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On the relationships between $H^p(\mathbb{T}, X/Y)$ and $H^p(\mathbb{T}, X)/H^p(\mathbb{T}, Y)^*$

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Abstract

First we show that for every $1 \leq p < \infty$ the space $H^p(\mathbb{T}, L^1(\lambda)/H^1)$ cannot be naturally identified with $H^p(\mathbb{T}, L^1(\lambda))/H^p(\mathbb{T}, H^1)$. Next we show that if Y is a closed locally complemented subspace of a complex Banach space X and $0 < p < \infty$, then the space $H^p(\mathbb{T}, X/Y)$ is isomorphic to the quotient space $H^p(\mathbb{T}, X)/H^p(\mathbb{T}, Y)$. This allows us to show that all odd duals of the James Tree space JT_2 have the analytic Radon-Nikodym property.

As usual, $\mathbb{D}$ and $\mathbb{T}$ will stand for the open unit disk and the unit circle in the complex plain $\mathbb{C}$. The normalized Lebesgue measure on $\mathbb{T}$ will be denoted by λ. Throughout, X will be a complex Banach space and Y its closed subspace. If Z is a Banach space, then $\|\cdot\|_Z$ denotes its norm, B_Z its closed unit ball and Z^* its topological dual. For $0 < p \leq \infty$, we denote by $h^p(\mathbb{D}, X)$ the space of all harmonic functions $F : \mathbb{D} \to X$ such that

$$\|F\| = \left(\sup_{0<r<1} \int_{\mathbb{T}} \|F(rt)\|^p \, d\lambda(t) \right)^{1/p} < \infty \qquad \text{if} \quad 0 < p < \infty$$
$$\|F\| = \sup \big\{ \|F(z)\| : z \in \mathbb{D} \big\} < \infty \qquad \text{if} \quad p = \infty \,,$$

equipped with the norm (when $1 \leq p \leq \infty$) or quasi-norm (when $0 < p < 1$) $\|\cdot\|$ defined above. The subspace of $h^p(\mathbb{D}, X)$ consisting of all holomorphic functions is

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denoted $H^p(\mathbb{D}, X)$. The spaces $h^p(\mathbb{D}, \mathbb{C})$ and $H^p(\mathbb{D}, \mathbb{C})$ are denoted briefly by h^p and H^p. The spaces $h^p(\mathbb{D}, X)$ and $H^p(\mathbb{D}, X)$ are Banach for $1 \le p \le \infty$, and p-Banach for $0 < p < 1$. For $0 < p < \infty$, we denote by $h^p(\mathbb{T}, X)$ the closure of the set of all harmonic polynomials in $h^p(\mathbb{D}, X)$, and by $H^p(\mathbb{T}, X)$ the closure of the set of all polynomials in $H^p(\mathbb{D}, X)$. Every function in $h^p(\mathbb{T}, X)$ as well as in $H^r(\mathbb{T}, X)$ for $1 \le p < \infty$ and $0 < r < \infty$ has radial limits λ-a.e. on $\mathbb{T}$. Furthermore, for $1 \le p < \infty$ every function in $h^p(\mathbb{T}, X)$ is representable as the Poisson integral of its boundary function (defined by radial limits) that belongs to $L^p(\lambda, X)$. Boundary functions of members of $H^p(\mathbb{T}, X)$ for $1 \le p < \infty$ belong to the set $\{f \in L^p(\lambda, X) : \hat{f}(n) = 0 \text{ for } n \in \mathbb{Z}_-\}$, where $\mathbb{Z}_-$ denotes the set of all negative integers.

In order to answer the question for which Banach spaces X the class $H^p(\mathbb{D}, X)$ coincides with $H^p(\mathbb{T}, X)$, Buchvalov and Danilevič introduced the notion of the analytic Radon-Nikodym property. Recall that a complex Banach space X has the *analytic Radon-Nikodym property* (aRNP) if every bounded holomorphic function $F : \mathbb{D} \to X$ has radial limits λ-a.e. on $\mathbb{T}$. By [5, Prop. 2] a complex Banach space X has the aRNP if and only if $H^p(\mathbb{T}, X) = H^p(\mathbb{D}, X)$ for all $0 < p < \infty$.

