_{n_2+1}[0, \frac{1}{u_0}]$ if $n_1 < n_2$ and $n_1, n_2 \in D$. The separability of $L^{(\Phi)}[0,1]$ implies that

$$\overline{\bigcup_{n+1\in D} X_{n+1}\left[0, \frac{1}{u_0}\right]} = L^{(\Phi)}\left[0, \frac{1}{u_0}\right].$$

Define $P_{n+1} : L^{(\Phi)}[0, \frac{1}{u_0}] \longmapsto X_{n+1}[0, \frac{1}{u_0}]$ by $P_{n+1}z(t) = \sum_{j=1}^{n+1} b_j \chi_{G_j}^{(n+1)}(t)$, where

$$b_j = \frac{1}{\mu(G_j^{(n+1)})} \int_{G_j^{(n+1)}} z(t)dt \,.$$

If $\|z\|_{(\Phi)} = 1$, we have from Jensen integral inequality

$$\begin{aligned} \int_0^{1/u_0} \Phi\big[P_{n+1}z(t)\big]dt &= \sum_{j=1}^{n+1} \mu(G_j^{(n+1)})\Phi\left[\frac{1}{\mu(G_j^{(n+1)})} \int_{G_j^{(n+1)}} z(t)dt\right] \\ &\le \int_0^{1/u_0} \Phi[z(t)]dt = 1 \,, \end{aligned}$$

i.e., $\|P_{n+1}\| \le 1$. On the other hand, if $z(t) = \chi_{[0,1/u_0]}(t) \in L^{(\Phi)}[0, 1/u_0]$ we have $P_{n+1}z = z$. Therefore, $\|P_{n+1}\| = 1$. It follows from Lemma 1.2 that for every $n+1 \in D$

$$JC\left(L^{(\Phi)}\Big[0, \frac{1}{u_0}\Big]\right) \ge JC\left(X_{n+1}\Big[0, \frac{1}{u_0}\Big]\right). \tag{30}$$

Now let us choose $n_0 + 1 \in D$ such that $1/n_0 < \epsilon$. Removing the first column of $H_{(n_0+1)\times(n_0+1)}$, we obtain $H_{n_0\times(n_0+1)}$. Define $A_{n_0+1} = co\,\{x_i : 1 \le i \le n_0 + 1\} \subset X_{n_0+1}[0, 1/u_0] = \text{span}\,\{\chi_{G_i} : 1 \le i \le n_0 + 1\}$, where

$$(x_1, x_2, \cdots, x_{n_0+1}) = \Phi^{-1}\left(\frac{n_0+1}{n_0}u_0\right)(\chi_{G_2}, \chi_{G_3}, \cdots, \chi_{G_{n_0+1}})H_{n_0\times(n_0+1)} \,,$$

and $G_i = G_i^{(n_0+1)}$ for simplicity. Then, for all $1 \le i \le n_0 + 1$

$$\|x_i\|_{(\Phi)} = \Phi^{-1}\left(\frac{n_0+1}{n_0}u_0\right)\|\chi_{\cup_{i=2}^{n_0+1}G_i}\|_{(\Phi)} = 1$$

and for $i \ne j$, by (29)

$$\|x_i - x_j\|_{(\Phi)} = \frac{2\Phi^{-1}(\frac{n_0+1}{n_0}u_0)}{\Phi^{-1}(2u_0)} < \left(1 + \frac{1}{n_0}\right)\frac{2\Phi^{-1}(u_0)}{\Phi^{-1}(2u_0)} < 2(1+\epsilon)(\alpha_\Phi + \epsilon) \,,$$

i.e., $d(A_{n_0+1}) < 2(1+\epsilon)(\alpha_\Phi + \epsilon)$. In view of Example 1.8, the Chebyshev center of A_{n_0+1} lies at 0 in $X_{n_0+1}[0, 1/u_0]$. One has from (1) and (16) that

$$JC\left(X_{n_0+1}\Big[0, \frac{1}{u_0}\Big]\right) \ge \frac{r\big(A_{n_0+1}, X_{n_0+1}[0, \frac{1}{u_0}]\big)}{d(A_{n_0+1})} > \frac{1}{2(1+\epsilon)(\alpha_\Phi + \epsilon)} \,. \tag{31}$$

Finally, we must prove

$$JC\big(L^{(\Phi)}[0,1]\big) \geq JC\left(L^{(\Phi)}\left[0, \frac{1}{u_0}\right)\right]. \tag{32}$$

Put $D' = (0,1], Y_s = L^{(\Phi)}[0,s]$ and define $Q_s : L^{(\Phi)}[0,1] \longmapsto Y_s$ by $Q_s z = z\chi_{[0,s]}$. Then $\|Q_s\| = 1$ and $\overline{\cup_{s\in D'} Y_s} = L^{(\Phi)}[0,1]$. It follows from Lemma 1.2 that $JC(L^{(\Phi)}[0,1]) \geq JC(Y_s)$ for every $s \in D'$. In particular, setting $s_0 = \frac{1}{u_0}$ we get (32). We have proved (19) by (32), (30) and (31) since ϵ is arbitrary. □

Corollary 2.2

Let Φ be an N-function and let $E^{(\Phi)}[0,1]$ be the closed separable subspase of $L^{(\Phi)}[0,1]$. Then

$$\beta_\Phi \leq JC\big(E^{(\Phi)}[0,1]\big). \tag{33}$$

Furthermore, if $\Phi \in \triangle_2(\infty)$, we also have

$$\frac{1}{2\alpha_\Phi} \leq JC\big(E^{(\Phi)}[0,1]\big). \tag{34}$$

Proof. It follows from the proof of Theorem 2.1. In addition, we give a short proof of (33). Let $v_k, k \geq 1$ satisfy (20). Without loss of generality, we may assume $\sum_{i=1}^{\infty} \frac{1}{2v_i} \leq 1$. Choose $G_i \subset [0,1]$ such that $G_i \cap G_j = \emptyset$ if $i \neq j$ and $\mu(G_i) = \frac{1}{2v_i}$ for all $i \geq 1$. Put $B = \{x_i : i \geq 1\}$, where $x_i(t) = \Phi^{-1}(v_i)\chi_{G_i}(t)$. Then $d(B) = 1$ since $\|x_i - x_j\|_{(\Phi)} = 1$ if $i \neq j$. Every $z \in E^{(\Phi)}[0,1]$ has absolutely continuous norm, which implies that $\lim_{i\to\infty} \|z\chi_{G_i}\|_{(\Phi)} = 0$ in virtue of $\lim_{i\to\infty} \mu(G_i) = 0$. Therefore, by (20),

$$\begin{aligned} r(B,z) &= \sup\big\{\|x - z\|_{(\Phi)} : x \in B\big\} \geq \limsup_{i\to\infty} \|x_i - z\|_{(\Phi)} \\ &\geq \limsup_{i\to\infty} \|(x_i - z)\chi_{G_i}\|_{(\Phi)} \geq \limsup_{i\to\infty} \{\|x_i\|_{(\Phi)} - \|z\chi_{G_i}\|_{(\Phi)}\} \\ &= \lim_{i\to\infty} \|x_i\|_{(\Phi)} = \beta_\Phi . \end{aligned}$$

Since $z \in E^{(\Phi)}[0,1]$ is arbitrary, we have $r(B, E^{(\Phi)}[0,1]) \geq \beta_\Phi$ which implies (33). □

Corollary 2.3

(i) If $\Phi \notin \triangle_2(\infty) \bigcap \nabla_2(\infty)$, then $JC(L^{(\Phi)}[0,1]) = JC(E^{(\Phi)}[0,1]) = 1$.
(ii) For every N-function Φ, we always have $JC(L^{(\Phi)}[0,1]) = JC(E^{(\Phi)}[0,1])$ and

$$\frac{1}{\sqrt{2}} \leq JC\big(L^{(\Phi)}[0,1]\big).$$

Proof. (i) By Theorem 1.5 (i), $\Phi \notin \triangle_2(\infty) \bigcap \nabla_2(\infty)$ implies that either $\beta_\Phi = 1$ or $\beta_\Phi < 1$ but $\alpha_\Phi = \frac{1}{2}$, i.e., $\max(\beta_\Phi, 1/2\alpha_\Phi) = 1$. Therefore, the conclusion follows from (18), (19), (33), (34) and (5).

