Collectanea Mathematica (electronic version): http://www.mat.ub.es/CM

Collect. Math. **48**, 4-6 (1997), 587–600

P-convexity of Musielak-Orlicz sequence spaces of Bochner type

Paweł Kolwicz

Institute of Mathematics, University of Technology, ul. Piotrowo 3a, 60-965 Poznań, Poland

Ryszard Płuciennik*

Institute of Mathematics, T. Kotarbiński Pedagogical University, Pl. Słowiański 9, 65-069 Zielona Góra, Poland

Abstract

It is proved that the Musielak-Orlicz sequence space $l_\varphi(X)$ of Bochner type is P-convex if and only if both spaces $l_\varphi(\mathbb{R})$ and X are P-convex. In particular, the Lebesgue-Bochner sequence space $l^p(X)$ is P-convex iff X is P-convex and $1 < p < \infty$.

I. Introduction

Relationships between various kinds of convexities of Banach spaces and the reflexivity were developed by many authors. D. Giesy [5] and R.C. James [10] raised the question whether B-convex Banach spaces are reflexive. James [11] settled the question negatively, constructing an example of a nonreflexive B-convex Banach space. It was natural to ask whether reflexivity is implied by some slightly stronger geometric property. Such a property was introduced by C.A. Kottman [18] and it was called P-convexity. Kottman proved that every P-convex Banach space is reflexive. D. Amir and C. Franchetti [2] showed that in Banach spaces P-convexity follows from uniform convexity as well as from uniform smoothness. It was natural to characterize P-convexity in some concrete Banach spaces. Y. Ye, M. He and

* Supported by KBN grant 2 P03A 031 10.

R. Płuciennik [21] proved that every Orlicz space is reflexive iff it is P-convex. The same result for Musielak-Orlicz function spaces was obtained by P. Kolwicz and R. Płuciennik [16] and for Musielak-Orlicz sequence spaces by Y. Ye and Y. Huang [22].

In this paper we show that $l_\varphi(X)$ is P-convex iff both l_φ and X are P-convex. This result implies immediately the main theorem from [22]. The similar characterization of P-convexity for Orlicz-Bochner function spaces was done in [17]. It is worth to mention that criteria for B-convexity of Orlicz-Bochner spaces were obtained in [4], [7] and [15].

Denote by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ the sets of natural, real and positive real numbers, respectively. Let $\mathcal{M}$ be the set of all real sequences $x = (u_n)_{n=1}^{\infty}$. A function φ is called an *Orlicz function*, if $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}_+$ is convex, even, $\varphi(u) = 0$ iff $u = 0$, and $\lim_{u\to 0}\ u^{-1}\varphi(u) = 0$. A sequence $\varphi = (\varphi_n)$ of Orlicz functions φ_n is called a *Musielak-Orlicz function.* Define on $\mathcal{M}$ a convex modular I_φ by

$$I_\varphi(x) = \sum_{n=1}^{\infty} \varphi_n(u_n)$$

for every $x \in \mathcal{M}$. By the *Musielak-Orlicz space* l_φ we mean

$$l_\varphi = \{x \in \mathcal{M} : I_\varphi(cx) < \infty \ \text{ for some } \ c > 0\}$$

equipped with the *Luxemburg norm*

$$\|x\|_\varphi = \inf\left\{\epsilon > 0 : I_\varphi\left(\frac{x}{\epsilon}\right) \le 1\right\}.$$

For every Musielak-Orlicz function φ we will denote by φ^* the sequence (φ_n^*) of functions φ_n^* that are *complementary to* φ_n *in the sense of Young,* i.e.

$$\varphi_n^*(v) = \sup_{u\ge 0}\left\{u\,|v| - \varphi_n(u)\right\}$$

for every $v \in \mathbb{R}$ and $n \in \mathbb{N}$.

We say that a Musielak-Orlicz function φ *satisfies the* δ_2*-condition* if there are constants $k_0, a_0 > 0$ and a sequence (c_n^0) of positive reals with $\sum_{n=1}^{\infty} c_n^0 < \infty$ such that

$$\varphi_n(2u) \le k_0\varphi_n(u) + c_n^0$$

for each $n \in \mathbb{N}$ and every $u \in \mathbb{R}$ satisfying $\varphi_n(u) \le a_0$. For more details we refer to [20].

Moreover, we can assume without loss of generality that $\varphi_n(1) = 1$ and $\varphi_n(u) = u^2$ for all $n \in \mathbb{N}$ and every $|u| > 1$. Otherwise we may define a new Musielak-Orlicz function $\psi = (\psi_n)$ by the following formula

$$\psi_n(u) = \begin{cases} \varphi_n(b_n u) & \text{for} \quad 0 \leq |u| \leq 1 \\ u^2 & \text{for} \quad |u| > 1, \end{cases}$$

where $\varphi_n(b_n) = 1$, for every $n \in \mathbb{N}$. The spaces l_φ and l_ψ are equal isometrically (see [13]). Under this assumption we should remember that every function φ_n ($n \in \mathbb{N}$) is non decreasing on $\mathbb{R}_+$ and it is convex on the interval $[0,1]$, but not necessarily convex on the whole $\mathbb{R}_+$. It is easy to prove that this modified function φ satisfies the δ_2-condition iff for every $\epsilon > 0$ there are a constant $k > 2$ and a sequence (c_n) of positive real numbers such that

$$\sum_{n=1}^{\infty} \varphi_n(c_n) < \epsilon \qquad \text{and} \qquad \varphi_n(2u) \leq k\varphi_n(u) \tag{1}$$

for each $n \in \mathbb{N}$ and every $u \in [c_n, 1]$ (see [13] and [3]).

