Collect. Math. **48**, 4-6 (1997), 733–742

Ultrapowers of Köthe function spaces

Yves Raynaud

Equipe d'Analyse (CNRS), Université Paris 6, 4, place Jussieu,

75252-Paris-Cedex 05, France

Abstract

The ultrapowers, relative to a fixed ultrafilter, of all the Köthe function spaces with non trivial concavity over the same measure space can be represented as Köthe function spaces over the same (enlarged) measure space. The existence of a uniform homeomorphism between the unit spheres of two such Köthe function spaces is reproved.

Introduction

Ultrapowers and ultraproducts were introduced in Banach Spaces theory quite a long time ago ([2]); their main interest in this field is to give a qualitative approach to the local structure of Banach spaces.

Here we focus on a description of the ultrapowers of Köthe function spaces with non trivial concavity. Let us recall some definitions.

Given a measure space $(\Omega, \mathcal{A}, \mu)$, a *Köthe function space over* $(\Omega, \mathcal{A}, \mu)$ is an order ideal (or solid linear subspace) of the space $L_0(\Omega, \mathcal{A}, \mu)$ (of classes of μ-measurable scalar functions), equipped with a norm for which it is a Banach lattice, and whose support is the whole of Ω. A Köthe function space (or simply a Banach lattice) X has *non trivial concavity* if it is q-concave for some $q < \infty$, i. e. (see [9], §1.d):

$$\exists\, C,\ \forall x_1, ...x_n \in X \qquad \Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q} \le C \left\| \Big(\sum_{i=1}^{n} |x_i|^q\Big)^{1/q} \right\|$$

Let I be an index set and $\mathcal{U}$ a non trivial ultrafilter over I. We shall denote by $\tilde{X}$ the ultrapower $X^I/\mathcal{U}$ (we refer to [7], [12] for basic facts and results about Banach space ultrapowers). Since $\tilde{X}$ is a quotient of $\ell_\infty(I;X)$, its elements are equivalence classes of bounded families of elements of X, indexed by I; we shall denote by $(x_i)^\bullet_{i\in I}$ the equivalence class of the family $(x_i)_{i\in I}$.

In the case where $X = L_1(\Omega,\mathcal{A},\mu)$, it is well known that $\tilde{X}$ is an abstract L_1-space, hence by Kakutani's theorem ([9], th. 1.b.2) can be identified with (i.e. is isometrically Riesz isomorphic to) a space $L_1(\tilde{\Omega},\tilde{\mathcal{A}},\tilde{\mu})$ over some measure space $(\tilde{\Omega},\tilde{\mathcal{A}},\tilde{\mu})$. On the other hand, the ultrapower $\tilde{X}$ of a q-concave Köthe function space over $(\tilde{\Omega},\tilde{\mathcal{A}},\tilde{\mu})$ is q-concave as well, hence order continuous, hence can be identified with some Köthe function space over some measure space (S,Σ,m) ([9], th. 1.b.14).

The aim of this note is to show that it is possible to identify $(\tilde{\Omega},\tilde{\mathcal{A}},\tilde{\mu})$ with (S,Σ,m), i. e. to represent $\tilde{X}$ as a Köthe function space over $(\tilde{\Omega},\tilde{\mathcal{A}},\tilde{\mu})$. In another paper ([11]), we obtained this identification using a result of [1] (generalizing an observation of [10]) on the existence of a support preserving homeomorphism between the unit spheres of X and of $L_1(\Omega,\mathcal{A},\mu)$. In this note we obtain directly this representation of $\tilde{X}$ as a K. f. s. over $(\tilde{\Omega},\tilde{\mathcal{A}},\tilde{\mu})$ (see th. 10 of §4), and we deduce very simply the result of [1] from this representation theorem (see th. 12 of §5 below). Of course the ransom to pay for this simplicity is the loss of the quantitative information (on moduli of uniform continuity) contained in [1], as is always the case for proofs using ultrapower (or non-standard) methods.

Recall that the result of [1] is also a corollary of a more general result on interpolation spaces given later by [3] (where however the preservation of the supports by the uniform homeomorphism seems to be less transparent).

1. The bilinear map $\tilde{X}^* \times \tilde{X} \to L_1(\tilde{\Omega},\tilde{\mathcal{A}},\tilde{\mu})$

If X is a q-concave Köthe function space over $(\Omega,\mathcal{A},\mu)$, it is in particular order continuous, hence X^* identifies with a Köthe function space over the same measure space (the Köthe dual X'). The bilinear map:

$$X^* \times X \to L_1(\Omega,\mathcal{A},\mu) \qquad (x^*,x) \mapsto x^*.x$$

is bounded, hence has an ultrapower extension:

$$\widetilde{X^*} \times \tilde{X} \to L_1(\tilde{\Omega},\tilde{A},\tilde{\mu}) \qquad (\tilde{x}^*,\tilde{x}) \mapsto \tilde{x}^*.\tilde{x}$$

defined by $\tilde{x}^*.\tilde{x} = (x_i^*.x_i)^\bullet_{i\in I}$ when $\tilde{x} = (x_i)^\bullet_{i\in I}$ and $\tilde{x}^* = (x_i^*)^\bullet_{i\in I}$.

