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Monotonicity, order smoothness and duality for convex functionals

W. Kurc

Faculty of Mathematics and Computer Science, University of Poznań,
Matejki 48/49, 60-769 Poznań, Poland.
E-Mail: wkurc@math.amu.edu.pl

Abstract

In the paper concepts of pointwise and uniform strict monotonicity and order-smoothness for convex and monotone functionals on locally convex-solid Riesz spaces are studied.

1. Introduction

This paper presents a number of general results concerning relations between monotonicity properties and its dual counterparts for convex and monotone functionals on solid Riesz spaces. As a result new concept of (order) subdifferential and concepts of modules of uniform monotonicity and order smoothness for convex and monotone functionals are introduced. Our first step was made in [8] in the context of geometry of the unit sphere in Banach lattices. In the paper under consideration, however, a general convex functions are considered.

In Section 2 it is proved that the monotonicity of a convex functional f reflects the same properties for the Young conjugate f^* and f^{**}. In Section 3 new concepts of order directional derivative $f_\vee(x,y)$ and of order subderivative are studied. Theorems 3.6 and 3.7 are main results in this section. It follows that there exists some analogy between the role of convexity and monotonicity. In Section 4 an order smoothness is proved to be an appropriate dual notion to strict monotonicity of convex and monotone functionals (Theorem 4.3). Next, in Section 5, we introduce modules of uniform monotonicity $\delta_f(\epsilon)$ and of order smoothness $\rho_{f^*}(\tau)$. Theorem 5.1 shows that these modules are closely related via duality formulas. Moreover, as a consequence, in Section 6 we get dual relations between uniform monotonicity and uniform order smoothness of convex and monotone functionals. Finally, in Section 7 the modules $\delta_f(\epsilon)$, $\rho_{f^*}(\tau)$ are estimated for the functional $f(x) = \int_T |x(t)|^p d\mu$ $(1 \leq p < +\infty)$ and in Section 8 an application to an optimization problem is considered.

We refer to [6], [2], [5] and [4] for background material concerning convex analysis and to [1] for locally convex-solid Riesz spaces.

2. Definitions

Let X be a locally convex-solid Riesz space ([1]) with the dual X^*. Equivalently, X is a Riesz space endowed with a locally convex topology on X generated by a family of lattice seminorms. A seminorm $p(\cdot)$ on X is said to be a lattice (or a Riesz) seminorm if $|y| \leq |x|$ implies $p(y) \leq p(x)$ (cf. [1]). If X is endowed with a lattice norm $\|\cdot\|$ then X is called a normed Riesz space. Let X_+ denotes the positive cone in X. The monotonicity of the lattice norm can be splitted into the the property (a) $\|x\|=\||x|\|$ and the property (b) $\|y\|\leq\|x\|$ for $0 \leq y \leq x$. If $f : X \to \overline{\mathbb{R}}$ is any function then let $f^*(x^*)= \sup_{x\in X}\{< x, x^* > -f(x)\}$ ([4]). The functional f^* is the conjugate (polar) to f in the Young sense. Similarly $f^{**}(x)$ $= \sup_{x^*\in X^*}\{< x^*, x > -f^*(x)\}$ defines the second conjugate f^{**} to f. It is well known that f^* and f^{**} are lsc and convex functionals as the pointwise suprema of affine functionals. If $x^* \geq 0$ then $f^*(x^*) = \sup_{x\geq 0}\{< x, x^* > -f(x)\}$. In what follows we will be concerned with proper functionals. This means that $f > -\infty$ and dom(f) $= \{x \in X : f(x) < +\infty\} \neq \emptyset$. In general $f^{**} \leq f$, however, if f is lsc then $f^{**}= f$.

Recall, the subdifferential $\partial f(x)$ of f at $x \in$dom(f) is defined by $\partial f(x) = \{x^* \in X^* : \forall_{h\in X} < h, x^* > \leq f(x + h) - f(x)\}$. The set $\partial f(x)$ is always convex and w^*-closed subset in X^* (perhaps empty). If f is continuous at x than $\partial f(x)$ is non-empty and w^*-compact set. An element $x^* \in \partial f(x)$ if and only if $< x, x^* >=$ $f(x) + f^*(x^*)$, i.e. the equality in the Young inequality is attained. Hence it follows that $x \in \partial f^*(x^*)$. The inverse implication holds true whenever f is lsc. Next, let $\partial f(x, \cdot) = \{x^* \in X^* : \forall_{h\in X} < h, x^* >\leq f(x, h)\}$. Then $\partial f(x) = \partial f(x, \cdot)$, where $f(x, y) = \inf_{t>0}(f(x + ty) - f(x))/t$ denotes the usual directional derivative (we will introduce below a different concept of directional derivative $f_\vee(x, y)$). Clearly $\partial_+ f(x) = \partial_+ f(x, \cdot)$ where $\partial_+ f(x) = \partial f(x) \cap X^*_+$ and $\partial_+ f(x, \cdot) = \partial f(x, \cdot) \cap X^*_+$. By X^*_+ we denote the positive cone in X^*.

Let $f : X \to \overline{\mathbb{R}}$ be any function. We call f (order) monotone if $f(y) \leq f(x)$ whenever $|y| \leq |x|$. It is easy to see that if f is a monotone functional then, equivalently, (a) $f(x) = f(|x|)$ and (b) $f(y) \leq f(x)$ whenever $0 \leq y \leq x$.

DEFINITION 1. Let f be monotone. Then, f is said to be strictly monotone (STM, for short) if $f(y) < f(x)$ whenever $0 \leq y < x$ (i.e. $y \leq x$ and $y \neq x$).

From the lemma below it follows that the monotonicity inherits to the Young conjugates f^* and f^{**} of monotone functionals f.

Lemma 2.1

If $f : X \to \overline{\mathbb{R}}$ is monotone then f^* andf^{**} are also monotone functionals on X^* and X, respectively.

Proof. Let us consider the case of f^* only since the proof for f^{**} runs analogously. If $0 \leq y^* \leq x^*$ then

$$\begin{aligned} f^*(y^*) &= \sup_{x \in X}\{< x, y^* > -f(x)\} \\ &\leq \sup_{x \in X}\{< |x|, y^* > -f(x)\} \\ &\leq \sup_{x \geq 0}\{< x, x^* > -f(x)\} = f^*(x^*) \end{aligned}$$

hence the monotonicity for positive elements follows.

To prove that $f^*(x^*) = f^*(|x^*|)$ let us first note that from the definition of f^* we have $f^*(x^*) \leq f^*(|x^*|)$. Recall, that for $x \geq 0$ we have $< x, |x^*| >= \sup_{|z| \leq x} < z, x^* >$ (e.g.[9], p. 49). Therefore,

$$\begin{aligned} f^*(|x^*|) &= \sup_{x \geq 0}\{< x, |x^*| > -f(x)\} \\ &= \sup_{x \geq 0} \sup_{|z| \leq x} \{< z, x^* > -f(x)\} \\ &\leq \sup_{x \geq 0} \sup_{|z| \leq x} \{< z, x^* > -f(z)\} \\ &\leq \sup_{z \in X}\{< z, x^* > -f(z)\} = f^*(x^*). \end{aligned}$$

Thus $f^*(x^*) = f^*(|x^*|)$ and the proof is finished. □

3. Order directional derivatives and order subdifferentials

In what follows let X be a locally convex Riesz space and let $f : X \to \overline{\mathbb{R}}$ be a proper convex functional which is monotone.