We say that φ *satisfies condition* $(*)$ if for every $\epsilon \in (0,1)$ there exists $\delta > 0$ such that $\varphi_n(u) < 1 - \epsilon$ implies $\varphi_n((1+\delta)u) \leq 1$ for all $u \in \mathbb{R}$ and every $n \in \mathbb{N}$.

Now, let us define the type of spaces to be considered in this paper. For a real Banach space $\langle X, \|\cdot\|_X \rangle$, denote by $\mathcal{M}(\mathbb{N}, X)$, or just $\mathcal{M}(X)$, the sequences $x = (x_n)$ such that $x_n \in X$ for all $n \in \mathbb{N}$. Define on $\mathcal{M}(X)$ a modular $\widetilde{I_\varphi}(x)$ by the formula

$$\widetilde{I_\varphi}(x) = \sum_{n=1}^{\infty} \varphi_n\left(\|x_n\|_X\right).$$

Let

$$l_\varphi(X) = \left\{ x \in \mathcal{M}(X) : x_0 = \left(\|x_n\|_X\right)_{n=1}^{\infty} \in l_\varphi \right\}.$$

Then $l_\varphi(X)$ equipped with the norm $\|x\| = \|x_0\|_\varphi$ becomes a Banach space which is called *a Musielak-Orlicz sequence space of Bochner type.*

A linear normed space X is called *P-convex* if there exist $\epsilon > 0$ and $n \in \mathbb{N}$ such that for all $x_1, x_2, ... x_n \in S(X)$

$$\min_{i \neq j} \|x_i - x_j\|_X \leq 2(1 - \epsilon),$$

where $S(X)$ denotes the unit sphere of X. The notion of P-convexity in Banach spaces can be characterized by the following lemma.

Lemma 1

A Banach space X is P-convex iff there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that for any elements $x_1, x_2, ..., x_{n_0} \in X \setminus \{0\}$ integers i_0, j_0 can be found such that

$$\left\| \frac{x_{i_0} - x_{j_0}}{2} \right\|_X \le \frac{\|x_{i_0}\|_X + \|x_{j_0}\|_X}{2} \left(1 - \frac{2\delta \min\left\{ \|x_{i_0}\|_X, \|x_{j_0}\|_X \right\}}{\|x_{i_0}\|_X + \|x_{j_0}\|_X} \right).$$

Proof. Suppose that X is P-convex. Then there exist $\delta > 0$ and $n \in \mathbb{N}$ such that for all $x_1, x_2, ..., x_{n_0} \in X \setminus \{0\}$ natural numbers i_0, j_0 can be found such that

$$\frac{1}{2} \left\| \frac{x_{i_0}}{\|x_{i_0}\|_X} - \frac{x_{j_0}}{\|x_{j_0}\|_X} \right\|_X < 1 - \delta.$$

We may assume without loss of generality that $\|x_{i_0}\| \ge \|x_{j_0}\|$. We have

$$1 - \delta > \frac{1}{2} \left\| \frac{x_{i_0}}{\|x_{i_0}\|_X} - \frac{x_{j_0}}{\|x_{j_0}\|_X} \right\|_X = \left\| \frac{x_{i_0} - x_{j_0}}{2\|x_{j_0}\|_X} + \left(\frac{1}{\|x_{i_0}\|_X} - \frac{1}{\|x_{j_0}\|_X} \right) \frac{x_{i_0}}{2} \right\|_X$$

$$\ge \frac{1}{\|x_{j_0}\|_X} \left\| \frac{x_{i_0} - x_{j_0}}{2} \right\|_X - \frac{1}{2} \|x_{i_0}\|_X \left| \frac{1}{\|x_{i_0}\|_X} - \frac{1}{\|x_{j_0}\|_X} \right|.$$

Hence

$$\frac{1}{\|x_{j_0}\|_X} \left\| \frac{x_{i_0} - x_{j_0}}{2} \right\|_X \le 1 - \delta + \frac{1}{2} \|x_{i_0}\|_X \left| \frac{1}{\|x_{i_0}\|_X} - \frac{1}{\|x_{j_0}\|_X} \right|$$

$$= \frac{1}{2} - \delta + \frac{1}{2} \frac{\|x_{i_0}\|_X}{\|x_{j_0}\|_X} = -\delta + \frac{1}{2} \left(1 + \frac{\|x_{i_0}\|_X}{\|x_{j_0}\|_X} \right)$$

$$= -\delta + \frac{1}{2} \frac{\|x_{i_0}\|_X + \|x_{j_0}\|_X}{\|x_{j_0}\|_X},$$

and finally

$$\left\| \frac{x_{i_0} - x_{j_0}}{2} \right\|_X \le \frac{\|x_{i_0}\|_X + \|x_{j_0}\|_X}{2} \left(1 - \frac{2\delta \|x_{j_0}\|_X}{\|x_{i_0}\|_X + \|x_{j_0}\|_X} \right).$$

Since the converse implication follows immediately by the definition of P-convexity, the proof is finished. □

Although the above lemma was proved in [17], we presented it here for the sake of convenience.

II. Results

Lemma 2

If φ satisfies the δ_2-condition, then for every $\alpha \in (0,1)$ there exists a non-decreasing sequence (A_m^α) of finite subsets of $\mathbb{N}$ such that

$$\bigcup_{m=1}^{\infty} A_m^\alpha = \mathbb{N}$$

and for every $m \in \mathbb{N}$ a number $k_m^\alpha > 2$ can be found such that

$$\varphi_n(2u) \le k_m^\alpha \varphi_n(u) \tag{2}$$

for each $n \in A_m^\alpha$ and every $u \in [\alpha c_n, 1]$, where c_n are from (1).