Let $\mathcal{Z} = L_\infty(\Omega, \mathcal{A}, \mu)$ and $\tilde{\mathcal{Z}} = \mathcal{Z}^I/\mathcal{U}$ (which identifies with a sub-C^*-algebra of $L_\infty(\tilde{\Omega}, \tilde{A}, \tilde{\mu})$). Then, similarly, the natural bilinear products:

$$\mathcal{Z} \times X \to X \quad (h, x) \mapsto h.x$$
$$\mathcal{Z} \times X^* \to X^* \quad (h, x^*) \mapsto h.x^*$$

extend to bilinear maps:

$$\tilde{\mathcal{Z}} \times \tilde{X} \to \tilde{X} \quad (\tilde{h}, \tilde{x}) \mapsto \tilde{h}.\tilde{x}$$
$$\tilde{\mathcal{Z}} \times \widetilde{X^*} \to \widetilde{X^*} \quad (\tilde{h}, \tilde{x}^*) \mapsto \tilde{h}.\tilde{x}^*$$

and we have clearly:

$$\tilde{h}.(\tilde{x}^*.\tilde{x}) = (\tilde{h}.\tilde{x}^*).\tilde{x} = \tilde{x}^*.(\tilde{h}.\tilde{x})\,.$$

As is well known ([12], p. 78), the ultrapower of the dual space, $\widetilde{X^*}$, isometrically identifies with a w^*-dense, norm closed subspace of the dual of the ultrapower space, $\tilde{X}^*$, by setting $\langle \tilde{x}^*, \tilde{x}\rangle = \lim_{i,\mathcal{U}} \langle x_i^*, x_i\rangle$ when $\tilde{x} = (x_i)^\bullet_{i\in I}$ and $\tilde{x}^* = (x_i^*)^\bullet_{i\in I}$. In fact the unit ball of $\widetilde{X^*}$ is w^*-dense in that of $\tilde{X}^*$. If X is superreflexive, then in fact $\widetilde{X^*} = \tilde{X}^*$.

For a Banach lattice, this embedding $j\colon \widetilde{X^*} \to \tilde{X}^*$ is a lattice homomorphism: it is clearly positive, and if $\tilde{x}^*, \tilde{y}^*$ are two positive disjoint elements of $\widetilde{X^*}$, then $j(\tilde{x}^*)$ and $j(\tilde{y}^*)$ are disjoint in $\tilde{X}^*$. (Use the fact that there are representing families $(x_i^*)_i$ and $(y_i^*)_i$ for $\tilde{x}^*$, resp. $\tilde{y}^*$, with $x_i^* \perp y_i^*$ for every $i \in I$; then for every $\tilde{z} \in \tilde{X}_+$, and $\varepsilon > 0$, choose any representing family $(z_i)_i$ of $\tilde{z}$ with nonnegative elements, and for every $i \in I$, choose $x_i, y_i \in X_+$, with $z_i = x_i + y_i$, $|\langle x_i^*, y_i\rangle| \le \varepsilon$ and $|\langle y_i^*, x_i\rangle| \le \varepsilon$; set $\tilde{x} = (x_i)^\bullet_i$ and $\tilde{y} = (y_i)^\bullet_i$. Then $\tilde{z} = \tilde{x} + \tilde{y}$ and $|\langle j(\tilde{x}^*), \tilde{y}\rangle| \le \varepsilon$, $|\langle j(\tilde{y}^*), \tilde{x}\rangle| \le \varepsilon$).

We equip the dual space $\tilde{X}^*$ with the locally convex topology τ, defined by the family of seminorms $(p_{\tilde{x}})_{\tilde{x}\in\tilde{X}_+}$, where $p_{\tilde{x}}(\xi) = \langle |\xi|, \tilde{x}\rangle$, for every $\xi \in \tilde{X}^*$. This topology is stronger than the w^*-topology, but the continuous linear forms are the same for the two topologies. For, every $F \in (\tilde{X}^*, \tau)^*$ is order continuous, since every downwards directed set $(\xi_\alpha)_\alpha \in \tilde{X}^*$ with g. l. b. 0 converges to 0 for the w^*-topology, hence for the τ-topology (since the ξ_α are nonnegative). Since $\tilde{X}$ is q-concave, hence a KB-space, the order-continuous dual of $\tilde{X}^*$ identifies with $\tilde{X}$ (see [13], p. 92–93).

