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Normal structure of Musielak-Orlicz spaces

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Abstract

We show that a Musielak-Orlicz function space L_ϕ has uniformly normal structure iff it is reflexive. We also give a criterion for normal structure of the Musielak-Orlicz sequence space ℓ_ϕ and under the assumption that ϕ is not linear around zero, a criterion for normal structure of L_ϕ.

Introduction

The concept of normal structure was introduced in 1948 by M.S. Brodskii and D.P. Milman [3] to study fixed points of isometries and it is a property shared by all uniformly convex spaces. In 1965, W.A. Kirk [10] observed that normal structure implies $C \subset X$ has the fixed point property if X is a reflexive Banach space (where C is closed, bounded and convex).

Necessary and sufficient conditions have been given for function and sequence Orlicz spaces to have (uniformly) normal structure (see [12], [5]). Also in [9] a criterion was given for Orlicz-Lorentz spaces to have uniformly normal structure.

Definition 1. Let X be a Banach space. We say that X has normal structure (NS) if each non-empty, bounded, closed, convex subset S of X with positive diameter, contains a point x such that:

$$\sup\left\{\|x-y\| : y \in S\right\} < \operatorname{diam} S := \sup\left\{\|z-y\| : z, y \in S\right\}.$$

In the case where there exists a constant $K \in (0,1)$ such that for all $x \in S$,

$$\sup \left\{ \|x-y\| : y \in S \right\} \leq K \mathrm{diam} S,$$

X is said to have uniformly normal structure (UNS).

DEFINITION 2. Let X be a Banach space and $\{x_n\}$ a sequence in X. If for any $x \in co\{x_n\}$, the convex hull of $\{x_n\}$, the limit $\Lambda(x) = \lim_{n\to\infty} \|x_n - x\| > 0$ exists and $\Lambda(x)$ is affine on $co\{x_n\}$, then $\{x_n\}$ is called a limit affine sequence.

If, in addition, $\Lambda(x)$ is constant on $co\{x_n\}$, then $\{x_n\}$ is called a limit-constant sequence [11].

DEFINITION 3. A Banach space is said to have the sum-property if it contains no non-constant limit-affine sequence $\{x_n\}$ for which $\{\Lambda(x_n)\}$ is non-decreasing [11].

Proposition 4 ([11])

A Banach space X has normal structure if and only if it contains no non-constant limit constant sequence.

It is easy to see that sum-property implies normal structure. In the case of Orlicz sequence spaces, normal structure is equivalent to sum-property [12]. We will see that this is also true for Musielak-Orlicz sequence spaces (see Theorem 12).

Let us now give some background for Musielak-Orlicz spaces. Set (T, Σ, μ) to be a non-atomic σ-finite, measure space. A non-negative, extended real-valued function $\phi : \mathbb{R}^+ \times T \longrightarrow \mathbb{R}_e^+$ is called a Young function (with parameter) if for almost all $t \in T$, $\phi(0,t) = 0$, $\phi(x,t) > 0$ for $x > 0$, $\phi(x,t)$ is convex with respect to x and for all $x \geq 0$, $\phi(x,t)$ is Σ-measurable with respect to t.

For any measurable function $f : T \longrightarrow \mathbb{R}$, we define the modular

$$\rho_\phi(f) = \int_T \phi(| f(t) |, t) d\mu \,.$$

The Musielak-Orlicz space L_ϕ is the set of all equivalence classes of measurable functions f such that $\rho_\phi(\lambda f) < \infty$, for some $\lambda > 0$.

Endowed with the Luxemburg norm defined by

$$\|f\|_\phi = \inf \left\{ r > 0 : \rho_\phi(f/r) \leq 1 \right\},$$

L_ϕ is a Banach space.

The conjugate function ϕ^*, of a Young function ϕ is defined by $\phi^*(x,t) = \sup_{y\geq 0}\{yx - \phi(y,t)\}$. It is easy to see that ϕ^* is a convex function and that $(\phi^*)^* = \phi$. We also notice that if ϕ assumes only finite values, then for a.a.$t \in T$, there exists $x > 0$ such that $\phi^*(x,t) < \infty$.

Recall that a Young function ϕ satisfies Δ_2-condition (written $\phi \in \Delta_2$) if there exists $K > 0$ and a non-negative integrable function h such that for almost all $t \in T$

$$\phi(2x,t) \leq K\phi(x,t) + h(t)$$

for all $x \geq 0$.

A function $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is called a Young function if ϕ is convex, $\phi(0) = 0$ and $\phi(x) > 0$ for $x > 0$. A sequence $\phi = \{\phi_n\}$ of Young functions, is called a Musielak-Orlicz function.

For any real sequence $x = \{x_n\}$, we define the modular

$$I_\phi(x) = \sum_{n=1}^{\infty} \phi_n(\mid x_n \mid).$$

The Musielak-Orlicz sequence space ℓ_ϕ is the set of all real sequences $x = \{x_n\}$ such that $I_\phi(\lambda x) < \infty$, for some $\lambda > 0$.

Under the Luxemburg norm defined by

$$\|x\|_\phi = \inf\left\{r > 0 : I_\phi(x/r) \leq 1\right\},$$

ℓ_ϕ is a Banach space.