Proof. Fix $\alpha \in (0,1)$. Denote $B_m = \{0, 1, ..., m\}$ and

$$C_m^\alpha = \left\{ n \in \mathbb{N} : \frac{1}{m} \le \alpha c_n \le c_n \right\} \quad (m = 1, 2, ...) \ .$$

Obviously, $C_m^\alpha \subset C_{m+1}^\alpha$ for every $m \in \mathbb{N}$ and $\bigcup_{m=1}^\infty C_m^\alpha = \mathbb{N}$. Define $A_m^\alpha = B_m \cap C_m^\alpha$ for every $m \in \mathbb{N}$. Then $A_m^\alpha \subset A_{m+1}^\alpha$ and $\operatorname{card} A_m^\alpha < \infty$ for every $m \in \mathbb{N}$. Moreover $\bigcup_{m=1}^\infty A_m^\alpha = \mathbb{N}$. Denote

$$k_m^\alpha = \frac{k}{\min\limits_{n \in A_m^\alpha} \varphi_n(\frac{1}{m})} \qquad (m = 1, 2, ...) \, ,$$

where k is from (1). Since φ_n $(n = 1, 2, ...)$ vanishes only at zero, $k < k_m^\alpha < \infty$ for $m \in \mathbb{N}$. Suppose that $n \in A_m^\alpha$. Then

$$\varphi_n(2u) \le \varphi_n(2c_n) \le k\varphi_n(c_n) \frac{\varphi_n(\alpha c_n)}{\varphi_n(\alpha c_n)}$$

$$\le k\varphi_n(1) \frac{\varphi_n(u)}{\varphi_n\left(\frac{1}{m}\right)} \le k_m^\alpha \varphi_n(u)$$

for $u \in [\alpha c_n, c_n]$ and

$$\varphi_n(2u) \le k\varphi_n(u) \le k_m^\alpha \varphi_n(u)$$

for $u \in [c_n, 1]$. Hence the proof is finished. □

Lemma 3

If φ and φ^* satisfy the δ_2-condition, then for every $\epsilon \in (0,1)$ there exist a number $\eta_\epsilon > 1$ and a sequence $d = (d_n)$ of positive real numbers with $\sum_{n=1}^\infty \varphi_n(d_n) < \epsilon$ such that

$$\varphi_n\left(\frac{\eta_\epsilon}{2} u\right) \le \frac{1}{2\eta_\epsilon} \varphi_n(u)$$

for each $n \in \mathbb{N}$ and every $u \in [d_n, 1]$.

Proof. We will apply the methods from [3] and [9]. Take an arbitrary $\epsilon > 0$. Since $\varphi_n(1) = 1$, $\varphi_n(u) = u^2$ for each $n \in \mathbb{N}$ and all $|u| > 1$, and φ_n is convex on $[-1, 1]$ for all $n \in \mathbb{N}$, using (17) from [3] and applying Lemma 2 from [3] we can find $b_0 > 1$ such that $\varphi_n^*(b_0 u) \leq 2\varphi_n^*(u)$ for each $n \in \mathbb{N}$ and every $|u| \geq b_n$, where φ_n^* is complementary to φ_n in the sense of Young and b_n is such that $\sum_{n=1}^{\infty} \varphi_n^*(b_n) \leq \epsilon/2$. The existence of such a sequence (b_n) follows by $\varphi^* \in \delta_2$. Furthermore a number $b \in (1, 2)$ can be found small enough to satisfy $\varphi_n^*(b^2 u) \leq 2b\varphi_n^*(u)$ for each $n \in \mathbb{N}$ and every $|u| \geq b_n$. Hence

$$\varphi_n^*(b^2 v) \leq 2b\varphi_n^*(v) + 2b\varphi_n^*(b_n)$$

for each $n \in \mathbb{N}$ and every $v \in \mathbb{R}$. Then

$$\begin{aligned}
\varphi_n\left(\frac{b}{2}u\right) &= \sup_{v \geq 0}\left\{\frac{b}{2}|u|\,v - \varphi_n^*(v)\right\} \\
&= \sup_{v \geq 0}\left\{\frac{b}{2}|u|\,v - \frac{1}{2b}\varphi_n^*(b^2 v)\right\} + \varphi_n^*(b_n) \\
&= \sup_{v \geq 0}\left\{\frac{1}{2b}|u|\,b^2 v - \frac{1}{2b}\varphi_n^*(b^2 v)\right\} + \varphi_n^*(b_n) = \frac{1}{2b}\varphi_n(u) + \varphi_n^*(b_n)
\end{aligned}$$

for every $u \in \mathbb{R}$ and $n \in \mathbb{N}$. Now take the sequence (w_n) such that $\varphi_n(w_n) = \varphi_n^*(b_n)$. Define

$$\overline{c}_n = \frac{2b}{\sqrt{b} - 1}\, w_n$$

for every $n \in \mathbb{N}$. Put $\xi = \sqrt{b}$. Since $\varphi_n(\alpha u) \leq \alpha\varphi_n(u)$ for each $n \in \mathbb{N}$, $u \in \mathbb{R}$ and $\alpha \in [0, 1]$, we get

$$\varphi_n\left(\frac{\xi}{2}u\right) \leq \varphi_n\left(\frac{b}{2}u\right) \leq \frac{1}{2b}\varphi_n(u) + \varphi_n\left(\frac{\sqrt{b} - 1}{2b}u\right) \leq \frac{1}{2\xi}\varphi_n(u)$$

for $u \geq \overline{c}_n$ and $n \in \mathbb{N}$.