In particular, the convex sets of $\tilde{X}^*$ have the same w^* and τ closures; hence the unit ball of $\widetilde{X^*}$ is τ-dense in the unit ball of $\tilde{X}^*$. On the other hand, for every $\tilde{x} \in \tilde{X}$, the linear map $\widetilde{X^*} \to L_1(\tilde{\Omega}, \tilde{A}, \tilde{\mu})$, $\tilde{x}^* \mapsto \tilde{x}^*.\tilde{x}$ is τ to norm continuous, since for every $\tilde{x}^* \in \widetilde{X^*}$ and $\tilde{x} \in \tilde{X}$, we have:

$$\|\tilde{x}^*.\tilde{x}\|_1 = \lim_{i,\mathcal{U}} \|x_i^*.x_i\|_1 = \lim_{i,\mathcal{U}} \big\langle\, |x_i^*|\,, |x_i|\,\big\rangle = \big\langle\, |\tilde{x}^*|\,, |\tilde{x}|\,\big\rangle$$

Hence this linear map has a unique τ to norm continuous extension $\xi \to \xi.\tilde{x}$ to the whole of of $\tilde{X}^*$.

Lemma 1

a) The map $\tilde{X}^* \times \tilde{X} \to L_1(\tilde{\Omega}, \tilde{A}, \tilde{\mu})$, $(\xi, \tilde{x}) \mapsto \xi.\tilde{x}$ is bilinear and bounded.
b) We have $\langle \xi, \tilde{x} \rangle = \langle \xi.\tilde{x}, \mathbb{1} \rangle$ for every $\xi \in \tilde{X}^*$ and $\tilde{x} \in \tilde{X}$.
c) We have $|\xi.\tilde{x}| = |\xi| \, . \, |\tilde{x}|$ for every $\xi \in \tilde{X}^*$ and $\tilde{x} \in \tilde{X}$.
d) We have $\xi.(\tilde{h}.\tilde{x}) = \tilde{h}.(\xi.\tilde{x})$ for every $\xi \in \tilde{X}^*, \tilde{h} \in \tilde{\mathcal{Z}}$ and $\tilde{x} \in \tilde{X}$.

Proof. a) The left linearity is clear, the right one is a consequence of the formula:

$$\xi.\tilde{x} = \lim_{\tilde{x}^* \underset{\tau}{\to} \xi} \tilde{x}^*.\tilde{x} \ .$$

Since in this limit we may suppose that $\|\tilde{x}^*\| \le \|\xi\|$, we have

$$\|\xi.\tilde{x}\| \le \|\xi\| \, \|\tilde{x}\| \ .$$

b), c), d): These formulas are true when $\xi \in \widetilde{X^*}$, and their two members are τ-continuous functions of ξ (resp. τ to norm continuous in the case c),d)). Note that the map $\tilde{X}^* \to \tilde{X}^*_+$, $\xi \mapsto |\xi|$ is τ-continuous. □

2. The action of $L_\infty(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$ on $\tilde{X}$

We define now an action of $L_\infty(\tilde{\Omega})$ on $\tilde{X}$: for every $f \in L_\infty(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$ and $\tilde{x} \in \tilde{X}$, let $f.\tilde{x}$ be the element of $\tilde{X}^{**}$ defined by:

$$\forall \, \xi \in \tilde{X}^*, \langle f.\tilde{x}, \xi \rangle = \langle f, \xi.\tilde{x} \rangle \, .$$

We have:

$$|\langle f.\tilde{x}, \xi \rangle| \le \|f\|_\infty \, \|\xi.\tilde{x}\|_1 = \|f\|_\infty \, \langle |\xi| \, , |\tilde{x}| \rangle$$

hence $f.\tilde{x}$ is a τ-continuous linear form, i. e. belongs to $\tilde{X}$.

Lemma 2

a) The map $(f, \tilde{x}) \mapsto f.\tilde{x}$, $L_\infty(\tilde{\Omega}) \times \tilde{X} \to \tilde{X}$ is a bounded bilinear map with norm one, which extends the bilinear map $\tilde{\mathcal{Z}} \times \tilde{X} \to \tilde{X}$, $(\tilde{h}, \tilde{x}) \mapsto \tilde{h}.\tilde{x}$.
b) For every $f, g \in L_\infty(\tilde{\Omega})$ and $\tilde{x} \in \tilde{X}$, $f.(g.\tilde{x}) = (f.g).\tilde{x}$.
c) For every $f \in L_\infty(\tilde{\Omega})$ and $\tilde{x} \in \tilde{X}$, $|f.\tilde{x}| = |f| \, . \, |\tilde{x}|$.
d) The map $(f, \tilde{x}) \mapsto f.\tilde{x}$ is continuous on the product of unit balls $B(L_\infty(\tilde{\Omega})) \times B(\tilde{X})$ for the topologies $\tau(L_\infty(\tilde{\Omega}), L_1(\tilde{\Omega}))$ and $\tau(\tilde{X}, \tilde{X}^*)$.