(ii) It is sufficient to show

$$\frac{1}{\sqrt{2}} \le \max\left(\beta_\Phi, \frac{1}{2\alpha_\Phi}\right).$$

If not, $\frac{1}{\sqrt{2}} > \beta_\Phi$ and $\frac{1}{\sqrt{2}} > \frac{1}{2\alpha_\Phi}$. We have thus reached a contradiction: $\alpha_\Phi > 1/\sqrt{2} > \beta_\Phi$, since $\alpha_\Phi \le \beta_\Phi$ always holds. □

Theorem 2.4

Let Φ be an N-function. Then the Jung constant of $L^{(\Phi)}[0,\infty) = (L^\Phi[0,\infty), \|\cdot\|_{(\Phi)})$ satisfies

$$\bar{\beta}_\Phi \le JC\big(L^{(\Phi)}[0,\infty)\big). \tag{35}$$

Furthermore, if $\Phi \in \triangle_2$, we have also

$$\frac{1}{2\bar{\alpha}_\Phi} \le J\big(E^{(\Phi)}[0,\infty)\big). \tag{36}$$

Proof. We first prove (35). By (8), for any given $\frac{1}{2} > \epsilon > 0$ there exists $0 < v_0 < \infty$ such that

$$\frac{\Phi^{-1}(v_0)}{\Phi^{-1}(2v_0)} > \bar{\beta}_\Phi - \epsilon. \tag{37}$$

Since $\lim_{n\to\infty} \Phi^{-1}(2v_0/n) = 0$, an integer n_0 can be found such that

$$\Phi^{-1}\Big(\frac{2v_0}{n_0}\Big) < \frac{\epsilon}{6}\Phi^{-1}(2v_0). \tag{38}$$

Put $e_i = [\frac{i-1}{2v_0}, \frac{i}{2v_0}) \subset [0,\infty)$ for $1 \le i \le n_0$ and define $A = \{x_i : 1 \le i \le n_0\}$, where $x_i(t) = \Phi^{-1}(2v_0)\chi_{G_i}(t)$. Then $A \subset S(L^{(\Phi)}[0,\infty))$ and $d(A) < 1/(\bar{\beta}_\Phi - \epsilon)$ by (37). Let $r_0 = r(A, L^{(\Phi)}[0,\infty))$. By the same way as in the proof of (18), we can find a $z_0 \in L^{(\Phi)}[0,\infty)$ such that $r(A, z_0) < r_0 + \epsilon/2 < 2$, where $z_0(t) = \sum_{i=1}^{n_0} \lambda_i x_i(t)$ with $0 \le \lambda_i \le 1$. Letting $\lambda_{i_0} = \min\{\lambda_i : 1 \le i \le n_0\}$, we have from (17) and (38) that

$$3 \ge \|z_0\|_{(\Phi)} = \frac{\lambda_{i_0}\Phi^{-1}(2v_0)}{\Phi^{-1}(\frac{2v_0}{n_0})} > \frac{6\lambda_{i_0}}{\epsilon},$$

i.e., $\lambda_{i_0} < \epsilon/2$. Therefore, $r(A, z_0) \ge 1 - \lambda_{i_0} > 1 - \epsilon/2$ and $r_0 > 1 - \epsilon$. Thus,

$$JC\big(L^{(\Phi)}[0,\infty)\big) \ge \frac{r_0}{d(A)} > (1-\epsilon)(\bar{\beta}_\Phi - \epsilon).$$

We have proved (35) since ϵ is arbitrary.

Next we show (36) under the assumption that $\Phi \in \triangle_2$. By (8), for any given $\epsilon > 0$, there is a $0 < u_0 < \infty$ such that

$$\frac{\Phi^{-1}(u_0)}{\Phi^{-1}(2u_0)} < \bar{\alpha}_\Phi + \epsilon\,.$$

Choose $n_0 \geq 3$ such that $\frac{1}{n_0} < \epsilon$ and the Hadamard matrix $H_{(n_0+1)\times(n_0+1)}$ exists. Divide the interval $[0, \frac{1}{u_0}) \subset [0, \infty)$ into $n_0 + 1$ parts: $G_i = [\frac{i-1}{(n_0+1)u_0}, \frac{i}{(n_0+1)u_0}), 1 \leq i \leq n_0 + 1$. Define $A_{n_0+1} \subset X_{n_0+1}[0, \frac{1}{u_0}]$ as in Example 1.8. It is easily seen that

$$JC\left(X_{n_0+1}\left[0, \frac{1}{u_0}\right]\right) \geq \frac{1}{2(1+\epsilon)(\bar{\alpha}_\Phi + \epsilon)}\,.$$

By using Lemma 1.2, we can verify

$$JC\big(L^{(\Phi)}[0,\infty)\big) \geq JC\left(L^{(\Phi)}\left[0, \frac{1}{u_0}\right)\right) \geq JC\left(X_{n_0+1}\left[0, \frac{1}{u_0}\right)\right)\,.$$

Therefore, we obtain (36). □

Corollary 2.5

Let Φ be an N-function and let $E^{(\Phi)}[0,\infty)$ be the closed separable subspase of $L^{(\Phi)}[0,\infty)$. Then

$$\bar{\beta}_\Phi \leq JC\big(E^{(\Phi)}[0,\infty)\big)\,. \tag{39}$$

Furthermore, if $\Phi \in \triangle_2$, we also have

$$\frac{1}{2\bar{\alpha}_\Phi} \leq JC\big(E^{(\Phi)}[0,\infty)\big). \tag{40}$$

Proof. The assertion follows from the proof of Theorem 2.4. In addition, we give a different proof of (39). By (8), there exist $0 < u_i < \infty$ such that

$$\lim_{i\to\infty} \frac{\Phi^{-1}(u_i)}{\Phi^{-1}(2u_i)} = \bar{\beta}_\Phi\,. \tag{41}$$

Choose $e_1 = [0, \frac{1}{2u_1})$ and $e_i = [\sum_{j=1}^{i-1} \frac{1}{2u_j}, \sum_{j=1}^{i} \frac{1}{2u_j})$ for $i \geq 2$. Put $B = \{x_i : i \geq 1\}$, where $x_i(t) = \Phi^{-1}(u_i)\chi_{e_i}(t)$. Then $d(B) = 1$ since $\|x_i - x_j\|_{(\Phi)} = 1$ if $i \neq j$. We must prove that for every $z \in E^{(\Phi)}[0,1]$

$$\lim_{i\to\infty} \|z\chi_{e_i}\|_{(\Phi)} = 0\,. \tag{42}$$

In the case $\sum_{i=1}^{\infty} \frac{1}{2u_i} < \infty$, we have $\mu(e_i) = \frac{1}{2u_i} \to 0$ as $i \to \infty$, which implies (42).