By $\varphi \in \delta_2$, we have $I_\varphi(\overline{c}) < \infty$. Hence we can find a number $\lambda > 0$ such that $I_\varphi(\lambda\overline{c}) < \epsilon/2$. Denote

$$A_k = \left\{n \in \mathbb{N} : \sup_{\lambda d_n \leq u \leq d_n} \frac{2(1 + \frac{1}{k})\varphi_n\left(\frac{1}{2}(1 + \frac{1}{k})u\right)}{\varphi_n(u)} \leq 1\right\}.$$

Since $\lim_{u \to 0} u^{-1}\varphi_n(u) = 0$ for every $n \in \mathbb{N}$, φ_n $(n \in \mathbb{N})$ is linear in no neighborhood of 0, and consequently $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$. Then there exists a number $l \in \mathbb{N}$ such that

$$\sum_{n \in \mathbb{N} \setminus A_l} \varphi_n(\overline{c}_n) < \frac{\epsilon}{2}\,.$$

Define

$$d_n = \begin{cases} \lambda \overline{c}_n & \text{for } n \in A_l \\ \overline{c}_n & \text{for } n \in \mathbb{N} \setminus A_l. \end{cases}$$

We get $I_\varphi(d) < \epsilon$. Taking $\eta_\epsilon = \min\{\xi, 1 + 1/l\}$, we obtain

$$\varphi_n\left(\frac{\eta_\epsilon}{2}u\right) \le \frac{1}{2\eta_\epsilon}\varphi_n(u)$$

for each $n \in \mathbb{N}$ and $u \ge d_n$ which finishes the proof. □

Lemma 4

If φ and φ^* satisfy the δ_2-condition, then for every $\epsilon \in (0,1)$ there are numbers $a = a(\epsilon) \in (0,1)$ and $\gamma = \gamma(a(\epsilon)) \in (0,1)$ such that

$$\varphi_n\left(\frac{u+v}{2}\right) \le \frac{1-\gamma}{2}\left(\varphi_n(u) + \varphi_n(v)\right) \tag{3}$$

for each $n \in \mathbb{N}$, every $u \in [d_n, 1]$ and $\left|\frac{v}{u}\right| < a$, where $d = (d_n)$ is from Lemma 3.

Proof. In the case of the Δ_2-condition for all $u \in \mathbb{R}$ the thesis of the lemma was proved for all $u, v \in \mathbb{R}$ with $\left|\frac{v}{u}\right| < a$ in Example 1.7 from [4]. The same method is applicable in our situation. Fix $\epsilon \in (0,1)$. Let $d = (d_n)$ and η_ϵ be as in Lemma 3. Then there exists a number $a \in (0,1)$ such that $1 + a \le \eta_\epsilon$. Taking $\gamma = a/a + 1$, we get $\gamma \in (0,1)$ and

$$\varphi_n\left(\frac{1+a}{2}u\right) \le \frac{1}{2(1+a)}\varphi_n(u) \le \frac{1}{2(1+a)}\left(\varphi_n(u) + \varphi_n(au)\right)$$
$$= \frac{1}{2}(1-\gamma)\left(\varphi_n(u) + \varphi_n(au)\right)$$

for each $n \in \mathbb{N}$ and every $u \in [d_n, 1]$. Since for every $u > 0$ the function

$$f(a) = 2\Phi\left(\frac{u+au}{2}\right) / (\Phi(u) + \Phi(au))$$

is nonincreasing, the above inequality holds true for every $a_0 < a$ with the same γ. Hence we obtain the thesis. □

Theorem 1

Let φ be the Musielak-Orlicz function satisfying condition $(*)$ and $(X, \|\cdot\|_X)$ be the Banach space. Then the following statements are equivalent:

(a) The Musielak-Orlicz sequence space $l_\varphi(X)$ of Bochner type is P-convex.

(b) Both l_φ and $(X, \|\cdot\|_X)$ are P-convex.

(c) l_φ is reflexive and $(X, \|\cdot\|_X)$ is P-convex.

(d) $(X, \|\cdot\|_X)$ is P-convex, $\varphi \in \delta_2$ and $\varphi^* \in \delta_2$.

Proof. (a)⇒ (b). Since the spaces l_φ and X are embedded isometrically into $l_\varphi(X)$ and P-convexity is inherited by subspaces, $L_\Phi(\mu)$ and X are P-convex.

(b)⇒ (c). Every P-convex Banach space is reflexive (see Theorem 3.2 in [18]). Hence l_φ is reflexive.

(c)⇒ (d). The reflexivity of Musielak-Orlicz sequence space l_φ is equivalent to the fact that $\varphi \in \delta_2$ and $\varphi^* \in \delta_2$ (see [8] and [12]).