For $0 < p \le \infty$ and a given complex Banach space Z, with every bounded linear operator $S : X \to Z$ we associate the operator

$$\tilde{S}_p : h^p(\mathbb{D}, X) \to h^p(\mathbb{D}, Z)\,,$$

where $\tilde{S}_p(F) = S \circ F$. In this paper we are going to examine properties of the operator $\tilde{Q}_p$, associated with the quotient map $Q : X \to X/Y$. Applying the representation of any function in $h^p(\mathbb{T}, X)$ with the aid of the Poisson integral of some function from $L^p(\lambda, X)$ we easily see that

$$B_{h^p(\mathbb{T}, X/Y)} \subset 2\,\tilde{Q}_p(B_{h^p(\mathbb{T}, X)}),$$

hence $\tilde{Q}_p(h^p(\mathbb{T}, X)) = h^p(\mathbb{T}, X/Y)$ for $1 \le p < \infty$. The following example shows that the equality $\tilde{Q}_p(H^p(\mathbb{T}, X)) = H^p(\mathbb{T}, X/Y)$ does not always hold.

EXAMPLE 1: Let $Q : L^1(\lambda) \to L^1(\lambda)/H^1$ be the quotient map (here we identify H^1 with the subspace of $L^1(\lambda)$ consisting of functions f such that $\hat{f}(n) = 0$ for $n \in \mathbb{Z}_-$). We are going to show that for any $1 \le p < \infty$ neither

$$\tilde{Q}_p\big(H^p(\mathbb{D}, L^1(\lambda))\big) \ne H^p\big(\mathbb{D}, L^1(\lambda)/H^1\big)$$

nor

$$\tilde{Q}_p\big(H^p(\mathbb{T}, L^1(\lambda))\big) \ne H^p\big(\mathbb{T}, L^1(\lambda)/H^1\big)\,.$$

The space $L^1(\lambda)$ as a Banach lattice without isomorphic copies of c_0 has the aRNP (see [5, Th. 1]), but the space $L^1(\lambda)/H^1$ does not have the aRNP. Hence for every $0 < p < \infty$ the space $H^p(\mathbb{D}, L^1(\lambda))$ is separable in contrast to $H^p(\mathbb{D}, L^1(\lambda)/H^1)$ (see [13, Cor. 5.9]). Hence

$$\tilde{Q}_p\big(H^p(\mathbb{D}, L^1(\lambda))\big) \neq H^p\big(\mathbb{D}, L^1(\lambda)/H^1\big)\,.$$

Consider now the second inequality. For $n \in \mathbb{Z}$ let $\varphi_n : \mathbb{T} \to \mathbb{C}$ be given by

$$\varphi_n(t) = t^n.$$

For every $n \in \mathbb{N}$ let $w_n : \mathbb{D} \to L^1(\lambda)$ be the polynomial of the form

$$w_n(z) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \cdot \varphi_{-k} \cdot z^k.$$

Since w_n is continuous on $\overline{\mathbb{D}}$,

$$\begin{aligned} \|\tilde{Q}_p(w_n)\|_{H^1(\mathbb{D}, L^1(\lambda)/H^1)} &= \int_{\mathbb{T}} \|\tilde{Q}_p(w_n)(t)\|_{L^1(\lambda)/H^1}\, d\lambda(t) \\ &\leq \int_{\mathbb{T}} \left\| \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \cdot \varphi_k \cdot t^{-k} \right\|_{L^1(\lambda)} d\lambda(t) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} |K_n(st^{-1})|\, d\lambda(s)\, d\lambda(t) = 1\,, \end{aligned}$$

where $K_n = \sum_{k=-n}^{n} \big(1 - \frac{|k|}{n+1}\big)\, \varphi_k$ is the n-th Fejer kernel. On the other hand for every polynomial u in $H^1(\mathbb{D}, H^1)$ of the form

$$u(z) = \sum_{k=0}^{l} a_k \cdot z^k,$$

where a_k for $k = 0, \ldots, l$ are scalar polynomials from H^1,

$$\|w_n - u\|_{H^1(\mathbb{D}, L^1(\lambda))} = \int_{\mathbb{T}} \|w_n(t) - u(t)\|_{L^1(\lambda)}\, d\lambda(t) = \int_{\mathbb{T}} \int_{\mathbb{T}} |v_s(t)|\, d\lambda(s)\, d\lambda(t),$$

where for every $s \in \mathbb{T}$ the polynomial v_s is given by

$$v_s(z) = w_n(z)(s) - u(z)(s) = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) \cdot \varphi_{-k}(s) \cdot z^k - \sum_{k=0}^{l} a_k(s) \cdot z^k.$$