In the case $\sum_{i=1}^{\infty} \frac{1}{2u_i} = \infty$, we have $\lim_{i\to\infty} \sum_{j=1}^{i-1} \frac{1}{2u_j} = \infty$. Let $E_i = [\sum_{j=1}^{i-1} \frac{1}{2u_j}, \infty)$. Then $e_i \subset E_i$ for all $i \geq 1$. Since $z \in E^{(\Phi)}[0,\infty)$, one has $\int_0^{\infty} \Phi[\lambda z(t)]dt < \infty$ for any given $\lambda > 1$ and so,

$$\rho_\Phi(\lambda z \chi_{e_i}) \leq \int_{E_i} \Phi\big[\lambda z(t)\big]dt \to 0$$

as $i \to \infty$. Therefore, (42) holds again by the fact that $\rho_\Phi(\lambda y_i) \to 0$ for any given $\lambda > 1$ if and only if $\|y_i\|_{(\Phi)} \to 0$.(see [14, p. 87])

It follows from (42) and (41) that

$$r(B,z) \geq \lim_{i\to\infty} \|x_i\|_{(\Phi)} = \bar{\beta}_\Phi$$

for every $z \in E^{(\Phi)}[0,\infty)$, which implies (39). □

Corollary 2.6

(i) If $\Phi \notin \triangle_2 \bigcap \nabla_2$ then $JC(L^{(\Phi)}[0,\infty)) = JC(E^{(\Phi)}[0,\infty)) = 1$.
(ii) For every N-function Φ, we always have $JC(L^{(\Phi)}[0,\infty)) = JC(E^{(\Phi)}[0,\infty))$ and

$$\frac{1}{\sqrt{2}} \leq JC\big(L^{(\Phi)}[0,\infty)\big).$$

Proof. Similar to Corollary 2.3. □

Lemma 2.7 (Chen and Sun [3])

If $L^{(\Phi)}(\Omega)$ is reflexive, then $\tilde{N}(L^{(\Phi)}(\Omega)) < 1$, where $\Omega = [0,1]$ or $[0,\infty)$ with the usual Lebesgue measure.

Theorem 2.8

$L^{(\Phi)}(\Omega)$ is reflexive if and only if $JC(L^{(\Phi)}(\Omega) < 1$, where Ω is as in Lemma 2.7.

Proof. The assertion follows from Corollary 2.2 (i), Corollary 2.6 (i), Lemma 2.7 and (5). □

§ 3. Lower Bounds of $JC(L^\Phi(\Omega))$

Now let us turn to the Orlicz function space $L^\Phi[0,1] = (L^\Phi[0,1], \|\cdot\|_\Phi)$ equipped with Orlicz norm.

Theorem 3.1

For every N-function Φ, we have

$$\frac{1}{2\alpha_\Psi} \le JC\big(L^\Phi[0,1]\big), \tag{43}$$

where Ψ is the complementary N-function to Φ. Furthermore, if $\Phi \in \triangle_2(\infty)$, we also have

$$\beta_\Psi \le JC\big(L^\Phi[0,1]\big)\,. \tag{44}$$

Proof. We first show (43). By (6), there exist $1 \le v_k \nearrow \infty$ such that $\lim_{k\to\infty} \frac{\Psi^{-1}(v_k)}{\Psi^{-1}(2v_k)} = \alpha_\Psi$. Note that $\lim_{v\to\infty} \frac{v}{\Psi^{-1}(v)} = \infty$. Therefore, for any given $\epsilon > 0$ there exists $v_0 \in \{v_k : k \ge 1\}$ such that

$$\frac{\Psi^{-1}(v_0)}{\Psi^{-1}(2v_0)} < \alpha_\Psi + \epsilon \tag{45}$$

and

$$\frac{2v_0}{\Psi^{-1}(2v_0)} > \frac{12}{\epsilon\Psi^{-1}(1)}\,. \tag{46}$$

Let n_0 be an integer satisfying $2v_0 - 1 < n_0 \le 2v_0$. Then

$$\frac{1}{2} < 1 - \frac{1}{2v_0} < \frac{n_0}{2v_0} \le 1\,. \tag{47}$$

Put $e_i = [\frac{i-1}{2v_0}, \frac{i}{2v_0})$ for all $1 \le i \le n_0$. Define $A = \{x_i : 1 \le i \le n_0\}$, where

$$x_i(t) = \frac{2v_0}{\Psi^{-1}(2v_0)}\chi_{e_i}(t)\,.$$

Then $A \subset S(L^\Phi[0,1])$ since $\mu(e_i) = \frac{1}{2v_0}$. If $i \ne j$, one has from (45) that

$$\|x_i - x_j\|_\Phi = \frac{2v_0}{\Psi^{-1}(2v_0)}\|\chi_{e_i\cup e_j}\|_\Phi = \frac{2\Psi^{-1}(v_0)}{\Psi^{-1}(2v_0)} < 2(\alpha_\Psi + \epsilon)\,,$$

i.e., $d(A) < 2(\alpha_\Psi + \epsilon)$.

Let $r_0 = r(A, L^\Phi[0,1])$. Then there exists $z \in L^\Phi[0,1]$ such that

$$\max \left\{ \|x_i - z\|_\Phi : 1 \le i \le n_0 \right\} = r(A, z) < r_0 + \frac{\epsilon}{2}\,.$$

Put $z_1(t) = z(t)\chi_e(t)$, where $e = \cup_{i=1}^{n_0} e_i = [0, \frac{n_0}{2v_0})$. Then $r(A, z_1) \le r(A, z)$. Secondly, let $F_i = \{t \in e_i : z_1(t) \le 2v_0/\Psi^{-1}(2v_0)\}$ and put

$$z_2(t) = \sum_{j=1}^{n_0} \left\{ z_1(t)\chi_{F_j}(t) + \Big[\frac{4v_0}{\Psi^{-1}(2v_0)} - z_1(t)\Big]\chi_{e_j - F_j}(t) \right\}.$$

Then $z_2(t) \le 2v_0/\Psi^{-1}(2v_0)$ and $|x_i(t) - z_2(t)| \le |x_i(t) - z_1(t)|$ for all $1 \le i \le n_0$ and $t \in e$. Thus, $r(A, z_2) \le r(A, z_1)$. Thirdly, set $F = \{t \in e : 0 \le z_2(t)\}$ and $z_3(t) = z_2(t)\chi_F(t)$. It is easily seen that $0 \le z_3(t) \le 2v_0/\Psi^{-1}(2v_0)$ and $|x_i(t) - z_3(t)| \le |x_i(t) - z_2(t)|$ for all $1 \le i \le n_0$ and $t \in e$. Therefore, $r(A, z_3) \le r(A, z_2)$. Finally, we define $z_0(t) = \sum_{j=1}^{n_0} b_j \chi_{e_j}(t)$, where $b_j = \frac{1}{\mu(e_j)} \int_{e_j} z_3(t)dt$ as in the proof of Theorem 2.1. In virtue of Theorem 13 in [14, p. 69], for each i if $\|x_i - z_3\|_\Phi \neq 0$, there exists $k_i > 0$ such that

$$\|x_i - z_3\|_\Phi = \frac{1}{k_i}\left[1 + \rho_\Phi(k_i(x_i - z_3))\right].$$

By Jensen integral inequality, we have

$$\begin{aligned}
&\|x_i - z_3\|_\Phi \\
&= \frac{1}{k_i}\left\{1 + \sum_{j\neq i} \int_{e_j} \Phi[k_i(0 - z_3(t))]dt + \int_{e_i} \Phi\Big[k_i\Big(\frac{2v_0}{\Psi^{-1}(2v_0)} - z_3(t)\Big)\Big]dt\right\} \\
&\ge \frac{1}{k_i}\left\{1 + \sum_{j\neq i} \mu(e_j)\Phi\Big[\frac{1}{\mu(e_j)} \int_{e_j} k_i z_3(t)dt\Big]\right. \\
&\qquad \left. + \mu(e_i)\Phi\Big[\frac{1}{\mu(e_i)} \int_{e_i} k_i\Big(\frac{2v_0}{\Psi^{-1}(2v_0)} - z_3(t)\Big)dt\Big]\right\} \\
&= \frac{1}{k_i}\left\{1 + \sum_{j\neq i} \mu(e_j)\Phi(k_i b_j) + \mu(e_i)\Phi\Big[k_i\Big(\frac{2v_0}{\Psi^{-1}(2v_0)} - b_i\Big)\Big]\right\} \\
&= \frac{1}{k_i}\left\{1 + \rho_\Phi(k_i(x_i - z_0))\right\} \\
&\ge \|x_i - z_0\|_\Phi\,.
\end{aligned}$$