Proof. a) is clear (the last sentence is a consequence of Lemma 1 (d)).

d): We observe first that, for every $f \in L_\infty(\tilde{\Omega})$ and $\tilde{x} \in \tilde{X}$, we have $|f.\tilde{x}| \leq |f|\,|\tilde{x}|$, since, for every $\xi \in \tilde{X}^*$, we have (using Lemma 1 (c)):

$$|\langle \xi, f.\tilde{x}\rangle| = |\langle \xi.\tilde{x}, f\rangle| \leq \langle |\xi.\tilde{x}|\,, |f|\rangle = \langle |\xi|\,.\,|\tilde{x}|\,, |f|\rangle = \langle |\xi|\,, |f|\,.\,|\tilde{x}|\rangle\,.$$

Now if $f, g \in B(L_\infty(\tilde{\Omega}))$ and $\tilde{x}, \tilde{y} \in B(\tilde{X})$, we have:

$$\begin{aligned}|f.\tilde{x} - g.\tilde{y}| &\leq |(f-g).\tilde{x}| + |g.(\tilde{x}-\tilde{y})|\\ &\leq |f-g|\,.\,|\tilde{x}| + |g|\,.\,|\tilde{x}-\tilde{y}| \leq |f-g|\,.\,|\tilde{x}| + |\tilde{x}-\tilde{y}|\end{aligned}$$

since $|g| \leq \mathbb{1}$, and $\mathbb{1}.\tilde{z} = \tilde{z}$ for every $\tilde{z} \in \tilde{X}$ by a). Hence, for every $\xi \in \tilde{X}^*_+$:

$$\langle \xi, |f.\tilde{x} - g.\tilde{y}|\rangle \leq \langle \xi, |f-g|\,.\,|\tilde{x}|\rangle + \langle \xi, |\tilde{x}-\tilde{y}|\rangle = \langle |\tilde{x}|\,.\xi, |f-g|\rangle + \langle \xi, |\tilde{x}-\tilde{y}|\rangle$$

which goes to zero when $g \to f$ and $\tilde{y} \to \tilde{x}$ for the τ-topologies.

b): The equality is true when $f, g \in \tilde{\mathcal{Z}}$, we extend it by using the τ-density of $B(\tilde{Z})$ in $B(L_\infty(\tilde{\Omega}))$ and the point d).

c): We showed above that $|f.\tilde{x}| \leq |f|\,.\,|\tilde{x}|$. Conversely, let $\varepsilon \in L_\infty(\tilde{\Omega})$ and $\tilde{h} \in \tilde{\mathcal{Z}}$, with $|\varepsilon| = \mathbb{1} = \left|\tilde{h}\right|$, $\varepsilon.f = |f|$ and $\tilde{h}.\tilde{x} = |\tilde{x}|$. Then, using the point b):

$$|f|\,.\,|\tilde{x}| = (\varepsilon.f).(\tilde{h}.\tilde{x}) = (\varepsilon.\tilde{h}).(f.\tilde{x}) \leq \left|\varepsilon.\tilde{h}\right|.\,|f.\tilde{x}| = |f.\tilde{x}|\,.\;\square$$

Lemma 3

For every $\tilde{x} \in \tilde{X}$, $\xi \in \tilde{X}^*$, and $f \in L_\infty(\tilde{\Omega})$, we have: $\xi.(f.\tilde{x}) = f.(\xi.\tilde{x})$.

Proof. For every $g \in L_\infty(\tilde{\Omega})$, we have, using Lemma 2 (b):

$$\langle \xi.(f.\tilde{x}), g\rangle \overset{\text{def}}{=} \langle \xi, g.(f.\tilde{x})\rangle = \langle \xi, (g.f).\tilde{x}\rangle \overset{\text{def}}{=} \langle \xi.\tilde{x}, g.f\rangle = \langle f.(\xi.\tilde{x}), g\rangle.\;\square$$

3. The support of an element of $\tilde{X}$

Definition 4. We call support of an element $\tilde{x} \in \tilde{X}$ (and we denote by $\sigma_{\tilde{x}}$) the g. l. b. of the set of idempotents of $L_\infty(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$ leaving $\tilde{x}$ invariant:

$$\sigma_{\tilde{x}} = \bigwedge \left\{e \in L_\infty(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu}) / \, e^2 = e; e.\tilde{x} = \tilde{x}\right\}.$$

The idempotent $\sigma_{\tilde{x}}$ is the indicator function of some element $\mathrm{Supp}\,(\tilde{x})$ of $\tilde{\mathcal{A}}$, which we call also support of $\tilde{x}$.