We say that a Musielak-Orlicz function $\phi = \{\phi_n\}$ satisfies δ_2-condition (written $\phi \in \delta_2$) if there are positive constants K and δ and a non-negative sequence $\{c_n\}$ in ℓ_1 such that for all $n \in \mathbb{N}$ and $x \geq 0$

$$\phi_n(2x) \leq K\phi_n(x) + c_n$$

whenever $\phi_n(x) \leq \delta$.

The following are useful results connected with $\Delta_2(\delta_2)$-condition. Let $f \in L_\phi, \{f_n\} \subset L_\phi$ and $\{x_n\} \subset \ell_\phi$.

1. If $\phi \in \Delta_2(\phi \in \delta_2)$ then,

$$\|f_n\|_\phi \longrightarrow 0 \Leftrightarrow \rho_\phi(f_n) \longrightarrow 0 \tag{1}$$

(respectively: $\|x_n\|_\phi \longrightarrow 0 \Leftrightarrow I_\phi(x_n) \longrightarrow 0$).

2. If $\phi \in \Delta_2$ then for every $\epsilon > 0$ there is $\eta(\epsilon) > 0$ such that

$$\|f\|_\phi \geq 1 - \eta(\epsilon) \Rightarrow \rho_\phi(f) \geq 1 - \epsilon. \tag{2}$$

3. If $\phi \in \Delta_2, (\phi \in \delta_2)$, then

$$\|f_n\|_\phi \longrightarrow 1 \Rightarrow \rho_\phi(f_n) \longrightarrow 1 \tag{3}$$

(respectively: $\|x_n\|_\phi \longrightarrow 1 \Rightarrow I_\phi(x_n) \longrightarrow 1$).

The proofs of the above results can be found in [15] or easily derived by imitating the proofs of analogous results in Orlicz spaces.

Preliminary results

Lemma 5

Let ϕ be a Young function with (parameter). The following statements are equivalent:

1. ϕ satisfies Δ_2-condition.

2. For every $\epsilon > 0$, there are $K > 0$ and a non-negative measurable function h with $\int_T h(t)d\mu < \epsilon$ such that for almost all $t \in T$

$$\phi(2x,t) \leq K\phi(x,t) + h(t) \tag{4}$$

for all $x \geq 0$.

3. For every $\epsilon > 0$, there are $K > 0$ and a non-negative measurable function f with $\int_T \phi(f(t),t)d\mu < \epsilon$ such that for almost all $t \in T$,

$$\phi(2x,t) \leq K\phi(x,t) \tag{5}$$

for all $x \geq f(t)$.

4. For every $\epsilon > 0$, and all $a > 1$, there are $b \in (1,2)$ and a non-negative measurable function f with $\int_T \phi(f(t),t)d\mu < \epsilon$ such that for almost all $t \in T$,

$$\phi(bx,t) \leq a\phi(x,t) \tag{6}$$

for all $x \geq f(t)$.

Proof. (1) $\Leftrightarrow$ (2) see [16].

(2) $\Rightarrow$ (3). Let $f(t) = \sup\{x \geq 0 : \phi(x,t) \leq h(t)\}$. Since $\phi(x,t) < \infty$ for each $x \geq 0$ and $x \longmapsto \phi(x,t)$ is convex, we have that $x \longmapsto \phi(x,t)$ is continuous and so $\phi(f(t),t) = h(t)$. Moreover $f(t)$ is measurable. Indded let $\phi^{-1}(x,t) = \phi_t^{-1}(x)$ where ϕ_t^{-1} is an inverse function of $\phi_t(x) = \phi(x,t)$. For any $a \in \mathbb{R}^+$, $\{t\colon \phi_t^{-1}(x) \leq a\} = \{t\colon x \leq \phi_t(a)\} \in \Sigma$, by the measurability of $t \longmapsto \phi(x,t)$, for any $x \geq 0$. Thus $t \longmapsto \phi^{-1}(x,t)$ is measurable and so $f(t) = \phi^{-1}(h(t),t)$ must be measurable, too. Thus, $\int_T \phi(f(t),t)d\mu < \epsilon$ and for almost all $t \in T$ and $x \geq f(t)$,

$$\phi(2x,t) \leq K\phi(x,t) + \phi(f(t),t) \leq (K+1)\phi(x,t)\,.$$

(3) $\Rightarrow$ (4). Let $\epsilon > 0$ and $a > 1$ be given. Take $M = \max\{a, K\}$. Let $l = \frac{a-1}{M-1}$, and $b = 1 + l$. Then, $b \in (1,2)$ and so for $x \geq f(t)$ we have:

$$\begin{aligned}\phi(bx,t) \leq{}& (1-l)\phi(x,t) + l\phi(2x,t)\\ \leq{}& (1-l)\phi(x,t) + lM\phi(x,t) = a\phi(x,t)\end{aligned}$$

(4) $\Rightarrow$ (2). Suppose (4) is true. Let $\epsilon > 0$ and $a = 2$. There are $b \in (1, 2)$ and a non-negative measurable function f with $\int_T \phi(f(t), t)d\mu < \epsilon$ such that

$$\phi(bx, t) \leq 2\phi(x, t)$$

for all $x \geq f(t)$.

