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Smoothness in Musielak-Orlicz spaces equipped with the Orlicz norm

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Abstract

A formula for the distance of an arbitrary element x in Musielak-Orlicz space L^Φ from the subspace E^Φ of order continuous elements is given for both (the Luxemburg and the Orlicz) norms. A formula for the norm in the dual space of L^Φ is given for any of these two norms. Criteria for smooth points and smoothness in L^Φ and E^Φ equipped with the Orlicz norm are presented.

0. Introduction

Throughout this paper $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{R}$ and $\mathbb{R}_+$ denote the sets of reals and nonnegative reals, respectively. The triple (T, Σ, μ) stands for a positive, nonatomic, σ-finite and complete measure space. By $L^0 = L^0(\mu)$ we denote the space of all (equivalence classes of) Σ-measurable real functions x defined on T. A mapping $\Phi : T \times \mathbb{R} \to \mathbb{R}_+$ is said to be a Musielak-Orlicz function if it satisfies the Caratheodory conditions, i.e. for any $u \in \mathbb{R}$, the function $\Phi(\cdot, u)$ is Σ-measurable and there is a set $T_0 \in \Sigma$ with $\mu(T_0) = 0$ such that for any $t \in T \setminus T_0$ the function $\Phi(t, \cdot)$ is an Orlicz function i.e. it is even, convex, vanishing at zero and satisfying $\Phi(t, u) \to +\infty$ as $u \to +\infty$. We assume in the whole paper (if it will be not excluded explicitly) that Φ satisfies the following condition

$$(\infty_1) \qquad \big(\Phi(t,u)/u\big) \to +\infty \quad \text{as} \quad u \to +\infty \quad \text{for} \quad \mu\text{-a.e.} \;\; t \in T.$$

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The complementary function of Φ in the sense of Young is defined by

$$\Phi^*(t,u) = \sup_{v>0}\big\{|u|v - \Phi(t,v)\big\} \quad \big(\forall\, t\in T,\ u\in\mathbb{R}\big)\,.$$

It is easy to see that Φ^* is also a Musielak-Orlicz function.

Given a Musielak-Orlicz function Φ we define on L^0 the functional I_Φ by

$$I_\Phi(x) = \int_T \Phi\big(t,x(t)\big)\,d\mu.$$

It is obvious that this functional is nonnegative, even and convex as well as that $I_\Phi(0)=0$ and if $x\in L^0$ and $I_\Phi(\lambda x)=0$ for all $\lambda>0$, then $x=0$. So, I_Φ is a convex modular (see [18]).

Every Musielak-Orlicz function Φ generates the Musielak-Orlicz space L^Φ defined as the set of these $x\in L^0$ that $I_\Phi(\lambda x)<+\infty$ for some $\lambda>0$ depending on x. We can define in L^Φ the following three norms (see [4], [16], [17] and [20]):

$$\|x\|_\Phi = \inf\big\{\lambda>0 : I_\Phi(x/\lambda)\le 1\big\} \quad \text{(the Luxemburg norm)},$$
$$\|x\|_\Phi^0 = \sup\left\{\Big|\int_T x(t)y(t)\,d\mu\Big| : I_{\Phi^*}(y)\le 1\right\} \quad \text{(the Orlicz norm)},$$
$$\|x\|_\Phi^A = \inf_{k>0}\frac{1}{k}\big(1+I_\Phi(kx)\big) \quad \text{(the Amemiya norm)}.$$

It is known that $\|x\|_\Phi \le \|x\|_\Phi^0 \le 2\|x\|_\Phi$ for any $x\in L^\Phi$ (see [18]). It can also be proved in an analogous way as for Orlicz spaces in [4], [16] and [20] that $\|x\|_\Phi^0=\|x\|_\Phi^A$ for any $x\in L^\Phi$ (see [8]).

We denote by $\Phi'_-(t,u)$ and $\Phi'_+(t,u)$ the left and the right derivatives of $\Phi(t,\cdot)$ at any fixed point $u\in\mathbb{R}$, respectively. We define $\partial\Phi(t,u)=[\Phi'_-(t,u),\Phi'_+(t,u)]$ for any $t\in T$ and $u\in\mathbb{R}$. It is easy to see that for μ-a.e. $t\in T$ and any $u\in\mathbb{R}, \partial\Phi(t,u) = \{v\in\mathbb{R} : uv=\Phi(t,u)+\Phi^*(t,v)\}$. We say for a given $t\in T$ that $\Phi(t,\cdot)$ is smooth at $u\in\mathbb{R}$ if $\Phi'_-(t,u)=\Phi'_+(t,u)$.