DEFINITION 2. Let $x \in \mathrm{dom}(f)$. We define an order directional derivative $f_\vee(x,y)$ of f at $x \in X$ in the direction $y \in X$ by means of the following formula

$$f_\vee(x,y) = \lim_{t \searrow 0} \frac{f(|x| \vee t|y|) - f(x)}{t}$$

where $y \in X$.

This formula is well defined since (see below) the difference quotient is a monotone function of t. In our study of $f_\vee(x,y)$ we will be mainly concerned with the case $x, y \geq 0$, since $f_\vee(x,y) = f_\vee(|x|,|y|)$.

Our goal in this section is to relate the order directional derivative $f_\vee(x,y)$ with a subdifferential $\partial_\vee(x,f)$ of special kind ((o)-subdifferential, cf. Definition 3). As a result the order (non-) smoothness of convex and monotone functionals can be expressed in terms of this subdifferential.

Let $x \in$ dom(f), $x \geq 0$, and let $y \geq 0$. Given $t > 0$ let $g(ty)/t = (\ f(x \vee ty) - f(x)\ )/t$. Applying the identity $x \vee u = (x + u + |x - u|)/2$ we conclude that $g(\cdot)$ is a proper convex function satisfying $g(0) = 0$. Moreover, a standard argumentation yields that the function $t \to g(ty)/t$ is nondecreasing for $t > 0$.

Theorem 3.1

Let $x \in dom(f)$. The following equality holds true for $y \in X$

$$f_\vee(x,y) = \inf_{t>0} \frac{f(|x| \vee t|y|) - f(x)}{t}$$

Moreover, $f_\vee(x, |\cdot|)$ is positively homogeneous, subadditive and proper functional which is sublinear whenever $dom(f) = X$. If f is continuous at x then $f_\vee(x, |\cdot|)$ is continuous on X.

Proof. Since $f_\vee(x,y) = f_\vee(|x|,|y|)$ and $f(x) = f(|x|)$ we can confine to $x, y \in X_+$. Applying that the function $t \to g(ty)/t$ is nondecreasing for $t > 0$ the first part of Theorem immediately follows.

From this formula it follows that $f_\vee(x, \alpha y) = \alpha\, f_\vee(x,y)$ for each $\alpha \geq 0$. Let u, v be in X_+. Applying that $x \vee w = (x + w + |x - w|)/2$ and that f is convex, we obtain

$$
\begin{aligned}
f_\vee(x,u+v) &= \inf_{t>0} \frac{f\big(x\vee t(u+v)\big)-f(x)}{t}\\
&\le \inf_{t>0}\frac{1}{t}\left\{f\left(\frac{1}{2}\ \frac{x+2tu+|x-2tu|}{2}+\frac{1}{2}\ \frac{x+2tv+|x-2tv|}{2}\right)-f(x)\right\}\\
&= \inf_{t>0}\frac{1}{t}\left\{f\left(\frac{x\vee 2tu+x\vee 2tv}{2}\right)-f(x)\right\}\\
&\le \inf_{t>0}\frac{1}{2t}\ \big(f(x\vee 2tu)-f(x)\big)+\inf_{t>0}\frac{1}{2t}\ \big(f(x\vee 2tv)-f(x)\big)\\
&= f_\vee(x,u)+f_\vee(x,v).
\end{aligned}
$$

Since $f_\vee(x,u+v)=f_\vee(x,|u+v|)\le f_\vee(x,|u|)+f_\vee(x,|v|)=f_\vee(x,u)+f_\vee(x,v)$, we conclude finally that $f_\vee(x,\cdot)$ is convex and proper. Moreover, if $\mathrm{dom}(f)=X$ then $f_\vee(x,\cdot)$ is finite on X and hence sublinear. Let f be continuous at $x\in\mathrm{int}\big(\mathrm{dom}(f)\big)$. Recall, a convex (proper) function is continuous at $x\in\mathrm{int}\big(\mathrm{dom}(f)\big)$ if and only if f is bounded from the above on some neighborhood V_x of x. Let the bounding constant be $K>0$. Since the mapping $y\to x\vee y$ is continuous at zero, there exists a neighborhood of zero $U\in X$ such that $x\vee U\subset V_x$. Therefore, if $y\ge 0$ and $y\in U$ then

$$0\le f_\vee(x,y)\le f(x\vee y)-f(x)\le K-f(x).$$

Consequently, $f_\vee(x,\cdot)$ is continuous on $D=\mathrm{int}\big(\mathrm{dom}(f_\vee(x,\cdot))\big)$. On the other hand U is an absorbing set, whence $D=X$, i.e. $f_\vee(x,\cdot)$ is finite and continuous on X. □

Proposition 3.2

The following inequalities hold true for all $x,y\in X_+$

$$f_\vee(x,y)\le\frac{f(x,y)+f(x,-y)}{2}\le f(x,y)$$

where $f(x,y)$ is the usual directional derivative of f at x in the direction y. Moreover, $f_\vee(0,y)=f(0,y)=f(0,-y)$ for $y\in X_+$.

Proof. It suffices to apply that $x\vee ty=\frac{(x+ty+|x-ty|)}{2}$. Hence the convexity of f and the inequality $f(x,-y)\le f(x,y)$ yield the desired inequalities. Let us point out that to get inequality $f(x,-y)\le f(x,y)$ we can apply in the definition of $f(x,-y)$ that $f(x-ty)=f(|x-ty|)\le f(x+ty)$, since f is monotone and $x,y\in X_+$. □

DEFINITION 3. Let $x \in \text{dom}\,(f)$. We call a functional $u^* \in X^*_+$ an order subderivative of f at the point x, if u^* is of the form $u^* = x^* - y^*$ with $x^* \geq y^* \geq 0$, x^*, $y^* \in X^*_+$ and such that $< x, y^* >= f(x) + f^*(x^*)$. If $\partial_\vee f(x)$ denotes a subset of X^* defined by

$$\partial_\vee f(x) = \{u^* = x^* - y^* : x^* \geq y^* \geq 0,\ < x, y^* >= f(x) + f^*(x^*)\}$$

then we call $\partial_\vee f(x)$ an order subdifferential ((o)-subdifferential, for short) of f at x.

Remarks. (a) Let $u^* \in \partial_\vee f(x)$ then the order interval $[0, u^*] \subset \partial_\vee f(x)$. Indeed, if $u^* = x^* - y^*$, $x^* \geq y^* \geq 0$ and $< x, y^* >= f(x) + f^*(x^*)$ then $< x, z^* >= f(x) + f^*(x^*)$ for each z^* satisfying $y^* \leq z^* \leq x^*$.

(b) If $x \in X$ then $\partial_\vee f(x) \subset \partial_\vee f(|x|)$. To prove this it suffices to remark that $< x, y^* >= f(x) + f^*(x^*) = f(|x|) + f^*(x^*)$ and apply the Young inequality $< |x|, y^* > \leq f(|x|) + f^*(x^*)$.