There exists $n \in \mathbb{N}$ such that $b^n \geq 2$. Thus for $x \geq f(t)$ we have

$$\phi(2x, t) \leq \phi(b^n x, t) \leq 2^n \phi(x, t)$$

and for $x \leq f(t)$,

$$\phi(2x, t) \leq \phi(2f(t), t) \leq 2^n \phi(f(t), t) .$$

Hence for any $x \geq 0$

$$\phi(2x, t) \leq 2^n \phi(x, t) + 2^n \phi(f(t), t) ,$$

which means Δ_2-condition and so (2). This completes the proof of the lemma. □

Lemma 6

Let ϕ be a Young function (with parameter). The following statements are equivalent.

1. ϕ satisfies Δ_2-condition.

2. For every $\epsilon > 0$ and $a > 1$, there are $b \in (4, 8), m \in \mathbb{N}$ and a nonnegative measurable function f with $\int_T \phi(f(t), t)d\mu < \epsilon$ such that for almost all $t \in T$:

$$\phi(bx, t) \leq a^m \phi(x, t) \tag{7}$$

for all $x \geq f(t)$.

Proof. (2) $\Rightarrow$ (1) Obvious.

(1) $\Rightarrow$ (2) Assume ϕ satisfies Δ_2-condition. Let $\epsilon > 0$ and $a > 1$ be given. Then by Lemma 5, if $x \geq f(t)$ then $\phi(4x, t) \leq K\phi(x, t)$ for some $K > 0$ and a non-negative measurable function f with $\int_T \phi(f(t), t)d\mu < \epsilon$. Set $M = \max\{a, K\}$. There is $m \in \mathbb{N}$ such that $m \geq 2$ and $a^m > M$. Let $l = \frac{a^m - M}{M^m - M}$ and $b = 4(1 + l)$. Then, $b \in (4, 8)$ and if we follow a similar argument as in the proof of (3) $\Rightarrow$ (4) in Lemma 5, we get that $\phi(bx, t) \leq a^m \phi(x, t)$ for $x \geq f(t)$. □

Lemma 7

Let ϕ be a Young function (with parameter). The following statements are equivalent.

1. ϕ satisfies Δ_2-condition.

2. For every $\epsilon > 0$ and $\eta \in (0, 1)$, there are $\xi \in (0, 1)$ and a non-negative measurable function h with $\int_T \phi^*(h(t), t)d\mu < \epsilon$ such that for almost all $t \in T$

$$\phi^*(\eta x, t) \leq \eta\xi\phi^*(x, t) \tag{8}$$

for all $x \geq h(t)$.

Proof. (1) $\Rightarrow$ (2). Since inequality (8) is obvious when $\phi^*(x,t) = \infty$, without loss of generality we assume that $\phi^*(x,t) < \infty$ for all x, t. Let $\epsilon > 0$ and $\eta \in (0,1)$ be given. Then, $a = \frac{1}{\eta} > 1$ and by Lemma 6 there are $b \in (4,8), m \in \mathbb{N}$ and a measurable function $g : T \longrightarrow \mathbb{R}^+$ with $\int_T \phi(g(t),t)d\mu < \epsilon\eta\frac{2-\sqrt{2}}{8\sqrt{2}}$ such that $\phi(bx,t) \leq a^m\phi(x,t)$ for all $x \geq g(t)$. If $0 \leq x \leq g(t)$ then $\phi(bx,t) \leq a^m\phi(g(t),t)$. So for $x \geq 0$, we have $\phi(bx,t) \leq a^m b[\phi(x,t) + \phi(g(t),t)]$, which implies

$$\frac{1}{a^m b}\phi(bx,t) - \phi(g(t),t) \leq \phi(x,t)\,,$$

for every $x \geq 0$.

Therefore

$$\begin{aligned}
\phi^*(\eta^m x,t) &= \sup_{y\geq 0}\left\{\eta^m xy - \phi(y,t)\right\} \\
&\leq \sup_{y\geq 0}\left\{\eta^m xy - \left[\frac{1}{a^m b}\phi(by,t) - \phi(g(t),t)\right]\right\} \\
&= \sup_{y\geq 0}\left\{\eta^m xy - \frac{\eta^m}{b}\phi(by,t)\right\} + \phi(g(t),t) \\
&= \frac{\eta}{b}\sup_{y\geq 0}\left\{\eta^{m-1}bxy - \eta^{m-1}\phi(by,t)\right\} + \phi(g(t),t) \\
&\leq \frac{\eta}{b}\sup_{y\geq 0}\left\{\eta^{m-1}byx - \phi(\eta^{m-1}by,t)\right\} + \phi(g(t),t) \\
&= \frac{\eta}{b}\phi^*(\eta^{m-1}x,t) + \phi(g(t),t)\,.
\end{aligned}$$