We can show in the same way as for Orlicz spaces in [4] and [20] that if we define for $x\in L^\Phi$ the following two constants:

$$k^* = k^*(x) = \inf\big\{k>0 : I_{\Phi^*}(\Phi'_+\circ k|x|)\ge 1\big\},$$
$$k^{**} = k^{**}(x) = \sup\big\{k>0 : I_{\Phi^*}(\Phi'_+\circ k|x|)\le 1\big\},$$

then $\|x\|_\Phi^A = \frac{1}{k}(1+I_\Phi(kx))$ for any $k\in[k^*,k^{**}]$.

We define a closed subspace E^Φ of L^Φ by

$$E^\Phi = \left\{x \in L^0 : I_\Phi(\lambda x) < +\infty \quad \text{for any} \quad \lambda > 0\right\}.$$

It is easy to see that E^Φ is the subspace of order continuous elements in L^Φ, i.e. $x \in L^\Phi$ belongs to E^Φ if and only if for any sequence (x_n) in L^0 such that $|x_n| \leq |x|$ for all $n \in \mathbb{N}$ and $|x_n| \to 0$ μ-a.e. in T there holds $\|x_n\|_\Phi \to 0$. For the definition of order continuous elements in Banach lattices see [1] and [15]. Since Φ is finitely valued, we have $E^\Phi \neq \{0\}$.

Kamińska [14] has constructed for any Musielak-Orlicz function Φ an ascending sequence $(T_n)_{n=1}^\infty$ of measurable sets with $0 < \mu(T_n) < +\infty$ for any $n \in \mathbb{N}$ such that $\sup_{t \in T_n} \Phi(t, \lambda) < +\infty$ for every $\lambda > 0$ and $n \in \mathbb{N}$ and $\bigcup_{n=1}^\infty T_n = T$. This yields that χ_{T_n} (the characteristic function of T_n) belongs to E^Φ for any $n \in \mathbb{N}$.

In the whole paper T_n denotes a set of this sequence of sets.

The spaces L^Φ and E^Φ equipped with everyone of these three norms are Banach spaces (see [18]). The spaces L^Φ and E^Φ coincide if and only if Φ satisfies the so-called Δ_2-condition. Recall that Φ satisfies the Δ_2-condition ($\Phi \in \Delta_2$ for short), if there are a set T_0 of measure zero, a constant $K > 0$ and a Σ-measurable nonnegative function h defined on T such that $\int_T h(t)\, d\mu < +\infty$ and

$$\Phi(t, 2u) \leq K\Phi(t, u) + h(t)$$

for every $t \in T \setminus T_0$ and $u \in \mathbb{R}$. For the consequences of $\Phi \in \Delta_2$ or $\Phi \notin \Delta_2$ we refer to [8], [11] and [14].

The dual space of L^Φ is represented in the following way (see [18] and for Banach function lattices also [15]):

$$(L^\Phi)^* = L^{\Phi^*} \oplus S,$$

i.e. every $x^* \in (L^\Phi)^*$ is uniquely represented in the form $x^* = \xi_v + \varphi$, where φ is a singular functional, i.e. $\varphi(x) = 0$ for any $x \in E^\Phi$ and ξ_v is the regular functional defined by a function $v \in L^{\Phi^*}$ by the formula

$$\xi_v(x) = \langle x, v\rangle = \int_T v(t)x(t)\, d\mu \qquad (\forall\ x \in L^\Phi).$$

For any $x^* \in (L^\Phi)^*$ we define

$$\|x^*\| = \sup\left\{x^*(x) : x \in L^\Phi \quad \text{and} \quad \|x\|_\Phi^0 \leq 1\right\},$$
$$\|x^*\|^0 = \sup\left\{x^*(x) : x \in L^\Phi \quad \text{and} \quad \|x\|_\Phi \leq 1\right\}.$$

Let us define for any $x \in L^\Phi$:

$$\begin{aligned} d(x) &= \inf\{\|x-y\|_\Phi : y \in E^\Phi\}, \\ d_0(x) &= \inf\{\|x-y\|_\Phi^0 : y \in E^\Phi\}, \\ \theta(x) &= \inf\{\lambda > 0 : I_\Phi(x/\lambda) < +\infty\}, \\ x_n(t) &= \begin{cases} x(t) & \text{if } |x(t)| \le n \text{ and } t \in T_n, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

for any $n \in \mathbb{N}$. It is obvious that $|x_n| \nearrow |x|$ and $0 \swarrow |x - x_n| \le |x|$ μ-a.e. in T, and $x_n \in E^\Phi$ for any $n \in \mathbb{N}$.