The following proposition shows that the (o)-subdifferential $\partial_\vee(x, f)$ can be expressed in terms of the subdifferential $\partial f(x)$ of f at x.

Proposition 3.3

Let $x \in$ dom(f), $x \geq 0$. Then, for $\partial_\vee f(x)$ we have
(a) $\partial_\vee f(x) = \{u^* = x^* - y^* : x^* \geq y^* \geq 0,\ x^* \in \partial_+ f(x), x^* - y^* \perp x\}$,
(b) $\partial_\vee f(x) = \{u^* = x^* - y^* : x^* \geq y^* \geq 0,\ y^* \in \partial_+ f(x), f^*(x^*) = f^*(y^*)\}$,
(c) $\partial_\vee f(x) = (\partial_+ f(x) - \partial_+ f(x))_+ \cap \{x\}^\perp$.

Proof. (a). If $u^* \in \partial_\vee(x, f)$ then $u^* = x^* - y^*$, $x^* \geq y^* \geq 0$ and $< x, y^* >= f(x) + f^*(x^*)$. In virtue of the Young inequality $< x, y^* > \leq < x, x^* > \leq\ f(x) + f^*(x^*) = < x, y^* >$. Thus $< x, x^* >= f(x) + f^*(x^*)$ and this is equivalent to $x^* \in \partial f(x)$. From these relations it follows that $< x, x^* >=< x, y^* >$. Hence $x^* - y^* \perp x$ and this proves the inclusion "$\subset$". To prove the converse, it suffices to observe that $x^* - y^* \perp x$, $x^* \geq y^* \geq 0$ and $x^* \in \partial f(x)$, i.e. $< x, x^* >= f(x) + f^*(x^*)$, yield $< x, y^* >= f(x) + f^*(x^*)$. Hence $u^* = x^* - y^* \in \partial_\vee f(x)$ as desired.

(b). The proof proceed by similar arguments. Let us point out only that in the right hand side of (b) one can put equivalently $x^*, y^* \in \partial_+ f(x)$ instead of $y^* \in \partial_+ f(x)$.

(c). We apply (a) and that $x^* \geq y^* \geq 0$, $x^* - y^* \perp x$, $x^* \in \partial f(x)$ imply $y^* \in \partial f(x)$. Hence the representation (c) easily follows. □

In virtue of the representation (c) from Proposition 3.3 and some well known facts concerning subdifferentials [5], Theorems 14B and 14C) as a corollary we get the following theorem.

Theorem 3.4

Let $x \in X_+$. The order subderivative $\partial_\vee f(x)$ is always convex and w^*-closed (perhaps empty) subset in X_+^*. If dom$(f) = X$, then $\partial_\vee f(x) \neq \emptyset$ (equivalently, $0 \in \partial_\vee f(x)$) if and only if the usual directional derivative $f(x, \cdot)$ is bounded from below on some neighborhood of zero in X. If f is continuous at $x \in$ dom(f), $x \geq 0$, then $\partial_\vee(x, f)$ is nonempty, convex and w^*-compact set.

Remarks. (a) Recall, f is continuous at some point if and only if f is continuous on $\mathrm{int}\big(\mathrm{dom(f)}\big) \neq \emptyset$. In this case $\partial f(y) \neq \emptyset$ for all $y \in \mathrm{int}\big(\mathrm{dom}(f)\big)$.

(b) Let us notice that if f is smooth at x, i.e. $\partial f(x) = \{\nabla f(x)\}$, then $\partial_\vee f(x) = \{0\}$.

In order to prove our main theorem in this section, concerning relation between the order directional derivative $f_\vee(x, y)$ and the (o)-subdifferential $\partial_\vee f(x)$, we need the following result.

Lemma 3.5

Let f be a proper convex and lsc functional which is monotone. Let $x \in$ dom(f), $x \geq 0$. The following formula holds true for all $y \in X_+$.

$$\begin{aligned} f_\vee(x, y) &= \inf_{t>0} \frac{f(x \vee ty) - f(x)}{t} \\ &= \inf_{t>0} \sup_{x^* \geq y^* \geq 0} \left\{ < y, x^* - y^* > + \frac{< x, y^* > - f(x) - f^*(x^*)}{t} \right\}. \end{aligned}$$

Proof. Since f is lsc and convex we have $f = f^{**}$. Therefore,

$$\begin{aligned} f(x \vee ty) &= f^{**}(x \vee ty) \\ &= \sup_{x^* \geq 0} (< x \vee ty, x^* > - f^*(x^*)). \end{aligned}$$

Applying that $< x \vee ty, x^* >= \sup_{x^* \geq y^* \geq 0}(< x, y^* > + t < x, x^* - y^* >)$ we obtain

$$\frac{f(x \vee ty) - f(x)}{t} = \sup_{x^* \geq y^* \geq 0} \left\{ < y, x^* - y^* > + \frac{< x, y^* > - f(x) - f^*(x^*)}{t} \right\}.$$

Hence the lemma follows. $\square$

Remark. If f is not lsc then in the formula from the lemma we have the inequality "$\geq$" instead of the equality. This follows from the inequality $f^{**} \leq f$.

Let us consider the positive part of the subdifferential for sublinear functional $f_\vee(x,\cdot)$, i.e. $\partial_+ f_\vee(x,\cdot) = \{u^* \geq 0 : \forall_{y\in X} < y, u^* >\leq f_\vee(x,y)\}$. Since $f_\vee(x,y) = f_\vee(x,|y|)$ we get

$$\partial_+ f_\vee(x,\cdot) = \{u^* \geq 0 : \forall_{y\in X_+} < y, u^* >\leq f_\vee(x,y)\}.$$

Indeed, the inclusion "$\supset$" is clear. Let $u^* \in \partial_+ f_\vee(x,\cdot)$. Thus $< h, u^* >\leq f_\vee(x,h)$ for all $h \in X_+$. For any $y \in X$ we obtain $< y, u^* > \leq < |y|, u^* > \leq f_\vee(x,|y|) = f_\vee(x,y)$ which proves the reverse inclusion.

Theorem 3.6

Let f be a proper, convex and monotone functional on X and let $x \in dom(f)$, $x \geq 0$. Then $\partial_\vee f(x) \subset \partial_+ f_\vee(x,\cdot)$. Moreover, the following equality holds true

$$\partial_\vee f(x) = \partial_+ f_\vee(x,\cdot),$$

if, additionally, $f(x,\cdot)$ is continuous on X.

Remark. The equality in the theorem is still true if any of the following conditions is satisfied instead of the assumption on $f(x,\cdot)$ in Theorem:
(a) f is continuous at x,
(b) X is barreled (eg. Banach lattice) and $f(x,\cdot)$ is finite and lsc on X,
(c) $\mathrm{int}(X_+) \neq \emptyset$ and $f(x,\cdot)$ is finite-valued.
Recall, $f(x,\cdot)$ is positively homogeneous, convex. Moreover $f(x,\cdot)$ is sublinear on X whenever it is finite-valued ([6]). If any assumption (a) or (b) is satisfied then $f(x,\cdot)$ is sublinear and continuous in X. Indeed, in the case (a) we apply that $f(x,y) \leq f(x+y) - f(x)$ for y in some neighborhood of zero. In the case (b) we refer to [4], Corollary 2.5. Finally, in the case (c) we apply in the proof that in the case under consideration any positive functional on X is automatically continuous ([5], 11D).