Therefore,

$$r(A, z_0) \leq r(A, z_3) < r_0 + \frac{\epsilon}{2} < 2\,. \tag{48}$$

Putting $\lambda_i = b_i\Psi^{-1}(2v_0)/2v_0$ and letting $\lambda_{i_0} = \min\{\lambda_i : 1 \leq i \leq n_0\}$, we have $0 \leq \lambda_i \leq 1$ and

$$z_0(t) = \sum_{i=1}^{n_0} \lambda_i x_i(t) \geq \lambda_{i_0} \frac{2v_0}{\Psi^{-1}(2v_0)} \chi_e(t)\,.$$

It follows from (17), (47) and (46) that

$$3 \geq \|z_0\|_\Phi \geq \frac{\lambda_{i_0} 2v_0}{\Psi^{-1}(2v_0)} \left[\frac{n_0}{2v_0}\Psi^{-1}\left(\frac{2v_0}{n_0}\right)\right] > \frac{6\lambda_{i_0}}{\epsilon}\,,$$

i.e., $\lambda_{i_0} < \frac{\epsilon}{2}$. Hence, by (48) one has $r_0 + \frac{\epsilon}{2} \geq r(A, z_0) \geq \|x_{i_0} - z_0\|_\Phi \geq \|(x_{i_0} - z_0)\chi_{e_{i_0}}\|_\Phi = 1 - \lambda_{i_0} > 1 - \frac{\epsilon}{2}$, i. e., $r_0 > 1 - \epsilon$. Finally,

$$JC\big(L^\Phi[0,1]\big) \geq \frac{r_0}{d(A)} > \frac{1-\epsilon}{2(\alpha_\Phi + \epsilon)}\,.$$

We have thus proved (43) since ϵ is arbitrary.

Next we prove (44) under the assumption $\Phi \in \triangle_2(\infty)$. In this case, $L^\Phi[0,1]$ is a separable dual space. By (6), there exist $1 \leq u_k \nearrow \infty$ such that $\lim_{k\to\infty} [\Psi^{-1}(u_k)/\Psi^{-1}(2u_k)] = \beta_\Psi$. Therefore, for any given $\frac{1}{2} > \epsilon > 0$, there is a $u_0 \in \{u_k : k \geq 1\}$ satisfying

$$\frac{\Psi^{-1}(u_0)}{\Psi^{-1}(2u_0)} > \beta_\Psi - \epsilon\,. \tag{49}$$

Choose n_0 such that $\frac{1}{n_0} < \epsilon$ and the Hadamard matrix $H_{(n_0+1)\times(n_0+1)}$ exists. Divide $[0, \frac{1}{v_0}] \subset [0,1]$ into $n_0 + 1$ parts $\{G_i : 1 \leq i \leq n_0 + 1\}$ such that $G_i \cap G_j = \emptyset$ if $i \neq j$ and $\mu(G_i) = \frac{1}{(n_0+1)u_0}$. Put $A_{n_0+1} = co\{x_i : 1 \leq i \leq n_0 + 1\}$, where

$$(x_1, x_2, \cdots, x_{n_0+1}) = \frac{(n_0+1)u_0}{n_0\Psi^{-1}(\frac{n_0+1}{n_0}u_0)}(\chi_{G_2}, \chi_{G_3}, \cdots, \chi_{G_{n_0+1}})H_{n_0\times(n_0+1)}\,.$$

Similarly to the proof of (19), A_{n_0+1} has 0 as its Chebyshev center in X_{n_0+1} $[0, \frac{1}{u_0}] = \operatorname{span}\{\chi_{G_i} : 1 \leq i \leq n_0 + 1\}$. Since $\|x_i\|_\Phi = 1$ for all $i \leq n_0 + 1$ and, by (49),

$$\|x_i - x_j\|_\Phi = \frac{(n_0+1)\Psi^{-1}(2u_0)}{n_0\Psi^{-1}(\frac{n_0+1}{n_0}u_0)} < \Big(1 + \frac{1}{n_0}\Big)\frac{\Psi^{-1}(2u_0)}{\Psi^{-1}(u_0)} < \frac{1+\epsilon}{\beta_\Psi - \epsilon}$$

if $i \neq j$, one has $r(A_{n_0+1}, X_{n_0+1}[0, \frac{1}{u_0}]) = 1$ and

$$JC\left(X_{n_0+1}\left[0, \frac{1}{u_0}\right]\right) > \frac{\beta_\Psi - \epsilon}{1+\epsilon}\,. \tag{50}$$

To complete the proof, we must show

$$JC\big(L^\Phi[0,1]\big) \geq JC\left(L^\Phi\left[0, \frac{1}{u_0}\right]\right) \geq JC\left(X_{n_0+1}\left[0, \frac{1}{u_0}\right]\right)\,. \tag{51}$$

Let $D, X_{n+1}[0, \frac{1}{u_0}]$ and $P_{n+1} : L^\Phi[0,1] \longmapsto X_{n+1}[0, \frac{1}{u_0}]$ be as in the proof of Theorem 2.1. If $z \in L^\Phi[0,1]$ with $\|z\|_\Phi \neq 0$, there exists $k > 0$ satisfying $\|z\|_\Phi = \frac{1}{k}[1 + \rho_\Phi(kz)]$. The Jensen integral inequality implies that

$$\begin{aligned}
\|P_{n+1}z\|_\Phi &\leq \frac{1}{k}\big[1 + \rho_\Phi(kP_{n+1}z)\big] \\
&= \frac{1}{k}\left\{1 + \sum_{j=1}^{n+1} \mu\big(G_j^{(n+1)}\big)\Phi\left[\frac{1}{\mu(G_j^{(n+1)})}\int_{G_j^{(n+1)}} kz(t)dt\right]\right\} \\
&\leq \frac{1}{k}\left\{1 + \sum_{j=1}^{n+1}\int_{G_j^{(n+1)}} \Phi\big[kz(t)\big]dt\right\} \\
&= \|z\|_\Phi\,,
\end{aligned}$$

i.e., $\|P_{n+1}\| \leq 1$. It is easily seen that $\|P_{n+1}\| = 1$. By Lemma 1.2 we have

$$JC\left(L^\Phi\left[0, \frac{1}{u_0}\right]\right) = \sup_{n+1 \in D} JC\left(X_{n+1}\left[0, \frac{1}{u_0}\right]\right) \geq JC\left(X_{n_0+1}\left[0, \frac{1}{u_0}\right]\right),$$

which is the right inequality of (51). The proof of the left inequality of (51) is similar to that of (32). We have thus proved (44) by (50) and (51) since ϵ is arbitrary. □

Corollary 3.2

Let Φ be an N-function and let $E^\Phi[0,1]$ be the closed separable subspase of $L^\Phi[0,1]$. Then

$$\frac{1}{2\alpha_\Psi} \leq JC\big(E^\Phi[0,1]\big)\,. \tag{52}$$

Furthermore, if $\Phi \in \triangle_2(\infty)$, we also have

$$\beta_\Psi \leq JC\big(E^\Phi[0,1]\big)\,. \tag{53}$$

Proof. The result follows from the proof of Theorem 3.1. □

Corollary 3.3

(i) If $\Phi \notin \triangle_2(\infty)\bigcap\nabla_2(\infty)$, *then* $JC(L^\Phi[0,1]) = JC\big(E^\Phi[0,1]\big) = 1$.
(ii) For every N-function Φ, *we have always* $JC(L^\Phi[0,1]) = JC(E^\Phi[0,1])$ *and*

$$\frac{1}{\sqrt{2}} \leq JC\big(L^\Phi[0,1]\big)\,.$$

Proof. Similar to that of Corollary 2.3. □

Theorem 3.4

Let Φ *be an N-function. Then the Jung constant of* $L^\Phi[0,\infty) = (L^\Phi[0,\infty), \|\cdot\|_\Phi)$ *satisfies*

$$\frac{1}{2\bar{\alpha}_\Psi} \leq JC\big(L^\Phi[0,\infty)\big)\,. \tag{54}$$

Furthermore, if $\Phi \in \triangle_2$, *we have also*

$$\bar{\beta}_\Psi \leq JC\big(L^\Phi[0,\infty)\big)\,. \tag{55}$$

Proof. We first show (54). For any given $\epsilon > 0$ there exists a $v_0 > 0$ such that

$$\frac{\Psi^{-1}(v_0)}{\Psi^{-1}(2v_0)} < \bar{\alpha}_\Psi + \epsilon\,. \tag{56}$$