By the Hardy inequality (see [7, p. 48]) we have

$$\int_{\mathbb{T}} |v_s(t)|\, d\lambda(t) \geq \frac{1}{\pi} \cdot \sum_{k=0}^{n} \frac{|\hat{v}_s(k)|}{k+1} = \frac{1}{\pi} \cdot \sum_{k=0}^{n} \frac{1}{k+1} \left| \left(1 - \frac{k}{n+1}\right) \cdot s^{-k} - a_k(s) \right|.$$

Moreover, for every $k \in \mathbb{N}$ we have

$$\int_{\mathbb{T}} \left| \left(1 - \frac{k}{n+1}\right) \cdot s^{-k} - a_k(s) \right| d\lambda(s) \geq \left(\left(1 - \frac{k}{n+1}\right) \cdot \varphi_{-k} - a_k \right)\widehat{\ }(-k) = 1 - \frac{k}{n+1}.$$

Applying the Fubini theorem we get

$$\|w_n - u\| \geq \frac{1}{\pi} \cdot \sum_{k=0}^{n} \frac{1}{k+1} \left(1 - \frac{k}{n+1}\right) > \frac{1}{\pi} \cdot \left(\sum_{k=1}^{n+1} \frac{1}{k} - 1 \right).$$

Since polynomials form a dense subset of H^1, for every polynomial h in $H^1(\mathbb{D}, H^1)$

$$\|w_n - h\| \geq \frac{1}{\pi} \cdot \left(\sum_{k=1}^{n+1} \frac{1}{k} - 1 \right).$$

Since the space H^1 possesses the aRNP, polynomials are dense in $H^1(\mathbb{D}, H^1) = H^1(\mathbb{T}, H^1)$. Hence

$$\lim_n \left\| w_n + H^1(\mathbb{T}, H^1) \right\|_{H^1(\mathbb{T}, L^1(\lambda))/H^1(\mathbb{T}, H^1)} = \infty\,.$$

As easily seen $\ker \tilde{Q}_1 = H^1(\mathbb{T}, H^1)$. If $Q_1(H^1(\mathbb{T}, L^1(\lambda)))$ were a closed subspace of $H^1(\mathbb{T}, L^1(\lambda)/H^1))$, then sequences $\big(\|\tilde{Q}_1(w_n)\|\big)$ and $\big(\|w_n + H^1(\mathbb{T}, H^1)\|\big)$ would be simultaneously bounded or unbounded. Therefore $\tilde{Q}_1(H^1(\mathbb{T}, L^1(\lambda)))$ is not a closed subspace of $H^1(\mathbb{T}, L^1(\lambda)/H^1)$.

If $s^{-1} + r^{-1} = 1$, $f \in H^s$ and $G \in H^r(\mathbb{D}, X)$, then $F = f \cdot G$ belongs to $H^1(\mathbb{D}, X)$ and $\tilde{Q}_p(F) = f \cdot Q \circ G$. Hence, applying the factorization theorem (see [14, Prop. 1.2] or [13, Th. 2.10]) we easily obtain

$$\begin{aligned} H^1\big(\mathbb{T}, L^1(\lambda)/H^1\big) &= \big\{ f \cdot G :\ f \in H^s, G \in H^r(\mathbb{T}, L^1(\lambda)/H^1) \big\} \\ &= H^s \cdot H^r\big(\mathbb{T}, L^1(\lambda)/H^1\big)\,. \end{aligned}$$

It follows that for every $1 \leq p < \infty$

$$\tilde{Q}_p\big(H^p(\mathbb{T}, L^1(\lambda))\big) \neq H^p\big(\mathbb{T}, L^1(\lambda)/H^1\big)\,.$$

Since $\tilde{Q}_p(H^p(\mathbb{T},L^1(\lambda)))$ contains all polynomials in $H^p(\mathbb{T},L^1(\lambda)/H^1)$, it is not closed.