Proof. Let $u^* \in \partial_\vee f(x)$, then $u^* = x^* - y^*$, $x^* \geq y^* \geq 0$ and $< x, y^* > -f(x) - f^*(x^*) = 0$. Referring to the remark below the proof of Lemma 3.5 we obtain that $f_\vee(x,y) \geq < y, u^* >$ for all $y \in X_+$ which means that $u^* \in \partial_+ f_\vee(x,\cdot)$.

To prove the converse let $f(x,\cdot)$ be continuous and let u^* be in $\partial_+ f_\vee(x,\cdot)$. For each $y \in X$ there holds $< y, u^* >\leq f_\vee(x,y)$. Moreover, in view of Proposition 3.2,

$$< h, u^* >\leq f_\vee(x,h) \leq f(x,h) \quad \text{for all} \quad h \in X_+.$$

We apply a lemma (Mazur-Orlicz) concerning an extension of a functional $u' \in X'$ satisfying $u' \leq p|_K$ for a cone $K \subset X$ to a (linear) functional $x' \in X'$ such that $u' \leq x'$ on K and $x' \leq p$ on X. On account of this lemma, setting $p(\cdot) = f(x, \cdot)$, $K = X_+$ and $u' = x^*$, we get a linear functional $x' \in X'$ such that

$$< h, u^* > \leq < h, x' > (\forall h \in X_+) \text{ and } < y, x' > \leq f(x, y) \ (\forall y \in X).$$

In fact the functional x' must be continuous since the majorant $f(x, \cdot)$ is continuous. Therefore, let us denote $x^* = x'$. Now, defining $y^* = x^* - u^*$ we obtain that $u^* = x^* - y^*$, $x^* \geq y^* \geq 0$ and $x^* - y^* \perp x$. Indeed, on account of our assumption we have $0 \leq < y, x^* - y^* > \leq f(x, y)$ for all $y \in X_+$. In particular, for $y = x$ we obtain $f(x, x) = 0$ and the orthogonality follows. Moreover, if $< y, x^* > \leq f(x, y)$ for all $y \in X$ then $x^* \in \partial f(x)$. Consequently, in virtue of Lemma 3.3 (a), we have proved that $u^* \in \partial_\vee f(x)$ which ends the proof. □

Now, we are ready to prove our main theorem in this section.

Theorem 3.7

Let f be a convex proper and monotone functional on X. Assume moreover that $f(x, \cdot)$ is continuous on X. Then the following representation for $f_\vee(x, \cdot)$ holds true for all $y \in X$.

$$f_\vee(x, y) = \sup_{u^* \in \partial_\vee f(x)} < y, u^* >$$

If f is continuous at x (and hence $f(x, \cdot)$ is continuous on X) then "sup" *in the formula can be replaced by* "max".

Proof. Under the assumptions of the theorem, in view of Theorem 3.6, we can deal equivalently with the usual subdifferential $\partial_+ f_\vee(x, \cdot) = \{u^* \in X_+^* : \forall_{y \in X} < y, u^* > \leq f_\vee(x, y)\}$ instead of the (o)-subdifferential $\partial_\vee f(x)$. Hence, the inequality "$\geq$" follows immediately. To prove the converse, we proceed in a standard way (eg. [5], 14D) assuming for a contrary that this inequality is strict. Namely, let there exists a real number $\alpha > 0$ such that

$$f_\vee(x, y) > \alpha > \sup_{u^* \in \partial_\vee f(x)} < y, u^* >$$

for some $y \in X$. Define a linear functional $\overline{u}^*$ on $M = \text{span}(y)$ by $< ty, \overline{u}^* >= t\alpha$ for $t \in$ R. Since $f_\vee(x, \cdot)$ is sublinear we have on M that $\overline{u}^* \leq f_\vee(x, \cdot)$. Applying Hahn-Banach theorem (cf. [5] 11G) it follows that there exists an extension u^* of $\overline{u}^*$ to all of X, in continuous way, such that for all $z \in X$ we have $< z, u^* > \leq f_\vee(x, z)$ with $< y, u^* >= \alpha$, a contradiction.

If f is continuous at $x \in$ dom(f), $x \geq 0$, then applying Theorem 3.4 we conclude that the (o)-subdifferential is w^*-compact and therefore the supremum is attained. Consequently, "sup" can be replaced by "max". □

Remark. If X is barreled, then to get the formula in Theorem (with "max") we need only that $f(x,\cdot)$ is finite and lsc. Also, the formula (with "sup") holds true whenever (b) $\mathrm{int}(X_+) \neq \emptyset$ and $f(x,\cdot)$ is finite-valued.

4. Order smoothness and strict monotonicity

Let X be a locally convex-solid Riesz space and let $f : X \to \overline{\mathbb{R}}$ be a proper convex functional which is monotone. Our goal here is to show that an appropriate dual notion to strict monotonicity (STM) for convex and monotone functional f is an order smoothness of the Young conjugate f^*.

DEFINITION 4. We say that $f : X_+ \to \overline{\mathbb{R}}$ is order smooth ((o)-SM for short) at x if $x \in \mathrm{dom}(f)$, and

$$f_\vee(x,y) = 0 \ \text{ for all } \ y \in X.$$

Moreover, f is said to be (o)-SM if f is (o)-SM at each point $x \in dom(f)$.

Remarks. Since $f_\vee(x,y) = f_\vee(|x|,|y|)$ it suffices to confine ourselves in the above definition to $x \in$ dom(f), $x \geq 0$ and $y \in X_+$. Also, recall that $f_\vee(0,y) = f(0,y)$ where $f(0,y)$ is the usual directional derivative. Thus the order smoothness at zero of convex and monotone functionals f coincide with the usual one, since in the case under consideration $f(0,y) = f(0,|y|)$.

On account of Theorem 3.7, the (o)-SM can be expressed in terms of the (o)-subdifferential $\partial_\vee(x,f)$.

Theorem 4.1

Let f be as above and let f be continuous at $x \in$ dom(f), $x \geq 0$. The following statements are equivalent.
(a) f is (o)-SM at x,
(b) $\partial_\vee f(x) = \{0\}$,
(c) For each order interval $[y^*, x^*] \subset \partial_+ f(x)$, with $f^*(y^*) = f^*(x^*)$, there holds $y^* = x^*$.

Proof. (a)⇒(b). If for all $y \in X_+$ we have $f_\vee(x,y) = 0$ then, by 3.7, for each $u^* \in \partial_\vee f(x)$, we have $u^* = 0$. Hence (b) follows.

(b)⇒(c). Let $[y^*, x^*] \subset \partial_+ f(x)$ with $f^*(y^*) = f^*(x^*)$ and let us assume for a moment that $y^* < x^*$. Hence $u^* = x^* - y^* > 0$, $x^* \geq y^* \geq 0$, $x^*, y^* \in \partial_+ f(x)$

and $f^*(y^*) = f^*(x^*)$. In virtue of Proposition 3.3 (b) it follows that a non-zero $u^* \in \partial_\vee f(x)$, a contradiction.