Since $\lim_{n\to\infty} \frac{n}{2v_0}\Psi^{-1}(\frac{2v_0}{n}) = \infty$, an integer n_0 can be found such that

$$\frac{n_0}{2v_0}\Psi^{-1}\left(\frac{2v_0}{n_0}\right) > \frac{3\Psi^{-1}(2v_0)}{\epsilon v_0}\,. \tag{57}$$

Put $e_i = [\frac{i-1}{2v_0}, \frac{i}{2v_0}) \subset [0,\infty), 1 \leq i \leq n_0$ and define $A = \{x_i : 1 \leq i \leq n_0\}$, where $x_i(t) = [2v_0/\Psi^{-1}(2v_0)]\chi_{e_i}(t)$. Then $A \subset S(L^\Phi[0,\infty))$ and $d(A) < 2(\bar{\alpha}_\Psi + \epsilon)$ by (56). If $r_0 = r(A, L^\Phi[0,\infty))$, similarly to the proof of Theorem 3.1, there exists a function $z_0 \in L^\Phi[0,\infty)$ in the form $z_0(t) = \sum_{i=1}^{n_0} \lambda_i x_i(t)$ with $0 \leq \lambda_i \leq 1$ such that $r(A, z_0) < r_0 + \frac{\epsilon}{2}$. Let $\lambda_{i_0} = \min\{\lambda_i : 1 \leq i \leq n_0\}$. Then we have from (17) and (57) that

$$3 \geq \|z_0\|_\Phi \geq \lambda_{i_0}\left\|\sum_{i=1}^{n_0} x_i\right\|_\Phi > \frac{6\lambda_{i_0}}{\epsilon}\,,$$

i.e., $\lambda_{i_0} < \frac{\epsilon}{2}$. Therefore, $r_0 + \frac{\epsilon}{2} > r(A, z_0) \geq \|(x_{i_0} - z_0)\chi_{i_0}\|_\Phi = 1 - \lambda_{i_0} > 1 - \frac{\epsilon}{2}$ and

$$JC\big(L^\Phi[0,\infty)\big) \geq \frac{r_0}{d(A)} > \frac{1-\epsilon}{2(\bar{\alpha}_\Psi + \epsilon)}\,,$$

which implies (54) since ϵ is arbitrary.

The proof of (55) is similar to that of (44). □

Corollary 3.5

Let Φ be an N-function and let $E^{\Phi}[0,\infty)$ be the closed separable subspase of $L^{\Phi}[0,\infty)$. Then

$$\frac{1}{2\bar{\alpha}_{\Psi}} \leq JC\big(E^{\Phi}[0,\infty)\big)\,. \tag{58}$$

Furthermore, if $\Phi \in \triangle_2$, we also have

$$\bar{\beta}_{\Psi} \leq JC\big(E^{\Phi}[0,\infty)\big)\,. \tag{59}$$

Proof. It follows from the proof of Theorem 3.4. □

Corollary 3.6

(i) If $\Phi \notin \triangle_2 \bigcap \nabla_2$, then $JC(L^{\Phi}[0,\infty)) = JC(E^{\Phi}[0,\infty)) = 1$.
(ii) For every N-function Φ, we always have $JC(L^{\Phi}[0,\infty)) = JC(E^{\Phi}[0,\infty))$ and

$$\frac{1}{\sqrt{2}} \leq JC\big(L^{\Phi}[0,\infty)\big)\,.$$

Lemma 3.7 (Wang and Shi [18])

If $L^{\Phi}(\Omega)$ is reflexive, then $\tilde{N}(L^{\Phi}(\Omega)) < 1$, where $\Omega = [0,1]$ or $[0,\infty)$ with the usual Lebesgue measure.

Theorem 3.8

$L^{\Phi}(\Omega)$ is reflexive if and only if $JC(L^{\Phi}(\Omega) < 1$, where Ω is as in Lemma 3.7.

Proof. The assertion follows from Corollary 3.2 (i), Corollary 3.6 (i), Lemma 3.7 and (5). □

Now we can sum up the main results on lower bound of the Jung constant of reflexive Orlicz function space $L^{(\Phi)}(\Omega)$ together with its dual space in the following.

Theorem 3.9

Let Φ and Ψ be a pair of complementary N-functions.
(a) If $\Phi \in \triangle_2(\infty) \bigcap \nabla_2(\infty)$, then

$$\max\left(\frac{1}{2\alpha_{\Phi}}, \beta_{\Phi}\right) \leq \min\left\{JC(L^{(\Phi)}[0,1]), JC(L^{\Psi}[0,1])\right\}. \tag{60}$$

(b) If $\Phi \in \triangle_2 \bigcap \nabla_2$, *then*

$$\max\left(\frac{1}{2\bar{\alpha}_\Phi}, \bar{\beta}_\Phi\right) \leq \min\left\{JC(L^{(\Phi)}[0,\infty)), JC(L^{\Psi}[0,\infty))\right\}. \tag{61}$$

Proof. Note that $\Phi \in \nabla_2(\infty) \iff \Psi \in \triangle_2(\infty)$ and $\Phi \in \nabla_2 \iff \Psi \in \triangle_2$. Hence, (a) follows from Theorem 2.1 and Theorem 3.1 while (b) follows from Theorem 2.4 and Theorem 3.4. □

§ 4. Main Theorems

In 1966, Rao [13] obtained Riesz-Thorin interpolation theorem between Orlicz spaces equipped with Orlicz norm (see also [14, p. 226]). In 1972, Cleaver [4] generalized Rao's interpolation theorem and obtained the L^Φ-inequalities (see also [5, Theorem 3.2] and [14, p. 240, Corollary 11]). In 1985, the first named author proved that these results are still valid for $L^{(\Phi)}$ spaces equipped with Luxemburg norm (see [14, p. 226, p. 256]). In fact, we have the following.

Lemma 4.1

Let Φ *be an N-function and* $\Omega = [0,1]$ *or* $\Omega = [0,\infty)$. *Suppose that* $\Phi_0(u) = u^2, 0 \leq s \leq 1$ *and* $\Phi_s(u)$ *is defined to be the inverse of*

$$\Phi_s^{-1}(u) = \left[\Phi^{-1}(u)\right]^{1-s}\left[\Phi_0^{-1}(u)\right]^s. \tag{62}$$

Then, for any collection $\{y_i : 1 \leq i \leq N\} \subset E^{(\Phi_s)}(\Omega)$ *and any* $\{c_i \geq 0\}_1^N$ *with* $\sum_{i=1}^N c_i = 1$, *we have*

$$\sum_{i=1}^N \sum_{j=1}^N c_i c_j \|y_i - y_j\|_{(\Phi_s)}^{2/(2-s)} \leq 2c^{2(1-s)/(2-s)} \sum_{i=1}^N c_i \|y_i\|_{(\Phi_s)}^{2/(2-s)}, \tag{63}$$

where $c = \max\left\{1 - c_i : 1 \leq i \leq N\right\}$. *Similarly, we have for* $\left\{y_i : 1 \leq i \leq N\right\} \subset E^{\Phi_s}(\Omega)$

$$\sum_{i=1}^N \sum_{j=1}^N c_i c_j \|y_i - y_j\|_{\Phi_s}^{2/(2-s)} \leq 2c^{2(1-s)/(2-s)} \sum_{i=1}^N c_i \|y_i\|_{\Phi_s}^{2/(2-s)}, \tag{64}$$

Lemma 4.2 (Ren [16, Lemma 3.3])

Let Φ *be an N-function and let* $\Phi_s(u)$ *be the inverse of (62). If* $0 < s \leq 1$, *then* $\Phi_s \in \triangle_2 \bigcap \nabla_2$.