For any Banach space X denote by $B(X)$ and $S(X)$ its unit ball and unit sphere and by $B(X^*)$ and $S(X^*)$ the unit ball and the unit sphere of the dual space X^* of X, respectively. By L^Φ and L_0^Φ we denote the Orlicz space L^Φ equipped with the Luxemburg and the Orlicz norm, respectively. Their unit spheres we denote by $S(L^\Phi)$ and $S(L_0^\Phi)$, respectively.

A functional $x^* \in X^*$ is said to be a support functional at $x \in X \setminus \{0\}$ if $\|x^*\| = 1$ and $x^*(x) = \|x\|$. The set of all support functionals at x is denoted by $\mathrm{Grad}\,(x)$ and $\mathrm{R\,Grad}\,(x)$ denotes the set of all regular functionals from $\mathrm{Grad}\,(x)$. We say that $x \in X \setminus \{0\}$ is a smooth point if $\mathrm{Grad}\,(x)$ contains exactly one element. We say a Banach space X is smooth if any $x \in S(X)$ (equivalently any $x \in X \setminus \{0\}$) is a smooth point (see [7] and [19]).

Criteria for smoothness of Orlicz function spaces equipped with the Orlicz norm were given in [3]. Smooth points in these spaces were characterized in [5]. Criteria for smooth points and smoothness of Orlicz function (and sequence) spaces endowed with the Luxemburg norm were presented in [9]. Smooth points and smoothness in Orlicz sequence spaces equipped with the Orlicz norm were characterized in [6]. Smoothness of Musielak-Orlicz spaces equipped with the Luxemburg norm was characterized under some restrictions in [21] and the problem was solved completely in [12]. Criteria for smooth points in Musielak-Orlicz sequence and function spaces endowed with the Luxemburg norm were given respectively in [13] and [22]. In this paper we describe smooth points and smoothness in Musielak-Orlicz spaces L^Φ and their subspaces E^Φ equipped with the Orlicz norm.

1. Introductory results

We start with the following lemma.

Lemma 1.1

If Φ is a Musielak-Orlicz function such that $\Phi(t,\cdot)$ vanishes only at zero for μ-a.e. $t \in T$, then $\|x\|_\Phi < \|x\|_\Phi^0$ for any $x \in L^\Phi \setminus \{0\}$.

Proof. We may assume without loss of generality that $\|x\|_\Phi = 1$. We need to show that $\|x\|^0_\Phi > 1$. It is obvious that $I_\Phi(x) \le 1$. Let us consider two cases.

1^0. $I_\Phi(x) = 1$. Then using the Amemiya formula for $\|x\|^0_\Phi$ (see [8]), we have

$$\|x\|^0_\Phi = \inf_{k>0} \frac{1}{k}\big(1 + I_\Phi(kx)\big) = \inf_{k\ge 1} \frac{1}{k}\big(1 + I_\Phi(kx)\big) = \frac{1}{k_0}\big(1 + I_\Phi(k_0 x)\big)$$

for some $k_0 \ge 1$. The last equality follows by the assumption that $(\Phi(t,u)/u) \to +\infty$ as $u \to +\infty$ for μ-a.e. $t \in T$ in an analogous way as for Orlicz spaces (see [9]). This yields $\|x\|^0_\Phi > 1$.

2^0. $I_\Phi(x) < 1$. Then it must be $I_\Phi(kx) = +\infty$ for any $k > 1$. Assume for the contrary that $I_\Phi(k_0 x) = +\infty$ for some $k_0 > 1$ and define the function $f(k) = I_\Phi(kx)$ for $k > 0$. Clearly, f is convex, $f(1) < 1$ and f is finite (so also continuous) on the interval $[0, k_0]$. Thus, there is $k_1 > 1$ such that $I_\Phi(k_1 x) \le 1$, whence $\|x\|_\Phi \le 1/k_1 < 1$, a contradiction which proves that $I_\Phi(kx) = +\infty$ for any $k > 1$. So, $\|x\|^0_\Phi = \frac{1}{k_0}(1 + I_\Phi(k_0 x))$ for some $0 < k_0 \le 1$. This yields again $\|x\|^0_\Phi > 1$. □

Lemma 1.2

For any $x \in L^\Phi$ there holds the equalities

$$\lim_{n\to+\infty} \|x - x_n\|^0_\Phi = \lim_{n\to+\infty} \|x - x_n\|_\Phi = \theta(x) = d_0(x) = d(x)\,.$$