Remark. Similar considerations allow us to show that $\tilde{Q}_p(H^p(\mathbb{T},ca(\mathbb{T})))$ is not a closed subspace of $H^p(\mathbb{T},ca(\mathbb{T})/H^1)$ for every $1 \le p < \infty$, where $ca(\mathbb{T})$ is the space of all complex Borel measures on $\mathbb{T}$.

Next remark was communicated to the author by 0. Blasco.

Remark. An alternative proof of the inequality

$$\tilde{Q}_1\big(H^1(\mathbb{T},L^1(\lambda))\big) \ne H^1\big(\mathbb{T},L^1(\lambda)/H^1\big)$$

follows from the Hardy property of the space $L^1(\lambda)$ which does not work for the space $L^1(\lambda)/H^1$ (see [2]). Recall, that a complex Banach space X has the *Hardy property* if

$$\sum_{n=1}^{\infty} \frac{\|\hat{F}(n)\|}{n} < \infty \qquad \text{for every} \quad F \in H^1(\mathbb{T},X)\,.$$

In the remaining part of the paper we are going to give some positive results.

For any $F : \mathbb{D} \to X$ and $0 < r < 1$, the function $W_r(F) : \mathbb{D} \to X$ is defined by

$$W_r(F)(z) = F(rz).$$

Clearly, for any F in $h^p(\mathbb{D},X)$ the function $W_r(F)$ belongs to $h^p(\mathbb{T},X)$ and $\|W_r(F)\| \le \|F\|$. If F belongs to $H^p(\mathbb{D},X)$, then of course $W_r(F)$ is in $H^p(\mathbb{T},X)$. In the sequel we will need the following analogue of a result of Drewnowski (see [6, Prop. 3]).

Proposition 2

Let Y be a closed subspace of X and let $0 < p < \infty$. Then the map

$$J : F + H^p(\mathbb{T},Y) \to F + H^p(\mathbb{T},X)$$

is an isomorphic embedding of the quotient space $H^p(\mathbb{D},Y)/H^p(\mathbb{T},Y)$ into the quotient space $H^p(\mathbb{D},X)/H^p(\mathbb{T},X)$.

Proof. Let $s = \min\{1, p\}$ and (r_n) be an increasing sequence of positive numbers converging to 1. Let us take any F in $H^p(\mathbb{D}, Y)$ and G in $H^p(\mathbb{T}, X)$. For every $n \in \mathbb{N}$ the function $W_{r_n}(G)$ belongs to $H^p(\mathbb{T}, X)$. Furthermore, by [14, Prop. 2.1], $\lim_n W_{r_n}(G) = G$ in $H^p(\mathbb{D}, X)$. Hence

$$\|F - G\| = \lim_n \|F - W_{r_n}(G)\|.$$

For every n we have

$$\begin{aligned}\|F - W_{r_n}(G)\|^s &\geq \|F - W_{r_n}(F)\|^s - \|W_{r_n}(G - F)\|^s \\ &\geq \|F - W_{r_n}(F)\|^s - \|G - F\|^s.\end{aligned}$$

Therefore

$$2 \cdot \|F - G\|^s \geq \limsup_n \|F - W_{r_n}(F)\|^s.$$

For every n the function $W_{r_n}(F)$ belongs to $H^p(\mathbb{T}, Y)$, so

$$2^{1/s}\|F + H^p(\mathbb{T}, X)\| \geq \|F + H^p(\mathbb{T}, Y)\| \geq \|F + H^p(\mathbb{T}, X)\|,$$

which completes the proof. □

For every $t \in \mathbb{T}$, the t-translate of a function $F : \mathbb{D} \to X$ is the function $F_t : \mathbb{D} \to X$ given by the formula

$$F_t(z) = F(zt^{-1}).$$

If $0 < p \leq \infty$ and F belongs to $H^p(\mathbb{D}, X)$, then also F_t belongs to $H^p(\mathbb{D}, X)$ and $\|F_t\| = \|F\|$. With every F in $H^p(\mathbb{D}, X)$ we associate a function $T_F : \mathbb{T} \to H^p(\mathbb{D}, X)$ defined by

$$T_F(t) = F_t.$$

The following proposition taken from [13, Th. 5.4] exhibits the relationships between properties of functions F and T_F.