Lemma 4.3 (Ren [16, Theorem 3.4])

Let Φ be an N-function and let Φ_s be the inverse of (62). If $0 < s \leq 1$ and $\Omega = [0,1]$ or $\Omega = [0,\infty)$, then

$$2^{s/2} \leq N\big(L^{(\Phi_s)}(\Omega)\big) \tag{65}$$

and

$$2^{s/2} \leq N\big(L^{\Phi_s}(\Omega)\big)\,. \tag{66}$$

Only the proof of (65) was given in [16]. By the same way of the proof of (65) we can verify (66).

Theorem 4.4

Let Φ be an N-function and let Φ_s be the inverse of (62). Further let Ψ_s^+ be the complementary N-function to Φ_s. If $0 < s \leq 1$ and $\Omega = [0,1]$ or $\Omega = [0,\infty)$, then we have

$$\max\big\{JC(L^{(\Phi_s)}(\Omega)), JC(L^{\Phi_s}(\Omega))\big\} \leq 2^{-s/2} \tag{67}$$

and

$$\max\big\{JC(L^{\Psi_s^+}(\Omega)), JC(L^{(\Psi_s^+)}(\Omega))\big\} \leq 2^{-s/2}\,. \tag{68}$$

Proof. (67) follows directly from (65), (66), (5) and the notation $\tilde{N}(X) = 1/N(X)$. To prove (68) we first show

$$JC\big(L^{\Psi_s^+}(\Omega)\big) \leq 2^{-s/2}\,. \tag{69}$$

By Lemma 4.2, $L^{\Psi_s^+}(\Omega)$ is reflexive, of course, it is a separable dual space. Let $\{z_i : i \geq 1\}$ be a dense set in $L^{\Psi_s^+}(\Omega)$ and put $X_n = \operatorname{span}\{z_i : 1 \leq i \leq n\}$. For any given bounded closed convex set $A \subset X_n$ with $r(A, X_n)$ being its Chebyshev radius and $d(A)$ being its diameter, there always exists some x as its Chebyshev center. In view of Lemma 1.3, there exist an integer $N \leq n$, $\{x_i : i \leq N\} \subset L^{\Psi_s^+}(\Omega)$, $\{y_i : i \leq N\} \subset S((L^{\Psi_s^+}(\Omega))^*) = S(L^{(\Phi_s)}(\Omega))$ and $\{c_i \geq 0 : i \leq N\}$ with $\sum_{i=1}^N c_i = 1$, which satisfy conditions (a), (b) and (c) in Lemma 1.3. Putting $\lambda = \frac{2}{2-s}$ in (3), we have from (63) that

$$\begin{aligned}
\frac{2^{2/(2-s)}[r(A,X_n)]^{2/(2-s)}}{\left(\frac{n}{n+1}\right)^{2/(2-s)-1}} &\leq \big[d(A)\big]^{2/(2-s)} \sum_{i=1}^{N}\sum_{j=1}^{N} c_i c_j \|y_i - y_j\|_{(\Phi_s)}^{2/(2-s)} \\
&\leq \big[d(A)\big]^{2/(2-s)}\; 2\sum_{i=1}^{N} c_i \|y_i\|_{(\Phi_s)}^{2/(2-s)} \\
&= 2\big[d(A)\big]^{2/(2-s)}
\end{aligned}$$

or

$$\frac{r(A, X_n)}{d(A)} \le 2^{-s/2}\left(\frac{n}{n+1}\right)^{s/2}.$$

Since A is arbitrary, we obtain

$$JC(X_n) \le 2^{-s/2}\left(\frac{n}{n+1}\right)^{s/2}.$$

Hence, (69) follows from (4). Similarly, by using (64) and (4) we can prove

$$JC\big(L^{(\Psi_s^+)}(\Omega)\big) \le 2^{-s/2}. \tag{70}$$

Finally, (68) follows from (69) and (70). □

Corollary 4.5 (Pichugov [12])

If $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\Omega = [0,1]$ or $\Omega = [0,\infty)$, then

$$JC\big(L^p(\Omega)\big) = JC\big(L^q(\Omega)\big) = \max\left(2^{1/p-1},\ 2^{-1/p}\right). \tag{71}$$

Proof. We first show

$$\max\left(2^{1/p-1}, 2^{-1/p}\right) \le \min\left\{JC(L^p(\Omega)), JC(L^q(\Omega))\right\}. \tag{72}$$

In fact, putting $M(u) = |u|^p$, we have $L^{(M)}(\Omega) = L^p(\Omega)$, $\|\cdot\|_{(M)} = \|\cdot\|_p$, $L^N(\Omega) = L^q(\Omega)$ and $\|\cdot\|_N = \|\cdot\|_q$, where

$$N(v) = \frac{(q-1)^{q-1}}{q^q}|v|^q$$

is the complementary N-function to $M(u)$. Since $\alpha_M = \beta_M = \bar{\alpha}_M = \bar{\beta}_M = 2^{-1/p}$, we obtain (72) from (60) and (61).

Next we prove

$$\max\left\{JC(L^p(\Omega)), JC(L^q(\Omega))\right\} \le \max\left(2^{1/p-1},\ 2^{-1/p}\right). \tag{73}$$

If $1 < p \le 2$, we choose $1 < a < p \le 2$. Putting $\Phi(u) = |u|^a$, $\Phi_0(u) = u^2$ and $s = \frac{2(p-a)}{p(2-a)}$ in Theorem 4.4, we have $0 < s \le 1$ and for $u \ge 0$

$$\Phi_s^{-1}(u) = u^{1-s/a+s/2} = u^{1/p},$$

i.e., $\Phi_s(u) = |u|^p$. Since $L^{(\Phi_s)}(\Omega) = L^p(\Omega)$, $L^{\Psi_s^+}(\Omega) = L^q(\Omega)$ and $\lim_{a\searrow 1}(-\frac{s}{2}) = \frac{1}{p} - 1$, we have from (67) and (68) that

$$\max\{JC(L^p(\Omega)), JC(L^q(\Omega))\} \leq 2^{1/p-1}\,. \tag{74}$$

If $2 \leq p < \infty$, we choose $2 \leq p < b < \infty$. Letting $\Phi(u) = |u|^b$ and $s = \frac{2(b-p)}{p(b-2)}$, again we have $0 < s \leq 1$ and $\Phi_s(u) = |u|^p$. Note that $\lim_{b\nearrow\infty}(-\frac{s}{2}) = -\frac{1}{p}$. By (67) and (68) we get

$$\max\{JC(L^p(\Omega)), JC(L^q(\Omega))\} \leq 2^{-1/p}\,. \tag{75}$$

Thus, (73) follows from (74) and (75). Finally, (71) follows from (72) and (73). □

EXAMPLE 4.6: If $1 < p < \infty$ and $\Phi(u) = |u|^{2p}+2|u|^p$, then $\Phi^{-1}(u) = (\sqrt{u+1}-1)^{1/p}$ for $u \geq 0$ and for $0 < s \leq 1$

$$\Phi_s^{-1}(u) = (\sqrt{u+1} - 1)^{1-s/p}u^{s/2}\,.$$

It is easily seen that

$$\alpha_{\Phi_s} = \beta_{\Phi_s} = \lim_{u\to\infty}\frac{\Phi_s^{-1}(u)}{\Phi_s^{-1}(2u)} = \left(\frac{1}{2}\right)^{1-s/2p+s/2},$$