Proof. If $x \in E^\Phi$, then all the values are equal to 0, so the equalities hold true. Assume now that $x \in L^\Phi \setminus E^\Phi$, i.e. $\theta(x) > 0$. Since the sequences $(\|x - x_n\|^0_\Phi)$ and $(\|x - x_n\|_\Phi)$ are nonincreasing, so the limits from the equalities exist. For any $\varepsilon \in (0, \theta(x))$, we have $I_\Phi(x/(\theta(x)-\varepsilon)) = +\infty$, so also $I_\Phi((x-x_n)/(\theta(x)-\varepsilon)) = +\infty$, whence $\|x - x_n\|_\Phi \ge \theta(x) - \varepsilon$ for any $n \in \mathbb{N}$. Consequently,

$$\lim_{n\to+\infty} \|x - x_n\|^0_\Phi \ge \lim_{n\to+\infty} \|x - x_n\|_\Phi \ge \theta(x)\,.$$

Now, we will show the inequality $\lim_{n\to+\infty} \|x - x_n\|^0_\Phi \le \theta(x)$. Let $\varepsilon > 0$ be arbitrary and denote for short $\theta(x) = \theta$. Then $I_\Phi(x/(\theta+\varepsilon)) < +\infty$, whence $\lim_{n\to+\infty} I_\Phi((x-x_n)/(\theta+\varepsilon)) = 0$. By the Amemiya formula for the Orlicz norm, we get

$$\|x - x_n\|^0_\Phi \le (\theta+\varepsilon)\big[1 + I_\Phi((x - x_n)/(\theta+\varepsilon))\big] \to \theta + \varepsilon$$

as $n \to +\infty$, whence the desired inequality follows. So,

$$\theta(x) = \lim_{n\to+\infty} \|x - x_n\|^0_\Phi \ge d_0(x) \ge d(x)\,.$$

To finish the proof we need only to show that $d(x) \ge \theta(x)$.

Let $\varepsilon \in (0,\theta)$ be arbitrary. Take any function $w \in E^\Phi$ and define the sequence of measurable sets:

$$F_n = \{t \in T : |w(t)| \le n\} \cap G_n,$$

where

$$G_n = G_n(x) = \{t \in T_n : |x(t)| \le n\}.$$

The sequence (F_n) is increasing and $\mu(T \setminus \bigcup_{n=1}^\infty F_n) = 0$. By $I_\Phi(w/(\varepsilon/2)) < +\infty$ and the Lebesgue dominated convergence theorem, we get for $w_n = w\chi_{F_n}$,

$$I_\Phi\big((w-w_n)/(\varepsilon/2)\big) = I_\Phi\big(w\chi_{T\setminus F_n}/(\varepsilon/2)\big) \to 0$$

as $n \to +\infty$. Thus, there is $n_0 \in \mathbb{N}$ such that

$$I_\Phi\big((w-w_n)/(\varepsilon/2)\big) \le 1, \quad \text{i.e. } \|w-w_n\| \le \varepsilon/2 \tag{1}$$

for any $n \ge n_0$. From the definition of $\theta(x)$, we have $I_\Phi(x/(\theta-\varepsilon/2)) = +\infty$. Since $F_{n_0} \subset G_{n_0}$, we have

$$I_\Phi\big(x\chi_{F_{n_0}}/(\theta-\varepsilon/2)\big) \le I_\Phi\big(x\chi_{G_{n_0}}/(\theta-(\varepsilon/2)\big) < +\infty,$$

whence

$$I_\Phi\big(x\chi_{T\setminus F_{n_0}}/(\theta-\varepsilon/2)\big) = +\infty.$$

By $\operatorname{supp} w_{n_0} = F_{n_0}$, we get

$$\begin{aligned} I_\Phi\big((x-w_{n_0})/(\theta-\varepsilon/2)\big) &= I_\Phi\big((x-w_{n_0})\chi_{F_{n_0}}/(\theta-\varepsilon/2)\big) \\ &\quad + I_\Phi\big(x\chi_{T\setminus F_{n_0}}/(\theta-\varepsilon/2)\big) \ge I_\Phi\big(x\chi_{T\setminus F_{n_0}}/(\theta-(\varepsilon/2)\big) = +\infty, \end{aligned}$$

whence $\|x-w_{n_0}\|_\Phi \ge \theta - \varepsilon/2$. Combining this with (1), we get

$$\|x-w\|_\Phi \ge \|x-w_{n_0}\|_\Phi - \|w_{n_0}-w\|_\Phi \ge \theta - \varepsilon/2 - \varepsilon/2 = \theta - \varepsilon.$$