Proposition 3

Let $0 < p < \infty$. Then for every $F \in H^p(\mathbb{D}, X)$ the following assertions are equivalent:

(a) $F \in H^p(\mathbb{T}, X)$,
(b) F has radial limits λ-a.e. on $\mathbb{T}$,
(c) $T_F : \mathbb{T} \to H^p(\mathbb{D}, X)$ is continuous,
(d) $T_F(\mathbb{T})$ is a separable subset of $H^p(\mathbb{D}, X)$,
(e) there exists a λ-measurable subset A of $\mathbb{T}$ such that $\lambda(A) > 0$ and $T_F(A)$ is a separable subset of $H^p(\mathbb{D}, X)$.

For a given Banach space Z denote by q_Z the quotient mapping of $H^p(\mathbb{D}, Z)$ onto $H^p(\mathbb{D}, Z)/H^p(\mathbb{T}, Z)$. The following corollary is a local version of the main result in [6] for Hardy classes of holomorphic functions.

Corollary 4

Let $0 < p < \infty$. If $F \in H^p(\mathbb{D}, X) \setminus H^p(\mathbb{T}, X)$, then $q_X \circ T_F(A)$ is a nonseparable subset of $H^p(\mathbb{D}, X)/H^p(\mathbb{T}, X)$ for every λ-measurable subset A of $\mathbb{T}$ such that $\lambda(A) > 0$.

Proof. Let A be a λ-measurable subset of $\mathbb{T}$ such that $\lambda(A) > 0$. The closed linear hull Y of $F(\mathbb{D})$ is separable and F belongs to $H^p(\mathbb{D}, Y)$. Since F does not belong to $H^p(\mathbb{T}, Y)$, in view of Proposition 3 $T_F(A)$ is a nonseparable subset of $H^p(\mathbb{D}, Y)$. Since the space $H^p(\mathbb{T}, Y)$ is separable, the set $q_Y(T_F(A))$ is also nonseparable. For every $t \in \mathbb{T}$,

$$J\big(q_Y(T_F(t))\big) = J\big(F_t + H^p(\mathbb{T}, Y)\big) = q_X\big(T_F(t)\big),$$

hence $q_X \circ T_F = J \circ q_Y \circ T_F$, where J is the the mapping defined in Proposition 2. Since J is an isomorphic embedding, the set $q_X(T_F(A))$ is nonseparable. □

Theorem 5

Let $0 < p < \infty$. Let Y be a closed subspace of a complex Banach space X and Q the quotient map of X onto X/Y. If Y has the aRNP and $\tilde{Q}_p(H^p(\mathbb{D}, X))$ contains $H^p(\mathbb{T}, X/Y)$, then $\tilde{Q}_p(H^p(\mathbb{T}, X)) = H^p(\mathbb{T}, X/Y)$.

Proof. Let $W = \tilde{Q}_p^{-1}(H^p(\mathbb{T}, X/Y))$. Consider the kernel of the mapping $\tilde{Q}_p$. If G belongs to $\ker \tilde{Q}_p$, then $Q(G(z)) = 0$ for every $z \in \mathbb{D}$. Hence G belongs to $H^p(\mathbb{D}, Y) = H^p(\mathbb{T}, Y)$ by the analytic Radon-Nikodym property of Y. Clearly, $\tilde{Q}_p$ generates an isomorphism I between $H^p(\mathbb{T}, X/Y)$ and the quotient space $W/H^p(\mathbb{T}, Y)$. Let $L : H^p(\mathbb{D}, X)/H^p(\mathbb{T}, Y) \to H^p(\mathbb{D}, X)/H^p(\mathbb{T}, X)$ be given by

$$L\big(F + H^p(\mathbb{T}, Y)\big) = F + H^p\big(\mathbb{T}, X\big).$$

It is easily seen that L is a continuous linear operator.

We show that W coincides with $H^p(\mathbb{T}, X)$. Let us take any F in W. Since $\tilde{Q}_p(F)$ belongs to $H^p(\mathbb{T}, X/Y)$, by Proposition 3 the function $T_{\tilde{Q}_p(F)}$ has separable range in $H^p(\mathbb{T}, X/Y)$. As easily checked

$$q_X \circ T_F = L \circ I \circ T_{\tilde{Q}_p(F)}.$$

Therefore $q_X \circ T_F(\mathbb{T})$ is a separable subset of $H^p(\mathbb{D},X)/H^p(\mathbb{T},X)$. In view of Corollary 4, F belongs to $H^p(\mathbb{T},X)$. Since F was taken arbitrary, $W = H^p(\mathbb{T},X)$, which completes the proof. □

Corollary 6

Let Y be a closed subspace of X and Q the quotient map of X onto X/Y. If Y has the aRNP and $\tilde{Q}_p(H^p(\mathbb{D},X)) = H^p(\mathbb{T},X/Y)$ for some $0 < p < \infty$, then X has the aRNP.