(c)⇒(a). Let for a contrary there exists $y > 0$ such that $f_\vee(x, y) > 0$ i.e. f is not (o)-smooth at x. Applying Theorem 3.7 we conclude that there exists $u^* = x^* - y^*$ in $\partial_\vee f(x)$ such that $< y, u^* > > 0$ where $x^* \geq y^* \geq 0$, $f^*(x^*) = f^*(y^*)$ and $x^*, y^* \in \partial_\vee f(x)$. Hence, the proper interval $[y^*, x^*] \subset \partial_\vee(x, f)$, a contradiction. □

Remark. Let $f(x) = \|x\|$ then $f^*(x^*) = \sigma_{B^*}(x^*)$ (the indicator function of the unit ball $B^* \subset X^*$). Since $\partial f(x) \subset B^*$, then given $[y^*, x^*] \subset \partial_+ f(x)$ we have automatically $f^*(x^*) = f^*(y^*)$. Therefore, from the above theorem it follows that $\| \cdot \|$ is (o)-SM at $x \geq 0$ if and only if $\partial_+\|x\|$ contains no proper order interval $[y^*, x^*]$ ([8]).

Lemma 4.2

Let f be a proper, convex, lsc and monotone functional on X such that $\partial f(x) \neq \emptyset$ for all $x \in$ dom(f), $x \geq 0$. The following statements hold true.
(a) If f^* is (o)-SM then f is STM.
(b) If f^* is STM then f is (o)-SM.

Proof. (a). Assume to the contrary that f is not STM, i.e. there exist $x > y \geq 0$ such that $f(x) = f(y)$. Since $y \in \text{dom}(f)$, there exists $z^* \in \partial_+ f(y)$. This is equivalent to $y \in \partial f^*(z^*)$, i.e. $< z^*, y >= f^*(z^*) + f^{**}(y)$. But $f^{**}(y) = f^{**}(x)$ so that

$$\begin{aligned} < z^*, y > = \; & f^*(z^*) + f^{**}(y) \\ & = < z^*, y >= f^*(z^*) + f^{**}(x) \\ & \geq < z^*, x > \geq < z^*, y > . \end{aligned}$$

This yields $x \in \partial f^*(z^*)$. Hence, $[y, x] \subset \partial f^*(z^*)$, where the order interval $[y, x]$ is nontrivial. Thus we arrived to a contradiction with the (o)-SM of f^*.

(b). Let f be not (o)-SM at some point $x \in$ dom(f), $x \geq 0$. Then there exist $x^* > y^* \geq 0$ such that $y^* \in \partial f(x)$ and $f^*(x^*) = f^*(y^*)$. However, this last equality contradicts to the STM of f^* and (b) follows. □

As a corollary we obtain the following duality relation concerning the STM and (o)-SM of convex and monotone functionals.

Theorem 4.3

Let f be a proper, convex, lsc and monotone functional on X such that $\partial f(x) \neq \emptyset$ for all $x \in$ dom(f), $x \geq 0$. Then, (a) f^* is (o)-SM if and only if f is STM and (b) f^* is STM if and only if f is (o)-SM.

Proof. It suffices to observe that on account of the lsc of f we have $f = f^{**}$. Consequently, applying the above lemma, if f^{**} is (o)-SM (on X) then f^* is STM and in turn this implies that f, and hence f^{**}, is (o)-SM. □

Remark. If one extends f^{**} to X^{**} by $f^{**}(x^{**}) = \sup_{x^* \in X^*}(< x^*, x^{**} > -f^*(x^*))$ then $f^{**}|_X = f$. In this case we cannot close the above chain of implications, however, the theorem is still true whenever X is reflexive.

5. Modules of uniform (o)-smoothness and uniform strict monotonicity

In this section we add some uniformity to the definitions of STM and (o)-SM. As a result two modules are introduced and dual relations are established. Let $f : X \to \overline{\mathbb{R}}$ be convex, proper and monotone functional on a normed lattice X with a monotone norm $\|\cdot\|$. Let us notice that everything below is also valid for locally-solid Riesz space X with a fixed seminorm $p(\cdot)$ instead of the norm $\|\cdot\|$.

DEFINITION 5. Let us consider the following functions for ϵ, $\tau \geq 0$ with values in $\overline{\mathbb{R}}_+$:

$$\delta_f(\epsilon) = \inf_{x \geq y \geq 0,\ \|u\| \geq \epsilon} \big(f(x) - f(x-u)\big),$$

$$\rho_f(\tau) = \sup_{x,y \geq 0,\ \|y\|=1} \big(f(x \vee \tau y) - f(x)\big).$$

We call $\delta_f(\cdot)$ a modulus of uniform monotonicity of f and $\rho_f(\cdot)$ a modulus of uniform (o)-smoothness (cf. [8]). In the same way one can define the modules δ_{f^*}, ρ_{f^*} and $\delta_{f^{**}}$, $\rho_{f^{**}}$. for f^* and f^{**}, respectively.

The following theorem concerns duality relations between modules under consideration (cf. Theorem 3 in [8]).

Theorem 5.1

Let f be convex, lsc and monotone on X with a point of continuity in dom(f). The following duality formulas hold true:
(a) $\rho_f(\tau) = \rho_{f^{**}}(\tau)$,
(b) $\delta_f(\epsilon) = \delta_{f^{**}}(\epsilon)$ *and*
(c) $\rho_{f^*}(\tau) = \sup_{\epsilon \geq 0} \big(\epsilon\tau - \delta_f(\epsilon)\big)$,
(d) $\delta_f(\epsilon) = \sup_{\tau \geq 0} \big(\tau\epsilon - \rho_{f^*}(\tau)\big)$
where $\epsilon \geq 0$ *and* $\tau \geq 0$.

Proof. (a) and (b). These assertions are evident, since f is lsc and $f = f^{**}$.

(c). Let x^*, $y^* \geq 0$ with $\|y^*\| = 1$. Let $\tau \geq 0$ and $x \geq 0$. Then

$$< x, x^* \vee \tau y^* > -f(x) = \sup_{x \geq u \geq 0} \big(< x-u, x^* > +\tau < u, y^* > -f(x) \big).$$

In virtue of the Young inequality we have $< x-u, x^* > \leq f(x-u) + f^*(x^*)$. Taking the supremum over all $x \geq 0$, by the definition of the Young conjugate f^*, we obtain

$$\begin{aligned} f^*(x^* \vee \tau y^*) - f^*(x^*) &\leq \sup_{x \geq y \geq 0} \big(\tau < u, y^* > +f(x-u) - f(x) \big) \\ &\leq \sup_{\epsilon \geq 0} \sup_{x \geq u \geq 0,\ \|u\| = \epsilon} \Big(\tau \|u\| - \big(f(x) - f(x-u) \big) \Big) \\ &\leq \sup_{\epsilon \geq 0} \big(\tau\epsilon - \delta_f(\epsilon) \big). \end{aligned}$$

Now, passing to "sup" over all $x^* \geq 0$ and $y^* \geq 0$ with $\|y^*\| = 1$ we obtain

$$\rho_{f^*}(\tau) \leq \sup_{\epsilon \geq 0} \big(\epsilon\tau - \delta_f(\epsilon) \big).$$