(d)⇒(a). Suppose that $\varphi \in \delta_2$, $\varphi^* \in \delta_2$ and $(X, \|\cdot\|_X)$ is P-convex. Let n_0 be a natural number from Lemma 1. Define

$$z_n = \max\{c_n, d_n\}$$

for every $n \in \mathbb{N}$, where c_n, d_n are from (1) and Lemma 3, respectively, and they correspond to $\epsilon = \frac{1}{4n_0}$. Hence

$$\sum_{n=1}^{\infty} \varphi(z_n) \le \sum_{n=1}^{\infty} \varphi(c_n) + \sum_{n=1}^{\infty} \varphi(d_n) < \frac{1}{2n_0}\,.$$

Let a be the number from Lemma 4 with z_n in place of d_n and let (A_m^a) be the ascending sequence of finite sets from Lemma 2 corresponding to $\alpha = a$ and z_n in place of c_n. Since $\varphi \in \delta_2$, $I_\varphi\left(\frac{1}{a}z\right) < \infty$. Then there exists a natural number m_0 such that

$$\sum_{n\in\mathbb{N}\setminus A_{m_0}^a} \varphi_n\left(\frac{1}{a}z_n\right) < \frac{1}{2n_0}. \tag{4}$$

We have

$$\varphi_n(2u) \le k_{m_0}^a \varphi_n(u)$$

for every $n \in A_{m_0}^a$ and $u \in [az_n, 1]$. Moreover, taking $p \in \mathbb{N}$ such that $1/a \le 2^p$ and applying p times inequality (2), we obtain

$$\varphi_n\left(\frac{1}{a}u\right) \le (k_{m_0}^a)^p \varphi_n(u)$$

for every $n \in A_{m_0}^a$ and $u \in [az_n, 1]$. Setting now $\frac{1}{a}u = v$ and $\frac{1}{(k_{m_0}^a)^p} = \beta_{m_0}$, we get

$$\varphi_n(av) \ge \beta_{m_0}\varphi_n(v) \tag{5}$$

for every $n \in A_{m_0}^a$ and $v \in [z_n, 1]$. Furthermore, if $z_n/a < 1$ for some $n \in \mathbb{N}$, repeating a similar argumentation, we obtain

$$\varphi_n(2u) \le k\varphi_n(u)$$

and

$$\varphi_n\left(\frac{1}{a}u\right) \leq k^p \varphi_n(u)$$

for every $u \in [z_n, 1]$. Consequently

$$\varphi_n(av) \geq \beta \varphi_n(v) \tag{6}$$

for every $v \in [\frac{z_n}{a}, 1]$, where $1/k^p = \beta$.

Now, we will show that there exists a number $r_1 \in (0,1)$ such that for every $x^1, x^2, ..., x^{n_0}$ from the unit ball $B(X)$, we have

$$\sum_{i=1}^{n_0}\sum_{j=i}^{n_0} \varphi_n\left(\left\|\frac{x^i - x^j}{2}\right\|_X\right) \leq \frac{n_0 - 1}{2} r_1 \sum_{i=1}^{n_0} \varphi_n\left(\left\|x^i\right\|_X\right) \tag{7}$$

for every $n \in \mathbb{N}$ such that $\max_{1\leq i\leq n_0}\left\{\left\|x^i\right\|_X\right\} \geq z_n/a$.

Take $x^1, x^2, ..., x^{n_0} \in B(X)$. Let k be an index such that

$$\left\|x^k\right\|_X = \max_{1\leq i\leq n_0}\left\{\left\|x^i\right\|_X\right\}.$$

For the clarity of the proof, we will divide it into two parts.

I. Suppose that there exists $i_1 \in \{1, 2, ..., n_0\}$ such that $\left\|x^{i_1}\right\|_X / \left\|x^k\right\|_X < a$. Since $\left\|x^k\right\| \geq z_n/a \geq z_n$, by inequality (3), we have

$$\begin{aligned}\varphi_n\left(\left\|\frac{x^{i_1} - x^k}{2}\right\|_X\right) &\leq \varphi_n\left(\frac{\left\|x^{i_1}\right\|_X + \left\|x^k\right\|_X}{2}\right) \\ &\leq \frac{1}{2}(1-\gamma)\left(\varphi_n\left(\left\|x^{i_1}\right\|_X\right) + \varphi_n\left(\left\|x^k\right\|_X\right)\right).\end{aligned}$$

Hence, by the convexity of φ_n ($n \in \mathbb{N}$) on the interval $[0,1]$, we get

$$\begin{aligned}&\sum_{i=1}^{n_0}\sum_{j=i}^{n_0} \varphi_n\left(\left\|\frac{x^i - x^j}{2}\right\|_X\right) \\ &\leq \frac{n_0-1}{2}\sum_{i=1}^{n_0}\varphi_n\left(\left\|x^i\right\|_X\right) - \frac{\gamma}{2}\left(\varphi_n\left(\left\|x^{i_1}\right\|_X\right) + \varphi_n\left(\left\|x^k\right\|_X\right)\right) \\ &\leq \frac{n_0-1}{2}\sum_{i=1}^{n_0}\varphi_n\left(\left\|x^i\right\|_X\right) - \frac{\gamma}{2n_0}\left(n_0\varphi_n\left(\left\|x^k\right\|_X\right)\right) \\ &\leq \frac{n_0-1}{2}\sum_{i=1}^{n_0}\varphi_n\left(\left\|x^i\right\|_X\right) - \frac{\gamma}{2n_0}\sum_{i=1}^{n_0}\varphi_n\left(\left\|x^i\right\|_X\right) \\ &= \frac{n_0-1}{2}\left(1 - \frac{\gamma}{n_0(n_0-1)}\right)\sum_{i=1}^{n_0}\varphi_n\left(\left\|x^i\right\|_X\right)\end{aligned} \tag{8}$$

for every $n \in \mathbb{N}$ such that the inequality $\max_{1\leq i\leq n_0}\{\left\|x^i\right\|_X\} \geq z_n/a$ holds true.