Remark 5. We have $\sigma_{\tilde{x}}.\tilde{x} = \tilde{x}$.

For, the set $\mathcal{E}_{\tilde{x}} = \{e \in L_\infty(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu}) / \, e^2 = e; e.\tilde{x} = \tilde{x}\}$ is downwards directed (as a consequence of Lemma 2 (b)) and τ-closed (by Lemma 2 (d)), hence contains its g. l. b..

Remark 6. If $|\tilde{z}| \leq |\tilde{x}|$, then $\sigma_{\tilde{z}} \leq \sigma_{\tilde{x}}$. In particular, the elements $\tilde{x}$ and $|\tilde{x}|$ have the same support.

For, there exists $\tilde{h} \in \tilde{\mathcal{Z}}$ such that $\tilde{z} = \tilde{h}.\tilde{x}$ (choose representing families $(z_i)_i$ and (x_i) for $\tilde{z}$ and $\tilde{x}$ respectively, with $|z_i| \leq |x_i|$, and $h_i \in \mathcal{Z}$ with $|h_i| \leq \mathbb{1}$, such that $z_i = h_i.x_i$, and set $\tilde{h} = (h_i)_i^\bullet)$. Then $\sigma_{\tilde{x}}.\tilde{z} = \sigma_{\tilde{x}}.(\tilde{h}.\tilde{x}) = \tilde{h}.(\sigma_{\tilde{x}}.\tilde{x}) = \tilde{h}.\tilde{x} = \tilde{z}$.

Lemma 7

Two elements of $\tilde{X}$ are disjoint iff they have disjoint supports.

Proof. a) Suppose that $\tilde{x} \perp \tilde{y}$. Then there exist idempotents $\tilde{h}, \tilde{k}$ of $\tilde{\mathcal{Z}}$, such that $\tilde{h} \perp \tilde{k}$ and $\tilde{h}.\tilde{x} = \tilde{x}$, $\tilde{k}.\tilde{y} = \tilde{y}$: choose representing families $(x_i)_i$ and $(y_i)_i$ for $\tilde{x}$, resp. $\tilde{y}$, with $x_i \perp y_i$ for all $i \in I$, let A_i and B_i be the supports of x_i, resp. y_i, set $h_i = \mathbb{1}_{A_i}, k_i = \mathbb{1}_{B_i}$ and finally $\tilde{h} = (h_i)_i^\bullet$, $\tilde{k} = (k_i)_i^\bullet$. Since $\sigma_{\tilde{x}} \leq \tilde{h}$ and $\sigma_{\tilde{y}} \leq \tilde{k}$, we have $\sigma_{\tilde{x}} \perp \sigma_{\tilde{y}}$.

b) Conversely, if $\sigma_{\tilde{x}} \perp \sigma_{\tilde{y}}$, then by Remark 6, for every $\tilde{z}$ with $0 \leq \tilde{z} \leq |\tilde{x}| \wedge |\tilde{y}|$, we have $\sigma_{\tilde{z}} = 0$, hence $\tilde{z} = 0$. □

Lemma 8

An idempotent $e \in L_\infty(\tilde{\Omega}, \tilde{a}, \tilde{\mu})$ is the support of an element $\tilde{x} \in \tilde{X}$ iff it is sigma-finite.

Proof. a) If $\tilde{x} \in \tilde{X}$ and $\sigma_{\tilde{x}}$ is not sigma-finite, we have $\sigma_{\tilde{x}} = \sum_\alpha e_\alpha$, where $(e_\alpha)_\alpha$ is an uncountable family of mutually disjoint, non null idempotents of $L_\infty(\Omega)$; then $\tilde{x} = \sum_\alpha e_\alpha \tilde{x}$, with $e_\alpha.\tilde{x} \neq 0$ for every α (by minimality of $\sigma_{\tilde{x}}$) and these elements are pairwise disjoint: this is impossible in a q-concave Banach lattice.

b) If e is a non null sigma-finite element of $L_\infty(\tilde{\Omega})$, there exists an element $\tilde{u} \in L_1(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$ whose support is e. There are $\tilde{x} \in \tilde{X}$ and $\tilde{x}^* \in \tilde{X}^*$ such that $\tilde{x}^*.\tilde{x} = \tilde{u}$: for, let $(u_i)_i$ be a representing family for $\tilde{u}$; using Lozanovskii's factorization theorem ([8]) we may write $u_i = x_i^*.x_i$, with $x_i^* \in X^*$, $x_i \in X$ and $\|x_i^*\| = \|x_i\| = \sqrt{\|u_i\|}$; set $\tilde{x}^* = (x_i^*)_i^\bullet$ and $\tilde{x} = (x_i)_i^\bullet$. Since $\sigma_{\tilde{x}}.\tilde{u} = \tilde{x}^*.(\sigma_{\tilde{x}}.\tilde{x}) = \tilde{x}^*.\tilde{x} = \tilde{u}$ we have $e \leq \sigma_{\tilde{x}}$; set $\tilde{y} = e.\tilde{x}$, then $\sigma_{\tilde{y}} = e$. □