To prove the inverse inequality let $x \geq u \geq 0$ be fixed but arbitrary where $x \in \operatorname{int}\big(\operatorname{dom}(f)\big)$. Let us choose $x^* \geq 0$ such that $< x-u, x^* >= f(x-u) + f^*(x^*)$. This is possible since f is continuous on $\operatorname{int}\big(\operatorname{dom}(f)\big)$ and $x \geq u \geq 0$. Let $y^* \geq 0$ be arbitrary such that $\|y^*\| = 1$. Then

$$\begin{aligned} \rho_{f^*}(\tau) &\geq f^*(x^* \vee \tau y^*) - f^*(x^*) \\ &= \sup_{z \geq 0} \big(< z, x^* \vee \tau y^* > -f(z) \big) - f^*(x^*) \\ &\geq < x, x^* \vee \tau y^* > -f(x) - f^*(x^*) \\ &= \sup_{x \geq y \geq 0} (< x-y, x^* > +\tau < y, y^* >) - f(x) - f^*(x^*) \\ &\geq f(x-u) + f^*(x^*) + \tau < u, y^* > -f(x) - f^*(x^*) \\ &= \tau < u, y^* > -\big(f(x) - f(x-u) \big). \end{aligned}$$

Passing to the supremum with this y^* and then with $x \geq u \geq 0$ we get finally that

$$\begin{aligned} \rho_{f^*}(\tau) &\geq \sup_{x \geq u \geq 0} \Big(\tau\|u\| - \big(f(x) - f(x-u) \big) \Big) \\ &\geq \sup_{\epsilon \geq 0} \big(\tau\epsilon - \delta_f(\epsilon) \big) \end{aligned}$$

which ends the proof of the assertion (c).

To prove the assertion (d) we apply (c). Hence it follows that

$$\delta_f(\epsilon) \geq \sup_{\tau \geq 0} \big(\tau\epsilon - \rho_{f^*}(\tau)\big).$$

Let x^*, $y^* \geq 0$ and $\|y^*\| = 1$ and let $\tau \geq 0$ be arbitrary. Then

$$\begin{aligned} f^*(x^* \vee \tau y^*) - f^*(x^*) &= \sup_{x \geq 0} \big(< x, x^* \vee \tau y^* > - f(x)\big) - f^*(x^*) \\ &= \sup_{x \geq 0} \big(\sup_{x \geq u \geq 0} (< x - u, x^* > - \tau < u, y^* >) \\ &\quad - \; f(x)\big) - f^*(x^*) \\ &\leq \sup_{x \geq u \geq 0} \big(f(x-u) - f(x) + \tau\|u\|\big), \end{aligned}$$

where it was applied that $< x - u, x^* > \leq f(x-u) + f^*(x^*)$ and that $\|y^*\| = 1$. Therefore,

$$\begin{aligned} \delta_f(\epsilon) &\geq \sup_{\tau \geq 0} \big(\epsilon\tau - \rho_{f^*}(\tau)\big) \\ &\geq \sup_{\tau \geq 0} \big(\epsilon\tau - \max\big\{\sup_{A_\epsilon} (\tau\|u\|)\big), \sup_{B_\epsilon} \big(f(x-u) - f(x) + \tau\|u\|\big)\big\} \\ &\geq \sup_{\tau \geq 0} \sup_{B_\epsilon} \big(f(x-u) - f(x) + \tau\|u\|\big)\big\} \\ &\geq \delta_f(\epsilon) \end{aligned}$$

where we have used the definition of $\rho_{f^*}(\tau)$ and that $A_\epsilon = \{(x,u) : x \geq u \geq 0, \|u\| < \epsilon\}$, $B_\epsilon = \{(x,u) : x \geq u \geq 0, \|u\| \geq \epsilon\}$. Hence the desired equality in (d) follows. $\square$

For the sake of completeness we list some fundamental properties of the modules under consideration.

Proposition 5.2

The following properties hold true for the modules δ_f, ρ_{f^*}.
(a) $\delta_f(\epsilon) = 0$ (resp. $+\infty$) for $\epsilon > 0$ if and only if $\rho_{f^*}(\tau) = +\infty$ (resp. 0) for $\tau > 0$.
(b) $\epsilon\,\tau \;\leq\; \delta_f(\epsilon) + \rho_{f^*}(\tau)$ for all ϵ, $\tau \geq 0$.
(c) The modules $\delta_f(\cdot)$, $\rho_{f^*}(\cdot)$ are convex, proper and lsc functions from $[0, +\infty)$ into $\overline{\mathbb{R}}_+$ and hence continuous on $\operatorname{int}\big(\operatorname{dom}\big(\delta_f(\cdot)\big)\big)$ and $\operatorname{int}\big(\operatorname{dom}\big(\rho_{f^*}(\cdot)\big)\big)$, respectively, with $\delta_f(0) = \rho_{f^*}(0) = 0$.
(d) The following function $\tau \longrightarrow \rho_{f^*}(\tau)/\tau$ is nondecreasing for $\tau > 0$.

Proof. Applying (c) and (d) from Theorem 5.1 the first and the second assertion follow immediately. Since $\delta_f(\cdot)$ and $\rho_{f^*}(\cdot)$ are pointwise suppremas of families of affine functions it follows that the modules $\delta_f(\cdot)$, $\rho_{f^*}(\cdot)$ are convex and lsc on $\mathbb{R}_+$ and hence continuous on the interiors of their effective domains so the proof of (c) is clear. Finally, to prove (d) it suffices to note that from Theorem 5.1 (c) it follows that

$$\frac{\rho_{f^*}(\tau)}{\tau} = \sup_{\epsilon\geq 0}\left(\epsilon - \frac{\delta_f(\epsilon)}{\tau}\right)$$

for $\tau > 0$. Hence it follows that $\rho_{f^*}(\tau)/\tau$ is nondecreasing for $\tau > 0$. □

Remark. From Theorem 5.1 it is clear that the modules $\delta_f(\epsilon)$, $\rho_{f^*}(\tau)$ are mutually conjugate in the Young sense functions. From this observation some further properties of these modules can be also derived.

EXAMPLE: Let μ be a $\sigma-$finite positive measure on a set T and let us consider the space $L_1(\mu)$ and its dual $L_\infty(\mu)$. Let $f(x) = \int_T |x(t)|dt$. From the definition of δ_f it is easy to see that $\delta_f(\epsilon) = \epsilon$ for $\epsilon \geq 0$. It is not clear how to determine $\rho_{f^*}(\tau)$ on the basis of the definition. However, applying the duality formulas from Theorem 5.1 we immediately obtain that for the functional f under consideration

$$\rho_{f^*}(\tau) = \begin{cases} +\infty & , \quad \tau > 1, \\ 0 & , \quad \tau \in [0,1]. \end{cases}$$

Let us observe, that for $\tau \in [0,1]$ there holds $\delta_f(\epsilon) + \rho_{f^*}(\tau) = \epsilon \geq \tau\epsilon$ where the equality is attained for $\tau = 1$. This shows that the inequality in the point (b) of the proposition cannot be improved in general.