Proof. Let us take any F in $H^p(\mathbb{D},X)$. By Theorem 5 there is $\tilde{F}$ in $H^p(\mathbb{T},X)$ such that $\tilde{Q}_p(F) = \tilde{Q}_p(\tilde{F})$. Therefore $F - \tilde{F}$ belongs to $H^p(\mathbb{T},Y) = \ker \tilde{Q}_p$. Hence F belongs to $H^p(\mathbb{T},X)$. Since F was taken arbitrary, X possesses the analytic Radon-Nikodym property. □

In the sequel we will need the following sequence of operators. For $1 \le p < \infty$ and $n \in \mathbb{N}$ the operator $\mathbb{K}_n : h^p(\mathbb{D},X) \to h^p(\mathbb{D},X)$ is given by

$$\mathbb{K}_n(F)(z) = \sum_{k=-n}^{n} \hat{F}(k) \cdot \left(1 - \frac{|k|}{n+1}\right) \cdot |z|^{|k|} s^k \qquad \text{for } z = |z|s \in \mathbb{D}.$$

As easily checked the operators $\mathbb{K}_n$ have the following properties:

Fact 7. Let $1 \le p < \infty$ and $n \in \mathbb{N}$.

(1) $\mathbb{K}_n(h^p(\mathbb{D},X)) \subset h^p(\mathbb{T},X)$,
(2) $\mathbb{K}_n(H^p(\mathbb{D},X)) \subset H^p(\mathbb{T},X)$,
(3) $\|\mathbb{K}_n(F)\| \le \|F\|$ for every F in $h^p(\mathbb{D},X)$,
(4) $\lim_m \mathbb{K}_m(F) = F$ in $h^p(\mathbb{D},X)$ for every F in $h^p(\mathbb{T},X)$.

A closed subspace Y of a Banach space X is *locally complemented* if there exists a constant κ such that for every finite dimentional subspace Z in X and for every $\varepsilon > 0$ there is an linear operator $T : Z \to Y$ such that $\|T\| \le \kappa$ and $\|T(y) - y\| \le \varepsilon\|y\|$ for every $y \in Y \cap Z$ (see [10]).

Theorem 8

Let Y be a closed locally complemented subspace of a complex Banach space X and Q the quotient map of X onto X/Y, and let $0 < p < \infty$. Then

$$\tilde{Q}_p\big(H^p(\mathbb{T},X)\big) = H^p\big(\mathbb{T},X/Y\big).$$

Proof. Consider first the case $1 \le p < \infty$. Let us take any F in $H^p(\mathbb{T},X/Y)$. Let (α_n) be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\alpha_0 = \|F\|$.

We construct a sequence of polynomials (w_n) in $H^p(\mathbb{T},X/Y)$ and a sequence of harmonic polynomials (u_n) in $h^p(\mathbb{T},X)$ such that for every $n\in\mathbb{N}$ we have:

(1) $\tilde{Q}_p(u_n)=w_n$,
(2) $\|u_{n+1}\|_{h^p(\mathbb{D},X)}\le 2\cdot(\kappa+2)\alpha_n$,
(3) $\hat{u}_n(k)=0 \qquad \text{for } k\in\mathbb{Z}_-$,
(4) $\|F-\sum_{k=0}^n w_k\|_{H^p(\mathbb{D},X/Y)}\le\alpha_n$.

By (3), u_n belongs to $H^p(\mathbb{T},X)$ for every n. If we put $u=\sum_{n=1}^\infty u_n$ then, as easily seen, $\tilde{Q}_p(u)=F$.