$\bar{\beta}_{\Phi_s} = \beta_{\Phi_s}$ and

$$\bar{\alpha}_{\Phi_s} = \alpha^0_{\Phi_s} = \lim_{u\to 0}\frac{\Phi_s^{-1}(u)}{\Phi_s^{-1}(2u)} = \left(\frac{1}{2}\right)^{1-s/p+s/2}.$$

Therefore, by Lemma 4.2, Theorem 3.9 and Theorem 4.4 we have

$$2^{-s/2} \geq \{JC(L^{(\Phi_s)}[0,1]), JC(L^{\Psi_s^+}[0,1])\} \geq 2^{-(1-s/2p)-s/2}$$

and

$$2^{-s/2} \geq \{JC(L^{(\Phi_s)}[0,\infty)), JC(L^{\Psi_s^+}[0,\infty))\} \geq \begin{cases} 2^{1-s/p+s/2-1}, & \text{if } 1 < p \leq \frac{3}{2} \\ 2^{-(1-s/2p)-s/2}, & \text{if } \frac{3}{2} \leq p < \infty\,. \end{cases}$$

In this paper we denote $a \leq b \leq d$ and $a \leq c \leq d$ by $a \leq \{c, b\} \leq d$ for simplicity.

Now we can find the exact values of Jung constants of a class of reflexive Orlicz function spaces equipped with Luxemburg norm and their dual spaces. The first main theorem of this paper is as follows.

Theorem 4.7

Let Φ *be an N-function and let* Φ_s *be the inverse of (62) with* $0 < s \le 1$.
(i) If $\Phi \notin \triangle_2(\infty) \bigcap \nabla_2(\infty)$*, then*

$$JC\big(L^{(\Phi_s)}[0,1]\big) = JC\big(L^{\Psi_s^+}[0,1]\big) = 2^{-s/2}\,. \tag{76}$$

(ii) If $\Phi \notin \triangle_2 \bigcap \nabla_2$*, then*

$$JC\big(L^{(\Phi_s)}[0,\infty)\big) = JC\big(L^{\Psi_s^+}[0,\infty)\big) = 2^{-s/2}\,. \tag{77}$$

Proof. (i) In virtue of (60), (67) and (68), one has

$$\max\left(\frac{1}{2\alpha_{\Phi_s}}, \beta_{\Phi_s}\right) \le \big\{JC(L^{(\Phi_s)}[0,1]), JC(L^{\Psi_s^+}[0,1])\big\} \le 2^{-s/2}. \tag{78}$$

Note that (62) implies that for $u > 0$

$$\frac{\Phi_s^{-1}(u)}{\Phi_s^{-1}(2u)} = \left[\frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}\right]^{1-s}\left[\frac{\sqrt{u}}{\sqrt{2u}}\right]^{s}. \tag{79}$$

If $\Phi \notin \triangle_2(\infty)$, then $\frac{1}{2} \le \alpha_\Phi \le \beta_\Phi = 1$ by Theorem 1.5 (i). Therefore, by (79) we have

$$2\alpha_{\Phi_s} = 2(\alpha_\Phi)^{1-s}\left(\frac{1}{2}\right)^{s/2} \ge \left(\frac{1}{2}\right)^{-s/2},$$

$$\beta_{\Phi_s} = (\beta_\Phi)^{1-s}\left(\frac{1}{2}\right)^{s/2} = 2^{-s/2}$$

and so,

$$\max\left(\frac{1}{2\alpha_{\Phi_s}}, \beta_{\Phi_s}\right) = 2^{-s/2}\,. \tag{80}$$

If $\Phi \notin \nabla_2(\infty)$, then $\frac{1}{2} = \alpha_\Phi \le \beta_\Phi \le 1$ by Theorem 1.5 (i). Because

$$\frac{1}{2\alpha_{\Phi_s}} = 2^{-s/2} \ge \beta_{\Phi_s}\,,$$

again (80) holds. Finally, (76) follows from (78) and (80).

(ii) The proof is similar to that of (i). □

To find the exact values of Jung constants of a class of reflexive Orlicz function spaces equipped with Orlicz norm and their dual spaces, we need the following two lemmas.

Lemma 4.8 (Ren [17, Lemma 4.5])

Let Φ be an N-function and let Φ_s be the inverse of (62). Then

$$\frac{1}{A_{\Phi_s}} = \frac{1-s}{A_\Phi} + \frac{s}{2}, \quad \frac{1}{B_{\Phi_s}} = \frac{1-s}{B_\Phi} + \frac{s}{2}, \tag{81}$$

$$\frac{1}{A^0_{\Phi_s}} = \frac{1-s}{A^0_\Phi} + \frac{s}{2}, \quad \frac{1}{B^0_{\Phi_s}} = \frac{1-s}{B^0_\Phi} + \frac{s}{2} \tag{82}$$

and

$$\frac{1}{\bar{A}_{\Phi_s}} = \frac{1-s}{\bar{A}_\Phi} + \frac{s}{2}, \quad \frac{1}{\bar{B}_{\Phi_s}} = \frac{1-s}{\bar{B}_\Phi} + \frac{s}{2}\,. \tag{83}$$

Lemma 4.9

Let Φ, Ψ be a pair of complementary N-functions. Suppose that

$$C_\Phi = \lim_{t\to\infty} \frac{t\phi(t)}{\Phi(t)} \tag{84}$$

exists. Then
(i) $\gamma_\Phi = \lim_{u\to\infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ *exists and* $\gamma_\Phi = 2^{-1/C_\Phi}$ *;*
(ii) $C_\Psi = \lim_{t\to\infty} \frac{t\psi(t)}{\Psi(t)}$ *exists and* $\gamma_\Psi = \lim_{v\to\infty} \frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)} = 2^{-1/C_\Psi}$ *;*
(iii) $2\gamma_\Phi\gamma_\Psi = 1$ *.*

Similarly, if $C^0_\Phi = \lim_{t\to 0} \frac{t\phi(t)}{\Phi(t)}$ *exists, then* $\gamma^0_\Phi = \lim_{u\to 0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} = 2^{-1/C^0_\Phi}$, $C^0_\Psi = \lim_{t\to 0} \frac{t\psi(t)}{\Psi(t)}$ *exists,* $\gamma^0_\Psi = 2^{-1/C^0_\Psi}$ *and* $2\gamma^0_\Phi\gamma^0_\Psi = 1$ *.*

Proof. The assertions follow from Propositions 1.6 and 1.7. □

The second main theorem of this paper is as follows.

Theorem 4.10

Let Φ, Φ_s and s be as in Theorem 4.7.
(i) If $\Phi \notin \triangle_2(\infty)\bigcap\nabla_2(\infty)$ *and C_Φ defined by (84) exists, then*

$$JC\big(L^{\Phi_s}[0,1]\big) = JC\big(L^{(\Psi_s^+)}[0,1]\big) = 2^{-s/2}\,. \tag{85}$$

(ii) If $\Phi \notin \triangle_2\bigcap\nabla_2$ *and C_Φ exists in the case that* $\Phi \notin \triangle_2(\infty)\bigcap\nabla_2(\infty)$ *or C^0_Φ exists in the case that* $\Phi \notin \triangle_2(0)\bigcap\nabla_2(0)$*, then*

$$J\big(L^{\Phi_s}[0,\infty)\big) = JC\big(L^{(\Psi_s^+)}[0,\infty)\big) = 2^{-s/2}\,. \tag{86}$$

Proof. (i) By Lemma 4.2, (60), (67) and (68), we have

$$\max\left(\frac{1}{2\alpha_{\Psi_s^+}}, \beta_{\Psi_s^+}\right) \le \left\{JC(L^{\Phi_s}[0,1]), JC(L^{(\Psi_s^+)}[0,1])\right\} \le 2^{-s/2}. \tag{87}$$

The conditions given in (i) imply that $C_\Phi = \infty$ or $C_\Phi = 1$.