6. Uniform strict monotonicity and uniform (o)-smoothness

As it was observed in the example concerning $L_1(\mu)$, the modulus $\delta_f(\epsilon) = \epsilon$, i.e. the functional $f(x) = \int_T |x(t)|dt$ has the modulus $\delta_f(\cdot)$ of STM which is uniformly far from zero. On the other hand, for the dual modulus $\rho_{f^*}(\cdot)$, we see that not only $\rho_{f^*}(\tau) = 0$ for τ small but also $\rho_{f^*}(\tau)/\tau = 0$ whenever $\tau \leq 1$. On the other hand let us observe that according to the definition of $\rho_f(\tau)$

$$\lim_{\tau\searrow 0}\frac{\rho_f(\tau)}{\tau} = \lim_{\tau\searrow 0}\sup_{x,y\geq 0,\ \|y\|=1}\frac{f(x\vee\tau y) - f(x)}{\tau}$$

is the "uniform" version of the order smoothness ($f_\vee(x,y) = 0$, for all $y \in X$, section 4).

Also, it can be observed some similarity between the modulus $\rho_f(\tau)$ and the modulus of uniform smoothness for Banach spaces [3], [8]). This leads to the concept of the uniform (o)-smoothness which in turn appears to be a dual property to the uniform monotonicity of functionals.

In this section let X denotes a normed lattice and let f be a proper, convex and monotone functional on X.

DEFINITION 6. We say that f is uniformly monotone (UM, for short) if for each $\epsilon > 0$ there holds $\delta_f(\epsilon) > 0$. Also, we say that f is order uniformly smooth ((o)-USM, for short), if

$$\inf_{\tau>0} \frac{\rho_{f^*}(\tau)}{\tau} = 0.$$

Let us point out that the function $\tau \to \rho_f(\tau)/\tau$, where

$$\frac{\rho_f(\tau)}{\tau} = \sup_{x,y\geq 0, \|y\|=1} \frac{f(x \vee \tau y) - f(x)}{\tau}$$

is nondecreasing, for $\tau > 0$. Therefore, we can replace "$\lim_{\tau\searrow 0}$" by "$\inf_{\tau>0}$".

Theorem 6.1

Let f be convex, proper lsc and monotone functional on a normed lattice X. The following statements hold.
(a) f is UM if and only if f^* is (o)-USM.
(b) f^* is UM if and only if f is (o)-USM.

Proof. (a). Assume to the contrary that f^* is not (o)-USM. According to the definition of (o)-SM there exists $\alpha > 0$ such that

$$\inf_{\tau>0} \frac{\rho_{f^*}(\tau)}{\tau} \geq \alpha.$$

Applying Theorem 5.1 it follows that

$$\delta_f(\epsilon) = \sup_{\tau>0} \tau \left(\epsilon - \frac{\rho_{f^*}(\tau)}{\tau} \right) \leq \sup_{\tau>0} \tau(\epsilon - \alpha) = 0$$

whenever $0 \leq \epsilon \leq \alpha$, which contradicts to the UM of f.

Conversely, let f^* be (o)-SM but f is not UM, i.e. there exists $\epsilon_0 > 0$ such that $\delta_f(\epsilon_0) = 0$. Then

$$\begin{aligned} 0 &= \inf_{\tau>0} \frac{\rho_{f^*}(\tau)}{\tau} \\ &= \inf_{\tau>0} \sup_{\epsilon\geq 0} \left(\epsilon - \frac{\delta_f(\epsilon)}{\tau}\right) \\ &\geq \epsilon_0 > 0, \end{aligned}$$

a contradiction.

(b). To prove (b) it suffices to put f^* instead of f in (a) and apply that $f = f^{**}$ since f is lsc. $\square$

7. An example

Let us consider a functional $f : L_p(\mu) \longrightarrow \overline{\mathbb{R}}$, where $1 \leq p \leq +\infty$ and

$$f_p(x) = \begin{cases} \int_T |x(t)|^p dt, & \text{if} \quad 1 \leq p < +\infty, \\ \sigma_{B_\infty}(x), & \text{if} \quad p = +\infty, \end{cases}$$

for $L_p(\mu)$ over a σ-finite positive measure space (T, Σ, μ) ($\sigma_{B_\infty}(x)$ denotes the indicator function for the unit ball B_∞ in $L_\infty(\mu)$). Let $\frac{1}{p} + \frac{1}{q} = 1$ where $1 \leq p, q \leq +\infty$. We have $(f_p)^*(x^*) = f_q(v)$ for $v \in L_q(\mu)$.

Proposition 7.1

The following estimations hold for the modulus of the uniform monotonicity $\delta_{f_p}(\cdot)$ and the modulus of the uniform (o)-smoothness $\rho_{f_q}(\cdot)$ of the functionals $f_p(x)$ and $f_q(x)$, respectively, where $\epsilon, \tau \geq 0$ and $1 \leq p, q < +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

(a) $\epsilon^p \leq \delta_{f_p}(\epsilon) \leq p\, \epsilon^{\,p}$,

(b) $\left(1 - \frac{1}{p}\right)\ (\frac{\tau}{p^2})^{1/p-1} \leq \frac{\rho_{f_q}(\tau)}{\tau} \leq \left(1 - \frac{1}{p}\right)\ (\frac{\tau}{p})^{1/p-1}$.

Moreover, $\delta_{f_1}(\epsilon) = \epsilon$, $\delta_{f_\infty}(\epsilon) = 0$ if $\epsilon \in [0, 1]$ and, by the definition, $\delta_{f_\infty}(\epsilon) = +\infty$, otherwise. Hence $\rho_{f_\infty}(\tau) = \sigma_{[0,1]}(\tau)$ (the indicator function of the interval $[0, 1]$) and $\rho_{f_1}(\tau) = \tau$ (on account of this definition).

Remark. One can also define $\delta_{f_\infty}(\epsilon) = 0$ for $\epsilon > 1$. In this case $\delta_{f_\infty}(\epsilon) = 0$ for all $\epsilon \geq 0$ and consequently $\rho_{f_1}(\tau) = +\infty$ for all $\tau > 0$ and $\rho_{f_1}(0) = 0$.

Proof. (a). To get the first part let us notice that $f_p(x) - f_p(x-u) = f_p(x-u+u) - f_p(x-u) \geq f_p(u)$ whenever $x \geq u \geq 0$. Now, if $\|u\|_p \geq \epsilon$ then for such x and u we obtain that $f_p(x) - f_p(x-u) \geq \epsilon^P$ which yields the first inequality.

To get the second one, let $1 < p < +\infty$ and $x \geq u \geq 0$. Since on supp $(x(\cdot))$ we have $0 \leq u(t)/x(t) \leq 1$, then

$$\big(x(t) - u(t)\big)^p = x(t)^p \left(1 - \frac{u(t)}{x(t)})^p\right) \geq x(t)^p - p\, u(t)x(t)^{p-1}$$

for $t \in \operatorname{supp}(x(\cdot))$. Consequently, with $\frac{1}{p} + \frac{1}{q} = 1$ $(q \geq 1)$, we obtain

$$\begin{aligned} f(x) - f(x-u) &\leq p \int_T u(t)x(t)^{p-1} d\mu \\ &\leq p \left(\int_T u(t)^p d\mu\right)^{1/p} \left(\int_T x(t)^{(p-1)q} d\mu\right)^{1/q} \\ &= \|u\|_p \|x\|_p^{p/q}. \end{aligned}$$

Taking the infimum over all $x \geq u \geq 0$, with $\|u\|_p \geq \epsilon$, we obtain that $\delta_{L_p}(\epsilon) \leq p\epsilon^{\,p}$. If $p = 1$, we already know that $\delta_{L_1}(\epsilon) = \epsilon$.