It remains to select sequences (w_n) and (u_n). Let us put $w_0=0$, $u_0=0$. Suppose that we have been able to construct polynomials w_n and u_n for all $0\le n<k$ with properties (1)–(4). Let $F_k=F-\sum_{n=0}^{k-1}w_n$. Since $\lim_m\mathbb{K}_m(F_k)=F_k$ in $H^p(\mathbb{D},X/Y)$, there exists $n_k\in\mathbb{N}$ such that $\|F_k-\mathbb{K}_{n_k}(F_k)\|_{H^p(\mathbb{D},X/Y)}\le 2^{-1}\alpha_k$. Since $B_{h^p(\mathbb{T},X/Y)}\subset 2\,\tilde{Q}_p(B_{h^p(\mathbb{T},X)})$, there exists v_k in $h^p(\mathbb{T},X)$ such that $\tilde{Q}_p(v_k)=\mathbb{K}_{n_k}(F_k)$ and $\|v_k\|_{h^p(\mathbb{T},X)}\le 2\,\|F_k\|_{H^p(\mathbb{D},X/Y)}$. In view of Fact 7 there is $m_k\in\mathbb{N}$ such that

$$\big\|F_k-\mathbb{K}_{m_k}\big(\mathbb{K}_{n_k}(F_k)\big)\big\|_{H^p(\mathbb{D},X/Y)}\le\alpha_k.$$

It is clear that $\tilde{Q}_p(\mathbb{K}_{m_k}(v_k))=\mathbb{K}_{m_k}(\mathbb{K}_{n_k}(F_k))$. Let us put $w_k=\mathbb{K}_{m_k}(\mathbb{K}_{n_k}(F_k))$. The polynomial w_k has property (4). Let Z_k be the linear hull of $\mathbb{K}_{m_k}(v_k)(\mathbb{T})$. As easily seen $\dim Z_k\le 2m_k+1$. Since Y is a locally complemented subspace of X, for Z_k and $\varepsilon_k=m_k^{-1}$ there is a linear operator $T_k:Z_k\to Y$ such that $\|T_k\|\le\kappa$ and $\|T_k(y)-y\|\le\varepsilon_k\|y\|$ for every $y\in Z_k\cap Y$. Hence $T_k\circ\mathbb{K}_{m_k}(v_k)$ belongs to $h^p(\mathbb{T},Y)$ and $\|T_k\circ\mathbb{K}_{m_k}(v_k)\|_{h^p(\mathbb{T},X)}\le\kappa\cdot\|v_k\|_{h^p(\mathbb{T},X)}$. Define $r_k:\mathbb{D}\to Y$ by the formula

$$r_k(z)=\sum_{n\in\mathbb{Z}_-}\big(\mathbb{K}_{m_k}(v_k)-T_k\circ\mathbb{K}_{m_k}(v_k)\big)\widehat{\ }(n)\cdot|z|^{|n|}s^n \qquad \text{for } z=|z|s\in\mathbb{D}.$$

Since $\tilde{Q}_p(\mathbb{K}_{m_k}(v_k))$ belongs to $H^p(\mathbb{D},X/Y)$, $\mathbb{K}_{m_k}(v_k)\widehat{\ }(n)$ is a member of Y for every $n\in\mathbb{Z}_-$. Hence r_k takes its values in Y. For every $l\in\mathbb{Z}$, $(T_k\circ\mathbb{K}_{m_k}(v_k))\widehat{\ }(l)=T_k(\mathbb{K}_{m_k}(v_k)\widehat{\ }(l))$ and

$$\begin{aligned}\|\mathbb{K}_{m_k}(v_k)\widehat{\ }(l)\|&\le\|\mathbb{K}_{m_k}(v_k)\|_{h^p(\mathbb{T},X)}\le\|v_k\|_{h^p(\mathbb{T},X)}\\&\le 2\cdot\|F_k\|_{H^p(\mathbb{D},X/Y)}\le 2\cdot\alpha_{k-1}.\end{aligned}$$

Hence for every $n\in\mathbb{Z}_-$

$$\big\|\big(\mathbb{K}_{m_k}(v_k)-(T_k\circ(\mathbb{K}_{m_k}(v_k))\big)\widehat{\ }(n)\big\|\le 2\cdot m_k^{-1}\cdot\alpha_{k-1}.$$