II. Assume that for all $i \neq k$ we have

$$\frac{\|x^i\|_X}{\|x^k\|_X} \geq a\,. \tag{9}$$

Then $\|x^i\| > 0$ for every $i \neq k$. Let i_0, j_0 be from Lemma 1. We may assume that

$$a \leq \frac{\|x^{i_0}\|_X}{\|x^{j_0}\|_X} \leq \frac{1}{a}. \tag{10}$$

Really, otherwise we have

$$a > \frac{\|x^{i_0}\|_X}{\|x^{j_0}\|_X} \geq \frac{\min\left\{\|x^{i_0}\|_X, \|x^{j_0}\|_X\right\}}{\max\left\{\|x^{i_0}\|_X, \|x^{j_0}\|_X\right\}} \geq \frac{\min\left\{\|x^{i_0}\|_X, \|x^{j_0}\|_X\right\}}{\|x^k\|_X},$$

which contradicts inequality (9). Hence applying Lemma 1 and inequality (10), we get

$$\begin{aligned}\left\|\frac{x^{i_0}-x^{j_0}}{2}\right\|_X &\leq \frac{\|x^{i_0}\|_X + \|x^{j_0}\|_X}{2}\left(1 - \frac{2\delta\min\left\{\|x^{i_0}\|_X, \|x^{j_0}\|_X\right\}}{\|x^{i_0}\|_X + \|x^{j_0}\|_X}\right)\\ &\leq \left(1-\frac{2\delta a}{1+a}\right)\frac{\|x^{i_0}\|_X + \|x^{j_0}\|_X}{2}\,.\end{aligned}$$

Therefore, by the convexity of each φ_n on $[0,1]$, we obtain

$$\varphi_n\left(\left\|\frac{x^{i_0}-x^{j_0}}{2}\right\|_X\right) \leq \frac{1}{2}(1-\alpha)\left(\varphi_n\left(\|x^{i_0}\|_X\right) + \varphi_n\left(\|x^{j_0}\|_X\right)\right), \tag{11}$$

where $\alpha = \frac{2\delta a}{1+a} \in (0,1)$. Consequently, by inequalities (11) and (6), we have

$$\begin{aligned}&\sum_{i=1}^{n_0}\sum_{j=i}^{n_0}\varphi_n\left(\left\|\frac{x^i-x^j}{2}\right\|_X\right)\\ &\leq \frac{n_0-1}{2}\sum_{i=1}^{n_0}\varphi_n\left(\|x^i\|_X\right) - \frac{\alpha}{2}\left(\varphi_n\left(\|x^{i_0}\|_X\right) + \varphi_n\left(\|x^{j_0}\|_X\right)\right)\\ &\leq \frac{n_0-1}{2}\sum_{i=1}^{n_0}\varphi_n\left(\|x^i\|_X\right) - \alpha\varphi_n\left(a\|x^k\|_X\right)\\ &\leq \frac{n_0-1}{2}\sum_{i=1}^{n_0}\varphi_n\left(\|x^i\|_X\right) - \frac{\alpha\beta}{n_0}\left(n_0\varphi_n\left(\|x^k\|_X\right)\right)\\ &\leq \frac{n_0-1}{2}\sum_{i=1}^{n_0}\varphi_n\left(\|x^i\|_X\right) - \frac{\alpha\beta}{n_0}\sum_{i=1}^{n_0}\varphi_n\left(\|x^i\|_X\right)\\ &= \frac{n_0-1}{2}\left(1-\frac{2\alpha\beta}{n_0(n_0-1)}\right)\sum_{i=1}^{n_0}\varphi_n\left(\|x^i\|_X\right) \end{aligned} \tag{12}$$

for every $n \in \mathbb{N}$ satisfying $\max_{1\le i\le n_0}\{\|x^i\|_X\} \ge z_n/a$. Define

$$r_1 = \max\left\{1 - \frac{\gamma}{n_0(n_0-1)}, 1 - \frac{2\alpha\beta}{n_0(n_0-1)}\right\}.$$

Combining inequalities (8) and (12), we get inequality (7). Repeating the same argumentation as in the proof of inequality (7), a number $r_2 \in (0,1)$ can be found such that

$$\sum_{i=1}^{n_0}\sum_{j=i}^{n_0} \varphi_n\left(\left\|\frac{x^i - x^j}{2}\right\|_X\right) \le \frac{n_0 - 1}{2} r_2 \sum_{i=1}^{n_0} \varphi_n\left(\|x^i\|_X\right) \tag{13}$$

for every $n \in A_{m_0}^{\alpha}$ satisfying $\max_{1\le i\le n_0}\{\|x^i\|_X\} \ge z_n$.

Let $\widetilde{x}^1, \widetilde{x}^2, ..., \widetilde{x}^{n_0} \in S(l_\varphi(X))$. Taking $\widetilde{x}^i = (x_n^i)$ for $i = 1, 2, ...n_0$, define

$$\mathcal{I} = \left\{n \in \mathbb{N} : \sum_{i=1}^{n_0} \varphi_n\left(\|x_n^i\|_X\right) \ge n_0 \varphi_n(z_n)\right\}.$$

Obviously

$$\max_{1\le i\le n_0}\left\{\|x_n^i\|_X\right\} \ge z_n$$

for every $n \in \mathcal{I}$. Decompose the set $\mathcal{I}$ into the following sets

$$\mathcal{I}_1 = \left\{n \in \mathcal{I} : \max_{1\le i\le n_0}\left\{\|x_n^i\|_X\right\} \ge \frac{z_n}{a}\right\},$$

$$\mathcal{I}_2 = \left\{n \in \mathcal{I} : z_n \le \max_{1\le i\le n_0}\left\{\|x_n^k\|_X\right\} < \frac{z_n}{a}\right\}.$$