Lemma 9

If $\tilde{x}, \tilde{y} \in \tilde{X}$ verify $|\tilde{y}| \leq \tilde{x}$, there is a unique $f \in L_\infty(\tilde{\Omega})$ with $\mathrm{Supp}\, f \subset \mathrm{Supp}\, \tilde{x}$ *and $\tilde{y} = f.\tilde{x}$.*

Proof. a) Choose representing families $(x_i)_i$, $(y_i)_i$ for $\tilde{x}$, resp. $\tilde{y}$, with $|y_i| \leq x_i$, for every $i \in I$; then choose for every i a $h_i \in L_\infty(\Omega)$ with $|h_i| \leq \mathbb{1}$ and $y_i = h_i.x_i$. We have clearly $\tilde{y} = \tilde{h}.\tilde{x}$. Then set $f = \sigma_{\tilde{x}}.\tilde{h}$.

b) Suppose that $g \in L_\infty(\tilde{\Omega})$ verifies $g.\tilde{x} = 0$. Then $|g|.\tilde{x} = 0$, hence, for every $n \geq 0$, $(\mathbb{1} \wedge n|g|).\tilde{x} = 0$. Set $\sigma_g = \mathbb{1}_{\mathrm{Supp}\, g}$. We have $\mathbb{1} \wedge n|g| \uparrow \sigma_g$ (in order, hence for the $\tau(L_\infty, L_1)$ topology), hence by lemma 2 (d), we see that $\sigma_g.\tilde{x} = 0$, which means that $\sigma_g \perp \sigma_{\tilde{x}}$. If $\sigma_g \leq \sigma_{\tilde{x}}$, we deduce that $g = 0$. □

4. The representation of $\tilde{X}$ as Köthe function space over $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$

At this stage it is easy to identify $\tilde{\mathcal{A}}_{\tilde{\mu}}$ (the quotient of $\tilde{\mathcal{A}}$ by the ideal of $\tilde{\mu}$-null sets) with the Boole algebra of band projections of $\tilde{X}$, and using [9], th. 1.b.14, to represent $\tilde{X}$ as a Köthe function space over $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$. We shall be more explicit by giving directly a Riesz homomorphism of $\tilde{X}$ onto an order ideal of $L_0(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$.

Theorem 10

There is an injective, order continuous Riesz homomorphism from $\tilde{X}$ onto an order ideal $\hat{X}$ of $L_0(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$, which commutes with the action of $L_\infty(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$. The support of $\hat{X}$ equals $\tilde{\Omega}$. Two such homomorphisms coincide up to the multiplication by a constant, a. e. positive element of $L_0(\tilde{\Omega})$.

Proof. Let $(e_\alpha)_{\alpha \in A}$ be a maximal system of mutually disjoint sigma-finite, non null idempotents of $L_\infty(\tilde{\Omega})$; we have $\bigvee_\alpha e_\alpha = \mathbb{1}$. For each $\alpha \in A$, let $\tilde{x}_\alpha$ be a nonnegative element of $\tilde{X}$ with support e_α (Lemma 8). Then $(\tilde{x}_\alpha)_\alpha$ is a system of mutually disjoint elements of $\tilde{X}$, and is maximal by Lemma 7. Let $\mathcal{I}$ be the (non closed) order ideal generated in the Banach lattice $\tilde{X}$ by the family $(\tilde{x}_\alpha)_\alpha$. Note that $\mathcal{I}$ is σ-order dense in $\tilde{X}$. More specifically, every element $\tilde{x}$ of $\tilde{X}_+$ can be written $\tilde{x} = \sum_k \tilde{y}_k$, for some sequence of mutually disjoint elements $\tilde{y}_k$ of $\mathcal{I}_+$ (for, by a standard exhaustion argument, it is sufficient to prove that $\tilde{x} = \tilde{y} + \tilde{z}$, with a non zero $\tilde{y} \in \mathcal{I}_+$ disjoint from $\tilde{z}$; it suffices then to choose α and $n \in \mathbb{N}$ such that $\tilde{u}_{\alpha,n} := (n\tilde{x}_\alpha - \tilde{x})_+ \neq 0$, and let $\tilde{y}$ be the projection of $\tilde{x}$ on the band generated by $\tilde{u}_{\alpha,n}$). By Lemma 9, every $\tilde{x} \in \mathcal{I}$ can be written uniquely:

$$\tilde{x} = \sum_\alpha f_\alpha.\tilde{x}_\alpha$$

where $f_\alpha \in L_\infty(\tilde{\Omega})$, $\sigma_{f_\alpha} \leq e_\alpha$ and the family $(f_\alpha)_\alpha$ has only finitely many nonzero elements. We set

$$\pi : \mathcal{I} \to L_\infty(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu}) \qquad \tilde{x} \mapsto \pi(\tilde{x}) = \sum_\alpha f_\alpha.e_\alpha$$