Since at most m_k of the summands of r_k do not vanish, $\|r_k\|_{h^p(\mathbb{T},X)} \le 2\cdot\alpha_{k-1}$. Let us put

$$u_k = \mathbb{K}_{m_k}(v_k) - T_k{\circ}\mathbb{K}_{m_k}(v_k) - r_k.$$

It is clear that the polynomial u_k has properties (1) and (3), furthermore

$$\begin{aligned}\|u_k\|_{h^p(\mathbb{T},X)} &\le \|\mathbb{K}_{m_k}(v_k) - T_k{\circ}\mathbb{K}_{m_k}(v_k)\|_{h^p(\mathbb{T},X)} + \|r_k\|_{h^p(\mathbb{T},X)}\\ &\le (\kappa+1)\cdot\|\mathbb{K}_{m_k}(v_k)\|_{h^p(\mathbb{T},X)} + 2\cdot\alpha_{k-1} \le 2\cdot(\kappa+2)\cdot\alpha_{k-1},\end{aligned}$$

which completes the proof in the case $1 \le p < \infty$.

Consider now the case $0 < p < 1$. Let us take $n \in \mathbb{N}$ so that $2^n p > 1$. From the factorization theorem for Hardy classes of holomorphic functions it follows that for every $0 < s < \infty$

$$H^s(\mathbb{T},X) = H^{2s}\cdot H^{2s}(\mathbb{T},X).$$

Hence

$$\tilde{Q}_s\big(H^s(\mathbb{T},X)\big) = H^{2s}\cdot\tilde{Q}_{2s}\big(H^{2s}(\mathbb{T},X)\big).$$

Immediately from the above considerations and the first part of the proof it follows that

$$\tilde{Q}_p\big(H^p(\mathbb{T},X)\big) = H^{2p}\cdot H^{4p}\cdot\ldots\cdot\tilde{Q}_{2^n p}\big(H^{2^n p}(\mathbb{T},X)\big) = H^p\big(\mathbb{T},X/Y\big). \ \square$$

Applying Corollary 6 we get

Corollary 9

Let Y be a closed locally complemented subspace of X. If Y has the aRNP and $\tilde{Q}_p(H^p(\mathbb{D},X)) \subset H^p(\mathbb{T},X/Y)$ for some $0 < p < \infty$, then X has the aRNP.

By the principle of local reflexivity (see [12]), every Banach space X is locally complemented in X^{**}. Therefore, as a straightfoward consequence of the above theorem we get

Corollary 10

For $0 < p < \infty$ and any complex Banach space X

$$\tilde{Q}_p\big(H^p(\mathbb{T},X^{**})\big) = H^p\big(\mathbb{T},X^{**}/X\big)\,.$$

Applying Corollary 6 we get

Corollary 11

If the complex Banach spaces X and X^{**}/X have the aRNP, then so does X^{**}.

EXAMPLE 12: Let $T = \bigcup_{k=0}^{\infty}\{0,1\}^k$. The James Tree space JT_2 consists of all functions $x : T \to \mathbb{C}$ such that

$$\|x\|_{JT_2} = \sup\left(\sum_{i=1}^{n}\left|\sum_{t\in S_i} x(t)\right|^2\right)^{1/2} < \infty\,,$$

where the supremum is taken over all families $(S_1, \ldots, S_n)$ of pairwise disjoint finite intervals of the tree T. The unit vectors $(e_t)_{t\in T}$ form a boundedly complete basis of JT_2. Let $\mathbb{B}$ be the closed subspace of JT_2^* spanned by the biorthogonal functionals associated with $(e_t)_{t\in T}$. Then $\mathbb{B}^*$ is isomorphic to JT_2. The space $\mathbb{B}$ does not have the Radon-Nikodym property but, as was shown in [8, Cor. 1.4], it has the analytic Radon-Nikodym property. Since the space $\mathbb{B}^{**}/\mathbb{B}$ is isomorphic to $l_2(\mathbb{R})$ (see [11]), applying Corollary 11 we obtain

Corollary 13

The space JT_2^* has the aRNP.

By [11, Cor. 1], for every $n > 1$ the space $\mathbb{B}^{(2n)*}$ is isomorphic to $\mathbb{B}^{**} \oplus l_2(\mathbb{R})$. Therefore the last corollary implies immediately.

Corollary 14

All odd duals $JT_2^{(2n+1)*}$ of the James Tree space JT_2 have the aRNP.

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