(b). To get the second inequality in (b) we apply that $\rho_{f_q}(\tau) = \sup_{\epsilon \geq 0} \big(\tau\epsilon - \delta_{f_p}(\epsilon)\big)$. Hence, $\rho_{f_q}(\tau) \leq \sup_{\epsilon \geq 0}(\tau\epsilon - \epsilon^p)$. An easy computation leads the desired inequality.

Again, in virtue of Theorem 6.1 (b), we get that for for $1 < p < +\infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned} \rho_{f_q}(\tau) &= \sup_{\epsilon \geq 0} \big(\epsilon\tau - \delta_{f_p}(\epsilon)\big) \\ &\geq \sup_{\epsilon \geq 0} \left(\epsilon\tau - p\, \epsilon^{\,p}\right), \end{aligned}$$

where the supremum is attained for $\epsilon = \big(\frac{\tau}{p^2}\big)^{1/(p-1)}$. Hence (b) follows.

We left the boundary cases for p and q since they easily follow from the definition of the module $\delta_\infty(\cdot)$ and the duality formulas in Theorem 6.1. The same estimations for the modules $\delta_{f_p}(\epsilon)$, $\rho_{f_q}(\tau)$, where $1/p + 1/q = 1$ and $1 \leq p < +\infty$, hold for the counting measure μ. $\square$

8. An application to optimization

Let us consider a convex proper and monotone functional $f : X \longrightarrow \overline{\mathbb{R}}$ over a locally convex-solid Riesz space X. Let $x \geq 0$ be fixed and let us consider the following optimization problem

$$(\mathrm{P}) \qquad \begin{cases} g(y) & \longrightarrow \quad \min \\ & y \geq 0 \end{cases}$$

where $g(y) = f(x \vee y)$. Let us consider a solution set P_g defined as follows

$$P_g = \left\{ y_0 \geq 0 : g(y_0) = \min_{y \geq 0} g(y) \right\}.$$

Clearly, $P_g \neq \emptyset$. In fact the order interval $[0, x] \subset P_g$. Moreover, $\min_{y\geq 0} g(y) = f(x)$, i.e. $y \in P_g$ if and only if $f(x \vee y) - f(x) = 0$. Therefore, applying Theorem 3.7 and the representations for $\partial_\vee f(x)$ from Theorem 3.3 we obtain a necessary condition for y being a solution of the problem (P).

Theorem 8.1

Let f be additionally continuous at x and $x \geq 0$ and let us consider the following statements.
(a) An element $y \geq 0$ is a solution of the minimization problem (P), i.e. $y \in P_g$.
(b) $\partial_\vee f(x) \perp y$, i.e. for each $x^*, y^* \in \partial f(x)$ such that $x^* \geq y^* \geq 0$ and $x^* - y^* \perp x$ there holds $x^* - y^* \perp y$. Then (a) implies (b).

Proof. Indeed, if $y \in P_g$ then in virtue of the definition of $f_\vee(x, y)$

$$0 \leq f_\vee(x, y) \leq f(x \vee y) - f(x).$$

Hence $f_\vee(x, y) = 0$ and applying Theorem 3.7 we get that for all $u^* \in \partial_\vee f(x)$ we have $< y, u^* >= 0$ which proves (b). □

Remark. This theorem can be applied to determine a constructive algorithm for finding candidates for (nontrivial) solutions of the problem (P).

Finally, we give a characterization theorem concerning solutions of the problem (P) (cf. [8], Theorem 8).

Theorem 8.2

Let $x \in X_+$ be fixed and let $y \geq 0$. Assume that f is continuous at x. The following statements are equivalent.
(a) $y \in P_g$.
(b) There exists $x^* \in {X_+}^*$ such that
(i) $x^* \in \partial_+ f(x) \cap \partial_+ f(x \vee y)$,
(ii) $< y - x, y^* > \leq \ 0$ for all $0 \leq y^* \leq x^*$.

Proof. (a) $\Rightarrow$ (b). Since f is continuous, there exists $x^* \in {X_+}^*$ such that $< x, x^* >= f(x) + f^*(x^*)$. Applying (a), since $f(x \vee y) = f(x)$, we obtain

$$\begin{aligned} < x, x^* > \leq < x \vee y, x^* > \leq & \ f(x \vee y) + f^*(x^*) \\ = & \ f(x) + f^*(x^*) =< x, x^* > . \end{aligned}$$

Hence $< x \vee y, x^* >= f(x \vee y) + f^*(x^*)$, i.e. $x^* \in \partial_+ f(x \vee y)$. Thus (b) (i) follows. Next,

$$\begin{aligned} f(x \vee y) + f^*(x^*) &= < x \vee y, x^* > \\ &= \sup_{0 \leq y^* \leq x^*} (< x, x^* - y^* > + < y, y^* > \\ &= < x, x^* > + \sup_{0 \leq y^* \leq x^*} < y - x, y^* > \\ &= f(x) + f^*(x^*) + \sup_{0 \leq y^* \leq x^*} < y - x, y^* > . \end{aligned}$$

Thus, $\sup_{0 \leq y^* \leq x^*} < y - x, y^* > \leq \ 0$ and (b)(ii) follows.

(b) $\Rightarrow$ (a). Assume that (b) is satisfied. Then the same equalities hold. Since, by (b)(ii), $\sup_{0 \leq y^* \leq x^*} < y - x, y^* > \leq \ 0$ we conclude that $f(x \vee y) = f(x)$, i.e. $y \in P_g$ which finishes the proof. $\square$

EXAMPLE: Let T be a compact Hausdorf space and let $X = C(T)$ be the space of all continuous real-valued functions on T with the maximum norm. The dual X^* consists of all functionals x^* such that $< x(\cdot), x^* >= \int_T x(t) d\mu$ with the norm $\|x^*\| = |\mu|(T)$. More precisely, these functionals can be identified with the Radon measures μ on T.

Let us consider the functional $f(x) = \max_{t \in T} |x(t)|$ and let $x \in C(T)$ be fixed such that $x(t) \geq 0$ for all $t \in T$. Following [6], section 4.5, the subdifferential of f at x consists of all nonnegative Radon measures μ such that

(a) $|\mu|(T) = 1$,
(b) $\int_T x(t) d\mu = f(x)$,
(c) μ is concentrated on the set $T_x = \{t \in T : x(t) = f(x)\}$.

Applying Theorem 8.2 we obtain the following characterization of the solutions of the optimization problem (P).

Corollary 8.3

Let f and X be as above and let $g(y) = f(x \vee y)$. The following statements are equivalent.
(a) $y \in P_g$.
(b) There exists a nonnegative Radon measure μ on T such that
(i) $|\mu|(T) = 1$,
(ii) μ *is concentrated on* T_x,
(iii) $T_x = T_{x \vee y}$,
(iv) $\int_T (y(t) - x(t)) d\eta \leq 0$ *for all Radon measures* η *satisfying* $0 \leq \eta \leq \mu$.

Something more concrete characterization of the solutions y can be derived from this corollary in the case $T = [0, 1]$, however we do not rise this question in the paper.

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