Then π is an injective Riesz homomorphism, which is ordercontinuous (if the net $\tilde{x}_\beta \downarrow 0$ then $\pi(\tilde{x}_\beta) \downarrow 0$). For every $\tilde{x} \in \tilde{X}_+$, set:

$$\pi(\tilde{x}) = \bigvee \{\pi(\tilde{y}) / \, \tilde{y} \in \mathcal{I}_+, \ 0 \le \tilde{y} \le \tilde{x}\}$$

where the supremum at the right hand is meant a priori in the space $L_0(\tilde{\Omega}; \overline{\mathbb{R}}_+)$; in fact it belongs to $L_0(\tilde{\Omega}; \mathbb{R}_+)$; for, let us write $\tilde{x} = \sum_k \tilde{y}_k$, for some sequence $(\tilde{y}_k)$ of mutually disjoint elements of $\mathcal{I}_+$; then, by order continuity of π, we have $\pi(\tilde{x}) = \sum_k \pi(\tilde{y}_k)$, and $(\pi(\tilde{y}_k))_k$ is a family of mutually disjoint elements of $L_\infty(\tilde{\Omega})$.

Finally, since π is clearly additive over $\tilde{X}_+$, it extends to $\tilde{X}$ by setting $\pi(\tilde{x}) = \pi(\tilde{x}_+) - \pi(\tilde{x}_-)$, where $\tilde{x}_+$ and $\tilde{x}_-$ are the positive and negative parts of $\tilde{x}$: we obtain thus the desired Riesz homomorphism. From its very definition, it is clear that $\pi(f.\tilde{x}) = f.\pi(\tilde{x})$ for every $f \in L_\infty(\tilde{\Omega})$ and $\tilde{x} \in \tilde{X}$.

Conversely, any such homomorphism conserves clearly the support. If π' is another such homomorphism, and $(\tilde{x}_\alpha)$ is the maximal family of nonzero nonnegative, mutually disjoint elements of $\tilde{X}$ used in the definition of π, let $\varphi_\alpha = \pi'(\tilde{x}_\alpha)$. These elements are disjoint, and their supports coincide with the e_α's. Set $\varphi = \sum_\alpha \varphi_\alpha$, it is easy to see that $\pi'(\tilde{x}) = \varphi.\pi(\tilde{x})$ for every $\tilde{x} \in \tilde{X}$. $\square$

Remark 11. Since $\tilde{X}$ is order continuous, the dual $\tilde{X}^*$ identifies with the Köthe dual $\hat{X}'$ of the Köthe function space $\hat{X}$. Then the bilinear map $\tilde{X}^* \times \tilde{X} \to L_1(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$ identifies with the pointwise multiplication $\hat{X}' \times \hat{X} \to L_1(\tilde{\Omega})$.

For, let $\pi_* : \tilde{X}^* \to L_0(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$ be the Riesz homomorphism resulting from the identification of $\tilde{X}^*$ and $\hat{X}'$. For every $\xi \in \tilde{X}^*, \tilde{x} \in \tilde{X}$ and $f \in L_\infty(\tilde{\Omega})$, we have:

$$\langle \xi.\tilde{x}, f\rangle = \langle \xi, f.\tilde{x}\rangle = \int_{\tilde{\Omega}} \pi_*(\xi).\pi(f.\tilde{x})\, d\tilde{\mu} = \int_{\tilde{\Omega}} \pi_*(\xi).\pi(\tilde{x}).f\, d\tilde{\mu}$$

whence $\xi.\tilde{x} = \pi_*(\xi).\pi(\tilde{x})$.

5. The uniform homeomorphism between the unit balls of two Köthe function spaces with non trivial concavity

Theorem 12 ([10], [1])

Let X be a Köthe function space over the measure space $(\Omega, \mathcal{A}, \mu)$, with non trivial concavity. There is a uniform homeomorphism from the unit sphere of X onto the unit sphere of $L_1(\Omega, \mathcal{A}, \mu)$ which preserves supports.

Proof. We may suppose, as in [1], that X is, say, 2-convex (since the Mazur map $x \to x^2$ maps uniformly homeomorphically the unit sphere of the 2-convexified space $X^{(2)}$ onto that of X) and that the 2-convexity and q-concavity constants of X equal one, hence that X is uniformly convex and uniformly smooth.