Next divide the set $\mathcal{I}_2$ into two subsets $\mathcal{I}_{21}$ and $\mathcal{I}_{22}$ defined by

$$\mathcal{I}_{21} = \mathcal{I}_2 \cap A_{m_0}^a \quad\text{and}\quad \mathcal{I}_{22} = \mathcal{I}_2 \setminus A_{m_0}^a.$$

By inequalities (7) and (13), we have

$$\sum_{i=1}^{n_0}\sum_{j=i}^{n_0} \varphi_n\left(\left\|\frac{x_n^i - x_n^j}{2}\right\|_X\right) \le \frac{1}{n_0}\binom{n_0}{2} r \sum_{i=1}^{n_0} \varphi_n\left(\|x_n^i\|_X\right) \tag{14}$$

for every $n \in \mathcal{I}_1 \cup \mathcal{I}_{21}$, where $r = \{r_1, r_2\}$. Moreover, by the definitions of the set $\mathcal{I}$ and the sequence (z_n), we have

$$\sum_{n\in\mathbb{N}\setminus\mathcal{I}} \sum_{i=1}^{n_0} \varphi_n\left(\|x_n^i\|_X\right) < \frac{1}{2}. \tag{15}$$

Now, let $n \in \mathcal{I}_{22}$. It follows, by inequality (4) that

$$\begin{aligned}\sum_{n\in\mathcal{I}_{22}}\sum_{i=1}^{n_0}\varphi_n\left(\left\|x_n^i\right\|_X\right) &= \sum_{n\in\mathcal{I}_2\setminus A_{m_0}^a}\sum_{i=1}^{n_0}\varphi_n\left(\left\|x_n^i\right\|_X\right) \\ &\le \sum_{n\in\mathcal{I}_2\setminus A_{m_0}^a} n_0\varphi_n\left(\left\|x_n^k\right\|_X\right) < \sum_{n\in\mathcal{I}_2\setminus A_{m_0}^a} n_0\varphi_n\left(\frac{z_n}{a}\right) \\ &\le \sum_{n\in\mathbb{N}\setminus A_{m_0}^a} n_0\varphi_n\left(\frac{z_n}{a}\right) < \frac{1}{2}\,. \qquad (16)\end{aligned}$$

Hence, by inequalities (15) and (16) we get

$$\begin{aligned}&\sum_{n\in\mathbb{N}\setminus(\mathcal{I}_1\cup\mathcal{I}_{21})}\sum_{i=1}^{n_0}\varphi_n\left(\left\|x_n^i\right\|_X\right) \\ &= \sum_{n\in\mathbb{N}\setminus\mathcal{I}}\sum_{i=1}^{n_0}\varphi_n\left(\left\|x_n^i\right\|_X\right) + \sum_{n\in\mathcal{I}_{22}}\sum_{i=1}^{n_0}\varphi_n\left(\left\|x_n^i\right\|_X\right) < 1\,.\end{aligned}$$

Since $\left\|\widetilde{x}^i\right\| = 1$ for $i = 1, 2, ...n_0$ and $\varphi \in \delta_2$, $\widetilde{I_\varphi}\left(\widetilde{x}^i\right) = 1$, for $i = 1, 2, ..., n_0$. Consequently

$$\sum_{n\in\mathcal{I}_1\cup\mathcal{I}_{21}}\sum_{i=1}^{n_0}\varphi_n\left(\left\|x_n^i\right\|_X\right) \ge n_0 - 1\,. \qquad (17)$$

Therefore, by inequalities (14) and (17), we have

$$\begin{aligned}&\sum_{i=1}^{n_0}\sum_{j=i}^{n_0}\widetilde{I_\varphi}\left(\frac{1}{2}\left(x^i - x^j\right)\right) \\ &= \sum_{i=1}^{n_0}\sum_{j=i}^{n_0}\sum_{n\in\mathcal{I}_1\cup\mathcal{I}_{21}}\varphi_n\left(\left\|\frac{x_n^i - x_n^j}{2}\right\|_X\right) + \sum_{i=1}^{n_0}\sum_{j=i}^{n_0}\sum_{n\in\mathbb{N}\setminus(\mathcal{I}_1\cup\mathcal{I}_{21})}\varphi_n\left(\left\|\frac{x_n^i - x_n^j}{2}\right\|_X\right) \\ &\le \frac{1}{n_0}\binom{n_0}{2} r\sum_{i=1}^{n_0}\sum_{n\in\mathcal{I}_1\cup\mathcal{I}_{21}}\varphi_n\left(\left\|x_n^i\right\|_X\right) + \frac{1}{n_0}\binom{n_0}{2}\sum_{i=1}^{n_0}\sum_{n\in\mathbb{N}\setminus(\mathcal{I}_1\cup\mathcal{I}_{21})}\varphi_n\left(\left\|x_n^i\right\|_X\right) \\ &= \frac{1}{n_0}\binom{n_0}{2} r\sum_{i=1}^{n_0}\sum_{n\in\mathcal{I}_1\cup\mathcal{I}_{21}}\varphi_n\left(\left\|x_n^i\right\|_X\right) \\ &\quad + \frac{1}{n_0}\binom{n_0}{2}\left(\sum_{i=1}^{n_0}\widetilde{I_\varphi}\left(x^i\right) - \sum_{i=1}^{n_0}\sum_{n\in\mathcal{I}_1\cup\mathcal{I}_{21}}\varphi_n\left(\left\|x_n^i\right\|_X\right)\right)\end{aligned}$$

$$= \binom{n_0}{2} \left(1 - \frac{1-r}{n_0} \sum_{i=1}^{n_0} \sum_{n \in \mathcal{I}_1 \cup \mathcal{I}_{21}} \varphi_n \left(\left\| x_n^i \right\|_X \right) \right)$$

$$\leq \binom{n_0}{2} \left(1 - \frac{(1-r)(n_0-1)}{n_0} \right).$$

Finally,

$$\sum_{i=1}^{n_0} \sum_{j=i}^{n_0} \widetilde{I_\varphi} \left(\frac{1}{2} \left(x^i - x^j \right) \right) \leq \binom{n_0}{2} (1-p),$$

where $p = \frac{(1-r)}{2}$. So there exist i_1, j_1 such that

$$\widetilde{I_\varphi} \left(\frac{1}{2} \left(x^{i_1} - x^{j_1} \right) \right) \leq 1 - p.$$