We can find a measurable function $h : T \longrightarrow \mathbb{R}^+$ such that

$$\phi^*(h(t),t) = \frac{b\sqrt{b}}{\eta(b-\sqrt{b})}\phi(g(t),t)$$

We have

$$\begin{aligned}
\int_T \phi^*(h(t),t)d\mu &< \frac{b\sqrt{b}}{\eta(b-\sqrt{b})}\epsilon\eta\frac{2-\sqrt{2}}{8\sqrt{2}} \\
&\leq \frac{8\sqrt{2}}{2-\sqrt{2}}\epsilon\frac{2-\sqrt{2}}{8\sqrt{2}} = \epsilon\,.
\end{aligned}$$

Now let $x \geq \frac{1}{\eta^{m-1}}h(t)$. Then, $\phi^*(\eta^{m-1}x,t) \geq \phi^*(h(t),t)$ and thus

$$\frac{\eta(b-\sqrt{b})}{b\sqrt{b}}\phi^*\left(\eta^{m-1}x,t\right) \geq \phi(g(t),t)\,.$$

Consequently, if $y \geq \frac{1}{\eta^{m-1}} h(t)$, we have

$$\begin{aligned}\phi^*(\eta^m y, t) \leq & \frac{\eta}{b}\phi^*(\eta^{m-1}y, t) + \frac{\eta(b-\sqrt{b})}{b\sqrt{b}}\phi^*(\eta^{m-1}y, t) \\ = & \ \eta\xi\phi^*(\eta^{m-1}y, t)\end{aligned}$$

where $\xi = \frac{1}{\sqrt{b}} < 1$.

Take $x \geq h(t)$. Then, $\frac{x}{\eta^{m-1}} \geq \frac{1}{\eta^{m-1}} h(t)$ and hence,

$$\phi^*(\eta x, t) \leq \eta\xi\phi^*(x, t)\,.$$

(2) $\Rightarrow$ (1). This proof is omitted since it is the same as the proof of (ii) implies (i) in Lemma 2 in [1]. $\square$

Lemma 8

Suppose that both ϕ and ϕ^* satisfy Δ_2-condition. Then for every $\epsilon > 0$, there exist a non-negative measurable function f, with $\int_T \phi(f(t), t)d\mu < \epsilon$, and constants $c \in (0,1)$ and $\gamma > 0$ such that if $\mid x \mid \geq f(t)$ and either $\mid x \mid \geq 1/c \mid y \mid$ or $xy \leq 0$ then,

$$\phi\left(\mid\frac{x+y}{2}\mid, t\right) \leq \frac{1-\gamma}{2}\left[\phi(\mid x \mid, t) + \phi(\mid y \mid, t)\right] \qquad (9)$$

for almost all $t \in T$.

Proof. Let $\epsilon > 0$ be given. Since ϕ^* satisfies Δ_2-condition, by Lemma 7, there are $\xi \in (0,1)$ and a measurable function h with $\int_T \phi(h(t), t)d\mu < \epsilon/3$ such that

$$\text{(i)} \qquad \phi\left(\frac{\mid x \mid}{2}, t\right) \leq \frac{1-\delta}{2}\phi(\mid x \mid, t) \quad \text{for all} \quad \mid x \mid \geq h(t),$$

where $0 < \delta = 1 - \xi < 1$.

Since $\phi \in \Delta_2$, by Lemma 5, there are $b \in (1,2)$ and a measurable function g with $\int_T \phi(g(t), t)d\mu < \epsilon/3$ such that

$$\text{(ii)} \qquad \phi(b \mid x \mid, t) \leq \frac{2}{2-\delta}\phi(\mid x \mid, t) \quad \text{for all} \quad \mid x \mid \geq g(t)\,.$$

Define $f : T \longrightarrow \mathbb{R}^+$ by $f = \max\{h, g\}$. Then f is measurable and $\int_T \phi(f(t), t)d\mu < \epsilon$. For all $\mid x \mid \geq f(t)$, (i) and (ii) hold. Set $c = b - 1$ and $\gamma = 1 - \frac{2-2\delta}{2-\delta}$. Let $\mid x \mid \geq f(t)$ and $\mid x \mid \geq 1/c \mid y \mid$ or $xy \leq 0$.

Then

$$\begin{aligned}\phi\Big(\big|\frac{x+y}{2}\big|,t\Big) &\leq \phi\Big(\frac{1}{2}(1+c)\mid x\mid,t\Big)\\ &\leq \frac{1-\delta}{2}\phi(b\mid x\mid,t)\\ &\leq \frac{1-\gamma}{2}\big(\phi(\mid x\mid,t)+\phi(\mid y\mid,t)\big)\,.\ \square\end{aligned}$$

Lemma 9 ([5])

If a Banach space X does not have uniformly normal structure, then for every $\epsilon > 0$ and $n \in \mathbb{N}$ there exists $\{x_i : 1 \leq i \leq n+1\}$ in X such that

$$\|x_j\| \leq 1, \quad \|x_i - x_j\| \leq 1, \quad 1 \leq i \leq j \leq n+1 \quad \textit{and}$$

$$\Big\|x_{k+1} - \frac{1}{k}\sum_{i=1}^{k} x_i\Big\| > 1-\epsilon, \quad 1 \leq k \leq n\,. \tag{10}$$

Main results

Theorem 10

The Musielak-Orlicz space L_ϕ has uniformly normal structure if and only if both ϕ and ϕ^* satisfy Δ_2-condition.