In the case that $C_\Phi = \infty$, we have $\frac{1}{C_{\Phi_s}} = \frac{1-s}{C_\Phi} + \frac{s}{2} = \frac{s}{2}$ by (81), $\frac{1}{C_{\Psi_s^+}} = 1 - \frac{1}{C_{\Phi_s}} = 1 - \frac{s}{2}$ by (12) and so, in view of Lemma 4.9,

$$\alpha_{\Psi_s^+} = \beta_{\Psi_s^+} = \gamma_{\Psi_s^+} = 2^{-1/C_{\Psi_s^+}} = 2^{s/2-1}.$$

Therefore,

$$\max\left(\frac{1}{2\alpha_{\Psi_s^+}}, \beta_{\Psi_s^+}\right) = 2^{-s/2}. \tag{88}$$

In the case that $C_\Phi = 1$, one has $\frac{1}{C_{\Phi_s}} = 1 - \frac{s}{2}$, $\frac{1}{C_{\Psi_s^+}} = \frac{s}{2}$ and $\alpha_{\Psi_s^+} = \beta_{\Psi_s^+} = \gamma_{\Psi_s^+} = 2^{-s/2}$. Again (88) holds. Thus, (85) follows from (87) and (88).

(ii) By Lemma 4.2, (61), (67) and (68), we have

$$\max\left(\frac{1}{\bar{\alpha}_{\Psi_s^+}}, \bar{\beta}_{\Psi_s^+}\right) \le \left\{JC(L^{\Phi_s}[0,\infty)), JC(L^{(\Psi_s^+)}[0,\infty))\right\} \le 2^{-s/2}. \tag{89}$$

Note that $\Phi \notin \triangle_2 \bigcap \nabla_2$ if and only if $\Phi \notin \triangle_2(\infty) \bigcap \nabla_2(\infty)$ or $\Phi \notin \triangle_2(0) \bigcap \nabla_2(0)$. In the case that $C_\Phi^0 = \infty$ or $C_\Phi^0 = 1$, one has

$$\max\left(\frac{1}{2\alpha^0_{\Psi_s^+}}, \beta^0_{\Psi_s^+}\right) = 2^{-s/2}. \tag{90}$$

Since $\max(\beta_{\Psi_s^+}, \beta^0_{\Psi_s^+}) \le \bar{\beta}_{\Psi_s^+}$ and $\bar{\alpha}_{\Psi_s^+} \le \min(\alpha_{\Psi_s^+}, \alpha^0_{\Psi_s^+})$ always hold, we have

$$\max\left(\frac{1}{2\alpha_{\Psi_s^+}}, \frac{1}{2\alpha^0_{\Psi_s^+}}, \beta_{\Psi_s^+}, \beta^0_{\Psi_s^+}\right) \le \max\left(\frac{1}{2\bar{\alpha}_{\Psi_s^+}}, \bar{\beta}_{\Psi_s^+}\right). \tag{91}$$

The conditions given in (ii) imply that (88) holds or (90) holds. Finally, (86) follows from (89), (91) and (88) or (90). □

Example 4.11: Let $1 < p < \infty$ and let $M(u)$ be the inverse of

$$M^{-1}(u) = [\ln(1+u)]^{1/2p} u^{1/4}, \quad u \ge 0.$$

Further, let $N(v)$ be the complementary N-function to $M(u)$. Then

$$JC(L^{(M)}[0,1]) = JC(L^N[0,1]) = 2^{-1/4}\,,$$
$$JC(L^{(M)}[0,\infty)) = JC(L^N[0,\infty)) = 2^{-1/4}\,,$$
$$JC(L^M[0,1]) = JC(L^{(N)}[0,1]) = 2^{-1/4}$$

and

$$J\big(L^M[0,\infty)\big) = J\big(L^{(N)}[0,\infty)\big) = 2^{-1/4}\,.$$

In fact, putting $\Phi(u) = e^{|u|^p} - 1$, we have $\Phi^{-1}(u) = [\ln(1+u)]^{1/p}$ for $u \geq 0$ and

$$\Phi_s^{-1}(u) = [\ln(1+u)]^{1-s/p}\, u^{s/2}\,.$$

Therefore, $M(u) = \Phi_s(u)|_{s=1/2}$ and $N(v) = \Psi_s^{+}(v)|_{s=1/2}$. Since $C_\Phi = \infty$, the conclusion follows from Theorem 4.7 and Theorem 4.10.

Acknowledgment. The authors would like to thank Professor Mikhail Ostrovskiĭ, who said "Estimation of the Jung constant is one of the directions of research in the geometric theory of normed spaces" in **MR** 92c: 46016. His review stimulated the authors to complete this work.

References

1. D. Amir, On Jung's constant and related constants in normed linear spaces, *Pacific J. Math.* **118** (1985), 1–15.
2. W.L. Bynum, Normal structure coefficients for Banach spaces, *Pacific J. Math.* **86** (1980), 427–435.
3. S.T. Chen and H. Y. Sun, Reflexive Orlicz spaces have uniformly normal structure, *Studia Math.* **109** (1994), 197–208.
4. C.E. Cleaver, On the extension of Lipschitz-Hölder maps on Orlicz spaces, *Studia Math.* **42** (1972), 195–204.
5. C.E. Cleaver, Packing spheres in Orlicz spaces, *Pacific J. Math.* **65** (1976), 325–335.
6. S.W. Golomb and L.D. Baumert, The search for Hadamard matrices, *Amer. Math. Monthly* **70** (1963), 12–17.
7. V.I. Ivanov and S.A. Pichugov, Jung's constants of ℓ_p^n-spaces, *Math. Notes* **48** (1990), 997–1004.
8. H.W. Jung, Über die kleinste Kugel, die eine räumliche Figur einschliesst *J. Reine Angew. Math.* **123** (1901), 241–257.
9. M.A. Krasnosel'skii and Ya. B. Rutickii, *Convex Functions and Orlicz Spaces,* P. Noordhoff Ltd., Groningen, 1961.

10. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces,* I and II, Springer, Berlin, 1977 and 1979.
11. E. Maluta, Uniformly normal structure and related coefficients, *Pacific J. Math.* **111** (1984), 357–369.
12. S.A. Pichugov, Jung's constant for the space L^p, *Math. Notes* **43** (1988), 348–353.
13. M.M. Rao, Interpolation, ergodicity and martigales, *J. Math. Mech.* **16** (1966), 543–567.
14. M.M. Rao and Z. D. Ren, *Theory of Orlicz Spaces,* Marcel Dekker, New York, 1991.
15. Z.D. Ren, Packing in Orlicz function spaces, Ph.D. Dissertation (Advisor: N. E. Gretsky), University of California, Riverside, 1994.
16. Z.D. Ren, Some geometric coefficients in Orlicz function spaces, Interaction between Functional Analysis, Harmonic Analysis, and Probability, edited by N. Kalton and E. Saab, Marcel Dekker, *Lecture notes in Pure and Applied Mathematics* **175** (1996), 391–404.
17. Z.D. Ren, Nonsquare constants of Orlicz spaces,*Stochastic Processes and Functional Analysis*, edited by J. A. Goldstein, N. E. Gretsky and J. J. Uhl, Jr., Marcel Dekker, Lecture Notes in Pure and Applied Mathematics **186** (1997), 179–197.
18. T.F. Wang and Z. R. Shi, On the uniformly normal structure of Orlicz spaces with Orlicz norm, *Comment. Math. Univ. Carolin.* **34** (1993), 433–442.