If $x \in S(X)$, let $J(x) \in S(X^*)$ be the unique normalized norming functional for x. Since X is order continuous, $J(x)$ is an element of the Köthe dual X', and $G(x) = |J(x)|.x$ an element of $L_1(\Omega, \mathcal{A}, \mu)$, in fact $G(x)$ belongs to the unit sphere of $L_1(\Omega)$. The map $x \mapsto J(x)$ is a norm-norm uniformly continuous map $S(X) \to S(X^*)$, ([5], p. 36), and so is G: $S(X) \mapsto S(L_1)$.

Conversely, for every $u \in S(L_1(\Omega))$ there are $x \in X, x^* \in X^*_+$ such that $u = x^*.x$, and $\|x\| = \|x^*\| = 1$ (by [6], th. 1; note that X has Fatou property, and that the hypothesis of sigma-finiteness of the measure is unnecessary). Moreover this factorization is unique under the supplementary condition that $\operatorname{Supp}(x) = \operatorname{Supp}(x^*) = \operatorname{Supp}(u)$ ([6], th. 3). (Note that in the case where X is uniformly convex and uniformly smooth, this condition is automatically verified). Then necessarily $G(x) = u$, i. e. $G^{-1} : S(L_1) \to S(X)$ is well defined.

Now let us show that G^{-1} is uniformly continuous. If not, there are a real number $\varepsilon > 0$ and sequences $(u_n)_n$ and $(v_n)_n$ in $S(L_1(\Omega))$ such that $\|u_n - v_n\| \to 0$ but $\left\|G^{-1}(u_n) - G^{-1}(v_n)\right\| \geq \varepsilon$. Let $x_n = G^{-1}(u_n)$ and $y_n = G^{-1}(v_n)$. Let $\mathcal{U}$ be a non trivial ultrafilter over $\mathbb{N}$, and consider in $\tilde{X} = X^{\mathbb{N}}/\mathcal{U}$ the points $\tilde{x} = (x_n)^\bullet_n$ and $\tilde{y} = (y_n)^\bullet_n$, and in $L_1(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu}) = L_1(\Omega)^{\mathbb{N}}/\mathcal{U}$ the point $\tilde{u} = (u_n)^\bullet_n = (v_n)^\bullet_n$. Consider also in $\tilde{X}^*$ the points $\tilde{x}^* = (J(x_n))^\bullet_n$ and $\tilde{y}^* = (J(y_n))^\bullet_n$. All these points have norm one, in their respective spaces. We have $\tilde{x}^*.\tilde{x} = \tilde{u} = \tilde{y}^*.\tilde{y}$. Since $\tilde{X}$ and $\tilde{X}^*$ can be represented as dual Köthe function spaces over $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$ in such a way that the pairing $\tilde{X}^* \times \tilde{X} \to L_1(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mu})$ is just the pointwise multiplication, we see that $\tilde{x} = G^{-1}(\tilde{u}) = \tilde{y}$, which contradicts the fact that $\|\tilde{x} - \tilde{y}\| \geq \varepsilon$. □

References

1. F. Chaatit, On Uniform Homeomorphisms of the Unit Spheres of certain Banach Lattices, *Pacific J. Math.* **168** (1995), 11–31.
2. D. Dacunha-Castelle and J.-L. Krivine, Application des ultraproduits à l'étude des espaces et algèbres de Banach, *Studia Math.* **41** (1972), 315–334.
3. M. Daher, Homéomorphismes uniformes entre les sphères unité des espaces d'interpolation, *Canad. Math. Bul.* **38** (1995), 286–294.
4. M. Daher, Homéomorphismes uniformes entre les sphères unité des espaces d'interpolation, *C. R. Acad. Sci. Paris, Sér. I Math.* **316** (1993), 1051–1054.
5. J. Diestel, *Geometry of Banach Spaces-Selected Topics,* Lecture Notes in Math. n° 485, Springer Verlag 1975.

6. T. A. Gillespie, Factorization in Banach function spaces, *Indag. Math.* **43** (1981), 287–300.
7. S. Heinrich, Ultraproducts in Banach Spaces Theory, *J. Reine Angew. Math.* **313** (1980), 72–104.
8. G. Ya. Lozanovskii, On some Banach lattices, *Siberian Math. J.* **10** (1969), 419–431.
9. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II: Function Spaces,* Ergebnisse der Math. n^{o} 97, Springer Verlag 1979.
10. E. Odell and T. Schlumprecht, The distortion problem, *Acta Math.* **173** (1994), 259–281.
11. Y. Raynaud, Ultrapowers of Calderón-Lozanovskii interpolation spaces, *Indag. Math.* to appear.
12. B. Sims, Ultra-techniques in Banach Space Theory, *Queen's Papers in Pure and Appl. Math.* **60** 1982.
13. P. Meyer-Nieberg, *Banach Lattices,* Springer Verlag, 1991.