Proof. ($\Rightarrow$) If L_ϕ has UNS then L_ϕ is reflexive [14] and thus $\phi, \phi^* \in \Delta_2$ (e.g. [7], [15]).

($\Leftarrow$) Suppose both ϕ and ϕ^* satisfy Δ_2-condition. By Lemma 8, there are a non-negative measurable function h with $\int_T \phi(8h(t),t)d\mu < 1/8$ and constants $\gamma > 0$ and $0 < c < 1$ such that if $\mid x \mid \geq \max\{h(t), 1/c \mid y \mid\}$ or $xy \leq 0$ then

$$\phi\Big(\big|\frac{x+y}{2}\big|,t\Big) \leq \frac{1-\gamma}{2}\big[\phi(\mid x\mid,t)+\phi(\mid y\mid,t)\big]\,.$$

We can find $n \in \mathbb{N}$ with $c(1+c)^{n-3} \geq 8$. Let $\epsilon = \frac{\gamma}{8n^2} > 0$. Then, by (2) there is $\eta(\epsilon) > 0$ such that if $\|f\|_\phi \geq 1-\eta(\epsilon)$ then $\rho_\phi(f) \geq 1-\epsilon$. Suppose that L_ϕ does not have UNS. Then by Lemma 9 there exists $\{f_i : 1 \leq i \leq n\}$ such that

$$\|f_i\|_\phi \leq 1, \quad \|f_i - f_j\|_\phi \leq 1, \quad 1 \leq i \leq j \leq n \ \text{ and}$$

$$\left\| f_{m+1} - \frac{1}{m} \sum_{i=1}^{m} f_i \right\|_\phi \geq 1 \; - \eta(\epsilon), \quad m < n\,. \tag{11}$$

We may assume that f_i's are simple functions with compact supports and that $f_1 = 0, f_2 \geq 0$.

Define $A = \{t : f_2(t) \geq 8h(t)\}$. Note that by (11) and since $f_1 = 0, \|f_2\| > 1 - \eta(\epsilon)$ and so $\rho_\phi(f_2) > 1 - \frac{\gamma}{8n^2} \geq \frac{3}{4}$.

The rest of the proof is similar to that of Theorem 6 in [9] so we will give only the basic steps.

For $2 < k \leq n$ and $t \in A$ consider the inequality

$$\sup \left\{ \mid f_k(t) - f_j(t) \mid : j < k \right\} > (1 + c) \sup \left\{ \mid f_{k-1}(t) - f_j(t) \mid : j < k - 1 \right\}. \tag{12}$$

Define
$B_i = \{t \in A :$ inequality (12) is true for $k < i$; but it is not true for $k = i\}$

The B_i's are disjoint measurable subsets of A. Observe that

$$\begin{aligned} \int_A \phi(f_2(t), t) d\mu &= \int_T \phi(f_2(t), t) d\mu - \int_{T-A} \phi(f_2(t), t) d\mu \\ &\geq \frac{3}{4} - \frac{1}{8} = \frac{5}{8}\,. \end{aligned}$$

Let $t \in B_i$ for some $3 \leq i \leq n$ and suppose that $k \leq i - 2$ be such that

$$\mid f_{i-1}(t) - f_k(t) \mid = \sup \left\{ \mid f_{i-1}(t) - f_j(t) \mid : j \leq i - 2 \right\}.$$

Applying Lemma 8, we get for $t \in B_i$

$$\begin{aligned} &\phi\left(\left| \frac{2f_i(t) - f_{i-1}(t) - f_k(t)}{2} \right|, t \right) \\ &\leq \frac{1 - \gamma}{2} \left[\phi(\mid f_i(t) - f_{i-1}(t) \mid, t) + \phi(\mid f_i(t) - f_k(t) \mid, t) \right]. \end{aligned} \tag{13}$$

If $t \in A \setminus \cup_{j=3}^{n} B_j$ then $\mid f_n(t) \mid \geq 8 f_2(t)$. Thus since $\|f_n\|_\phi \geq \rho_\phi(f_n)$,

$$\|f_n\|_\phi \geq 8 \left[\frac{5}{8} - \sum_{j=3}^{n} \int_{B_j} \phi(f_2(t), t) d\mu \right].$$

We have that $\|f_n\|_\phi \leq 1$. So we can find $3 \leq i \leq n$ such that

$$\int_{B_i} \phi(f_2(t), t) d\mu \geq \frac{1}{2n}\,. \tag{14}$$

Let i be as in (14). For $3 \leq j \leq i$ define

$$C_j = \Big\{ t \in B_i : j = \sup \big\{ k < i : \big|f_k(t) - f_{i-1}(t)\big| = \sup_{l<i} \big|f_l(t) - f_{i-1}(t)\big| \big\} \Big\} .$$

By (13), if $t \in C_j$, then

$$\begin{aligned} &\phi\left(\Big|\frac{2f_i(t) - f_{i-1}(t) - f_j(t)}{2}\Big|, t\right) \\ &\leq \frac{1-\gamma}{2}\big[\phi(| f_i(t) - f_{i-1}(t) |, t) + \phi(| f_i(t) - f_j(t) |, t)\big] . \end{aligned}$$