Hence, by δ_2-condition and $(*)$ (see [13], Lemma 9), we get

$$\left\| \frac{1}{2} \left(x^{i_1} - x^{j_1} \right) \right\| \leq 1 - q(p), \qquad 0 < q(p) < 1,$$

i.e. $l_\varphi(X)$ is $P-$convex. □

Corollary 1

The Lebesgue-Bochner sequence space l^p $(1 < p < \infty)$ is P-convex iff X is P-convex.

Proof. The Lebesgue space l^p is a Musielak-Orlicz space generated by the Orlicz function $\varphi_n(u) = |u|^p$ for every $n \in \mathbb{N}$ satisfying all the assumptions of Theorem 1. □

The following characterization of P-convexity, proved directly in [22] in a long way, is an immediate consequence of Theorem 1.

Corollary 2

The following statements are equivalent:

(a) l_φ is P-convex.
(b) l_φ is reflexive.
(c) $\varphi \in \delta_2$ and $\varphi^* \in \delta_2$.

Proof. It is enough to apply Theorem 1 with $X = \mathbb{R}$. □

The authors wish the thank Professor H. Hudzik for helpful comments.

References

1. G. Alherk and H. Hudzik, Uniformly non-l_n^1 Musielak-Orlicz spaces of Bochner type, *Forum Math.* **1** (1989), 403–410.
2. D. Amir and C. Franchetti, The radius ratio and convexity properties in normed linear spaces, *Trans. Amer. Math. Soc.* **282** (1984), 275–291.
3. M. Denker and H. Hudzik, Uniformly non-l_n^1 Musielak-Orlicz sequence spaces, *Proc. Indian Acad. Sci. (Math. Sci.)* **101** (1991), 71–86.
4. S. Chen and H. Hudzik, On some convexities of Orlicz and Orlicz-Bochner spaces, *Comment. Math. Univ. Carolin.* **29.1** (1988).
5. D.P. Giesy, On a convexity condition in normed linear spaces, *Trans. Amer. Math. Soc.* **125** (1966), 114–146.
6. H. Hudzik, Uniformly non-$l_n^{(1)}$ Orlicz spaces with Luxemburg norm, *Studia Math.* **81** (1985), 271–283.
7. H. Hudzik, Some classes of uniformly non-square Orlicz-Bochner spaces, *Comment. Math. Univ. Carolin.* **26** (1985), 269–274.
8. H. Hudzik and A. Kamińska, On uniformly convexifiable and B-convex Musielak-Orlicz spaces, *Comment. Math.* **25** (1985), 59–75.
9. H. Hudzik, A. Kamińska and W. Kurc, Uniformly non-$l_n^{(1)}$ Musielak-Orlicz spaces, *Bull. Polish Acad. Sci. Math.* **35** 7–8 (1987), 441–448.
10. R.C. James, Uniformly non-square Banach spaces, *Ann. of Math.* **2** (1964), 542–550.
11. R.C. James, A nonreflexive Banach space that is uniformly nonoctathedral, *Israel J. Math.* **18** (1974), 145–155.
12. A. Kamińska, Flat Orlicz-Musielak sequence spaces, *Bull. Polish Acad. Sci. Math.* **30** 7–8 (1982), 347–352.
13. A. Kamińska, Uniform rotundity of Musielak-Orlicz sequence spaces, *J. Approx. Theory* **47** (1986), 302–322.
14. A. Kamińska, Rotundity of Orlicz-Musielak sequence spaces, *Bull. Polish Acad. Sci. Math.* **29** 3–4 (1981), 137–144.
15. A. Kamińska and B. Turett, Uniformly non-$l_n^{(1)}$ Orlicz-Bochner spaces, *Bull. Polish Acad. Sci. Math.* **35** 3–4 (1987), 211–218.
16. P. Kolwicz and R. Pł uciennik, On P-convex Musielak-Orlicz spaces, *Comment. Math. Univ. Carolin.* **36** 4 (1995), 655–672.
17. P. Kolwicz and R. Płuciennik, *P-convexity of Orlicz-Bochner spaces,* to appear.
18. C.A. Kottman, Packing and reflexivity in Banach spaces, *Trans. Amer. Math. Soc.* **150** (1970), 565–576.
19. I.E. Leonard, Banach sequence spaces, *J. Math. Anal. Appl.* **54** (1976), 245–265.
20. J. Musielak, Orlicz Spaces and Modular Spaces, *Lecture Notes in Math.* **1034** (1983), 1–222.
21. Ye Yining, He Miaohong and R. Pł uciennik, P-convexity and reflexivity of Orlicz spaces, *Comment. Math.* **31** (1991), 203–216.
22. Ye Yining and Huang Yafeng, P-convexity property in Musielak-Orlicz sequence spaces, *Collect. Math.* **44** (1993), 307–325.