Thus

$$\begin{aligned} \phi\left(\Big|f_n(t) - \frac{1}{n-1}\sum_{k=1}^{n-1} f_k(t)\Big|, t\right) \leq \frac{1}{n-1}\sum_{k=1}^{n-1} \phi\big(| f_n(t) - f_k(t) |, t\big) \\ - \frac{\gamma}{2(n-1)}\phi\big(f_2(t), t\big)\chi_{B_n}(t) \end{aligned}$$

which implies that $\rho_\phi(f_n - \frac{1}{n-1}\sum_{k=1}^{n-1} f_k) \leq 1 - \frac{\gamma}{4n^2}$. On the other hand, we have that $1 - \frac{\gamma}{8n^2} \leq \rho_\phi(f_n - \frac{1}{n-1}\sum_{k=1}^{n-1} f_k)$. This is a contradiction. Therefore, L_ϕ must have UNS. □

A Young function ϕ is said to be linear around zero if there are $u > 0$ and $\alpha > 0$ such that $\phi(x) = \alpha x$ for all $x \in [0, u]$.

Theorem 11

Let ϕ be a Young function (with parameter) which is not linear around zero with respect to x for a.a. $t \in T$. Then, the Musielak-Orlicz space L_ϕ has normal structure if and only if ϕ satisfies Δ_2-condition.

Proof. ($\Rightarrow$) Suppose that L_ϕ has NS and that $\phi \notin \Delta_2$. Then L_ϕ contains an isometric copy of ℓ_∞ [5]. Since ℓ_∞ does not have NS neither does L_ϕ.

($\Leftarrow$) Assume $\phi \in \Delta_2$ and suppose that L_ϕ does not have NS. Then, there is a nonconstant unit limit-constant sequence $\{f_n\}$ in L_ϕ, i.e. $\lim_{n\to\infty} \|f_n - f\|_\phi = 1$ for every $f \in co\{f_n\}$. Thus by (3) we have that $\lim_{n\to\infty} \rho_\phi(f_n - f) = 1$, for every $f \in co\{f_n\}$. For i, j and n in N, define

$$\begin{aligned} F_n^{ij}(t) = \frac{1}{2}\phi\big(| f_n(t) - f_i(t) |, t\big) + \frac{1}{2}\phi\big(| f_n(t) - f_j(t), t | \big) \\ - \phi\Big(\frac{1}{2} | 2f_n(t) - f_i(t) - f_j(t) |, t\Big) . \end{aligned}$$

Then, $F_n^{ij}(t) \geq 0$ for a.a. $t \in T$ and F_n^{ij} is integrable for all i, j, n in N. Also, $\lim_{n\to\infty} \int_T F_n^{ij}(t) d\mu = 0$ for all i, j in N. Therefore, for all i, j in N, F_n^{ij} converges to zero in measure, on all sets with finite measure as $n \longrightarrow \infty$. Hence without loss of generality we may assume that $\lim_{n\to\infty} F_n^{ij}(t) = 0 \quad \mu$-a.e., for all $i, j \in \mathbb{N}$.

Let $\mid u(t) \mid = \lim_n \inf \mid f_n(t) \mid$. For each $t \in T$ choose $\{n_k = n_k(t)\}$ such that $\lim_k f_{n_k}(t) = u(t)$. Since ϕ is convex we have that for all i, j in N and a.a. $t \in T$

$$0 = \lim_k F_{n_k}^{ij}(t) = \frac{1}{2}\phi\big(\mid u(t) - f_i(t) \mid, t\big) + \frac{1}{2}\phi\big(\mid u(t) - f_j(t) \mid, t\big) - \phi\Big(\frac{1}{2} \mid 2u(t) - f_i(t) - f_j(t) \mid, t\Big) \tag{15}$$

Replacing j with n_k in (15) and taking $k \longrightarrow \infty$ we have that for all $i \in \mathbb{N}$ and *a.a.* $t \in T$

$$\frac{1}{2}\phi\big(\mid u(t) - f_i(t) \mid, t\big) = \phi\Big(\frac{1}{2} \mid u(t) - f_i(t) \mid, t\Big).$$

Thus, since ϕ is not linear around zero we must have that $u(t) = f_i(t)\mu$-a.e. for all $i \in \mathbb{N}$. Then $\lim_k \|f_{n_k} - f_i\|_\phi = 0$ which contradicts the fact that $\{f_n\}$ is a unit limit-constant sequence. □

Theorem 12

Let $\phi = \{\phi_i\}$ be a Musielak-Orlicz function such that ϕ_i is not linear around zero for every $i \in \mathbb{N}$. Then the following statements are equivalent

1. *ϕ satisfies δ_2-condition.*
2. *ℓ_ϕ has normal structure.*
3. *ℓ_ϕ has the sum property.*

Proof. (3) ⇒ (2) Obvious.

(2) ⇒ (1) Suppose ϕ does not satisfy δ_2-condition. Then ℓ_ϕ contains an isometric copy of ℓ_1 [2] and thus ℓ_ϕ cannot have NS.

(1) ⇒ (3) The proof that (1) implies (3) is omitted since it can be derived analogously as Theorem 3.8 in [4].

We conclude this paper by giving necessary and sufficient conditions for ℓ_ϕ to have normal structure in the case where ϕ_i may be linear around zero. So let $\phi = \{\phi_i\}$ be a Musielak-Orlicz function. We may assume that $\phi_i(1) = 1$ for every $i \in \mathbb{N}$ [8].

For every $i \in \mathbb{N}$, define

$$u_i = \sup\big\{u \geq 0 : \phi_i \text{ is linear on } [0, u]\big\}$$

and

$$\lambda_i = \sup\left\{u \geq 0 : \phi_i \text{ is linear on } [0,u] \text{ and } \phi_i(u) \leq 1\right\}.$$

Suppose that $\sum_{i=1}^{\infty} \phi_i(\lambda_i) < \infty$ and consider the set $D = \{i \in \mathbb{N} : u_i > 1\}$. Then, since $\phi_i(\lambda_i) = 1$ for $i \in D$, D must be finite. Therefore,

$$\begin{aligned}\sum_{i=1}^{\infty} \phi_i(u_i) &= \sum_{i\in\mathbb{N}-D} \phi_i(u_i) + \sum_{i\in D} \phi_i(u_i) \\ &= \sum_{i\in\mathbb{N}-D} \phi_i(\lambda_i) + \sum_{i\in D} \phi_i(u_i) < \infty\end{aligned}$$

i.e.

$$\sum_{i=1}^{\infty} \phi_i(\lambda_i) < \infty \Rightarrow \sum_{i=1}^{\infty} \phi_i(u_i) < \infty\,. \tag{16}$$

Theorem 13

The Musielak-Orlicz sequence space ℓ_ϕ has normal structure if and only if $\sum_{i=1}^{\infty} \phi_i(\lambda_i) < \infty$ and ϕ satisfies δ_2-condition.

Proof. The necessity is true since if $\phi \notin \delta_2$ or if $\sum_{i=1}^{\infty} \phi_i(\lambda_i) = \infty$, then ℓ_ϕ contains an isometric copy of ℓ_1 ([2]).

To show the sufficiency, assume that ℓ_ϕ does not have NS. There is a unit limit-constant sequence $\{x_n\}$ in ℓ_ϕ, i.e. $\lim_{n\to\infty} \|x_n - x\| = 1$, for all $x \in co\{x_n\}$, which by (3) implies that $\lim_{n\to\infty} I_\phi(x_n - x) = 1$for all $x \in co\{x_n\}$.

Since $\{x_n\}$ is bounded, by diagonal method we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and y such that $x_{n_k} \to y$ in coordinates. Then, by Fatou's Lemma we have that $y \in \ell_\phi$. Observe that $\{x_{n_k} - y\}$ is again a unit limit-constant sequence and thus without loss of generality we may assume that $x_n \longrightarrow 0$ in coordinates.

Since $\phi_i(1) = 1$ for each $i \in \mathbb{N}$ and $\{x_n\}$ is a unit-limit constant sequence, for every $i \in \mathbb{N}$ there is L>0 such that $\mid x_n(i) - x_m(i) \mid <$ L for all $n, m \in \mathbb{N}$.

Claim:

$$\mid x_n(i) - x_m(i) \mid \leq u_i \quad \text{for all} \quad i, m, n \in \mathbb{N} \tag{17}$$

Proof of Claim: Suppose that the claim is not true. Then there are $i_0, m_0, k_0 \in \mathbb{N}$ such that

$$\mid x_{k_0}(i_0) - x_{m_0}(i_0) \mid > u_{i_0}\,.$$

We can find $\delta, \mathrm{L} > 0$ such that

$$u_{i_0} + \delta < \mid x_{k_0}(i_0) - x_{m_0}(i_0) \mid \leq \mathrm{L}\,.$$

Since $[0, \mathrm{L}]$ is compact and ϕ_{i_0} is continuous, by Lemma 2 in [13] there are $\epsilon > 0$ and $\lambda > 0, (\epsilon = 1/3 \min\{u_{i_0}, \delta\})$ with

$$\phi_{i_0}\Big(\big|d_2 - \frac{d_1}{2}\big|\Big) < \frac{1}{2}\phi_{i_0}\big(\mid d_2 - d_1 \mid\big) + \frac{1}{2}\phi_{i_0}\big(\mid d_2 \mid\big) - \lambda$$

whenever $u_{i_0} + \delta < d_1 \leq \mathrm{L}$ and $0 < d_2 < d_1 + \epsilon$.

Without loss of generality we may assume that $x_{m_0}(i_0) < \epsilon, x_{k_0}(i_0) > 0$ and $x_{k_0}(i_0) > x_{m_0}(i_0)$. Since $\lim_{n \to \infty} x_n(i_0) = 0$ the following is true:

There are $n_1, n_2 \in \mathbb{N}$ such that $\mid x_n(i_0) \mid < x_{k_0}(i_0)$, for all $n \geq n_1$ and $\mid x_n(i_0) \mid < \epsilon - x_{m_0}(i_0)$, for all $n \geq n_2$.

Let $n \geq \max\{n_1, n_2\}$. Define

$$d_2 = x_{k_0}(i_0) - x_n(i_0), \qquad d_1 = x_{k_0}(i_0) - x_{m_0}(i_0).$$

Then $u_{i_0} + \delta < d_1 \leq \mathrm{L}$ and $0 < d_2 < d_1 + \epsilon$. Therefore

$$\begin{aligned} &\phi_{i_0}\left(\left|\frac{1}{2}\big(x_{k_0}(i_0) + x_{m_0}(i_0)\big) - x_n(i_0)\right|\right) \\ &= \phi_{i_0}\left(\left|x_{k_0}(i_0) - x_n(i_0) - \frac{x_{k_0}(i_0) - x_{m_0}(i_0)}{2}\right|\right) \\ &< \frac{1}{2}\phi_{i_0}\big(\mid x_{k_0}(i_0) - x_n(i_0) \mid\big) \\ &\qquad + \frac{1}{2}\phi_{i_0}\big(\mid x_{m_0}(i_0) - x_n(i_0) \mid\big) - \lambda\,. \end{aligned}$$

Thus

$$\mathrm{I}_\phi\Big(\frac{1}{2}\big(x_{k_0} + x_{m_0}\big) - x_n\Big) < \frac{1}{2}\mathrm{I}_\phi\big(x_{k_0} - x_n\big) + \frac{1}{2}\mathrm{I}_\phi\big(x_{m_0} - x_n\big) - \lambda.$$

Taking limits in the above inequality as $n \longrightarrow \infty$ we get a contradiction. Hence the claim holds.

By assumption we have that $\sum_{i=1}^{\infty} \phi_i(u_i) < \infty$. Thus there is $i_0 \in \mathbb{N}$ such that

$$\sum_{i=i_0+1}^{\infty} \phi_i(u_i) < \frac{1}{4}\,.$$

We can find $\epsilon > 0$ such that

$$\sum_{i=1}^{i_0} \phi_i(\epsilon) < \frac{1}{4}\,.$$

Since $\{x_n(i)\}_{n=1}^{\infty}$ is a Cauchy sequence for every $1 \leq i \leq i_0$, there is $k \in \mathbb{N}$ such that

$$| x_n(i) - x_m(i) | < \epsilon \quad \text{for} \quad n, m > k \quad \text{and} \quad 1 \leq i \leq i_0\,.$$

Then for any $n, m > k$

$$\mathrm{I}_\phi(x_n - x_m) < \sum_{i=1}^{i_0} \phi_i(\epsilon) + \sum_{i=i_0+1}^{\infty} \phi_i(u_i) < \frac{1}{2}\,.$$

Hence $\lim_{n\to\infty} \mathrm{I}_\phi(x_n - x_m) < 1/2$ which is a contradiction. This completes the proof of Theorem 13. □

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References

1. G. Alherk and H. Hudzik, Uniformly non-$\ell_n^{(1)}$ Musielak-Orlicz spaces of Bochner Type, *Forum Math* **1** (1989), 403–410.
2. G. Alherk and H. Hudzik, Copies of ℓ_1 and c_0 in Musielak-Orlicz Sequence Spaces, *Comment. Math. Univ. Carolin.* **35**:1 (1994), 9–19.
3. M.S. Brodski and D.P. Milman, On the center of a convex set, *Dokl. Akad. Nauk. SSSR* **59** (1948), 837–840 (Russian).
4. S. Chen, Geometric properties of Orlicz spaces, Preprint.
5. S. Chen and H. Sun, Reflexive Orlicz spaces have uniformly normal structure, *Studia Math.* **109** (1994), 197–208.
6. H. Hudzik, On some equivalent conditions in Musielak-Orlicz spaces, *Comment. Math. Univ. Carolin.* **24** (1984), 57–64.
7. H. Hudzik and A. Kamińska, On uniformly convexifiable and B-convex Musielak-Orlicz sequence spaces, *Comment. Math. Univ. Carolin.* **25** (1985), 59–75.
8. A. Kamińska, Uniform Rotundity of Musielak-Orlicz sequence spaces, *J. Approx. Theory* **47**:4 (1986), 302–322.
9. A. Kamińska, P.K. Lin and H. Sun, Uniformly normal structure of Orlicz-Lorentz spaces, Proceedings, Conference in Functional Analysis, Harmonic Analysis and Probability, Columbia, Missouri 1994.

10. W.A. Kirk, A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly* **72** 1004–1006.
11. T. Landes, Permanence properties of Normal Structure, *Pacific J. Math.* **110**:1 (1984), 125–143.
12. T. Landes, Normal Structure and Weakly Normal Structure of the Orlicz Sequence Spaces, *Trans. Amer. Math. Soc.* **285**:2 (1984), 523–534.
13. P. K. Lin and H. Sun, Normal Structure of Lorentz-Orlicz Spaces, Preprint.
14. E. Maluta, Uniformly Normal Structure and related coefficients, *Pacific J. Math.* **111**:2 (1984), 357–369.
15. J. Musielak, *Orlicz Spaces and Modular Spaces*, Lecture Notes in Mathematics, **1038**, Springer-Verlag 1983.
16. M. Wisla, Convergence in Musielak-Orlicz Spaces, *Bull. Polish Acad. Sci. Math.* **33**:9–10 (1985), 517–529.