By the arbitrariness of $w \in E^\Phi$ and $\varepsilon > 0$, we get $d(x) \ge \theta(x)$, which finishes the proof. □

Lemma 1.3

For any singular functional φ there hold the equalities

$$\|\varphi\| = \|\varphi\|^0 = \sup\{\varphi(x) : I_\Phi(x) < +\infty\} = \sup_{x\in L^\Phi} \varphi(x)/\theta(x).$$

Proof. We have for any $x \in L^\Phi$ and $n \in \mathbb{N}$, $\varphi(x) = \varphi(x - x_n) \leq \|\varphi\| \, \|x - x_n\|_\Phi$, whence in virtue of Lemma 1.2, we get $\varphi(x) \leq \theta(x)\|\varphi\|$. Moreover, if $I_\Phi(x) < +\infty$, then $I_\Phi(x - x_n) \to 0$ as $n \to +\infty$ and so $\theta(x) = \lim_{n\to+\infty} \|x - x_n\|_\Phi \leq 1$. Therefore,

$$\begin{aligned}\|\varphi\| &= \sup_{x\in L^\Phi} \varphi(x)/\|x\|^0_\Phi \leq \sup_{x\in L^\Phi} \varphi(x)/\|x\|_\Phi = \|\varphi\|^0 \\ &\leq \sup\big\{\varphi(x) : I_\Phi(x) < +\infty\big\} \leq \sup\big\{\varphi(x)/\theta(x) : I_\Phi(x) < +\infty\big\} \\ &\leq \sup_{x\in L^\Phi} \varphi(x)/\theta(x) \leq \|\varphi\|\,,\end{aligned}$$

whence it follows that all non-sharp inequalities are equalities in fact, which finishes the proof. □

Lemma 1.4

For any functional $x^* = \xi_v + \varphi \in (L^\Phi)^*$, we have

$$\|x^*\|^0 = \|v\|^0_{\Phi^*} + \|\varphi\|^0,$$
$$\|x^*\| = \inf\big\{\lambda > 0 : I_{\Phi^*}(v/\lambda) + \|\psi\|/\lambda \leq 1\big\}.$$

Proof. To prove the first formula, it is enough to show that $\|x^*\|^0 \geq \|v\|^0_{\Phi^*} + \|\varphi\|^0$. For any $\varepsilon > 0$ we can find $x_1, x_2 \in S(L^\Phi)$ such that

$$\|v\|^0_{\Phi^*} - \varepsilon < \xi_v(x_1) \quad \text{and} \quad \|\varphi\|^0 - \varepsilon < \varphi(x_2)\,.$$

We may assume that $x_1 \in E^\Phi$. Let (G_n) be the sequence of measurable sets defined as in the proof of Lemma 1.2, but corresponding to x_2 in place of x. By the Lebesgue dominated convergence theorem, we have

$$\int\limits_{T\setminus G_n} |v(t)x_1(t)|\, d\mu \to 0 \quad \text{as} \quad n \to +\infty,$$
$$\int\limits_{T\setminus G_n} |v(t)x_2(t)|\, d\mu \to 0 \quad \text{as} \quad n \to +\infty,$$
$$\int\limits_{T\setminus G_n} \Phi(t, x_2(t))d\mu \to 0 \quad \text{as} \quad n \to +\infty\,.$$

Moreover, $x_2\chi_{G_n} \in E^\Phi$ for any $n \in \mathbb{N}$. Take $m \in \mathbb{N}$ large enough such that

$$\int_{T\setminus G_m} |v(t)x_1(t)|\, d\mu < \varepsilon, \tag{2}$$

$$\int_{T\setminus G_m} |v(t)x_2(t)|\, d\mu < \varepsilon, \tag{3}$$

$$I_\Phi(x_2\chi_{T\setminus G_m}) < \varepsilon\,. \tag{4}$$

Define

$$x(t)=\begin{cases} x_1(t) & \text{if } t\in G_m\\ x_2(t) & \text{if } t\in T\setminus G_m. \end{cases}$$

In view of (4), we have

$$I_\Phi(x)\le I_\Phi(x_1)+I_\Phi(x_2\chi_{T\setminus G_m})\le 1+\varepsilon\,.$$

Hence $I_\Phi(x/(1+\varepsilon))\le 1$ and consequently $\|x\|_\Phi\le 1+\varepsilon$. Since φ is singular, we have $\varphi(x_1)=0$ and $\varphi(x_2)=\varphi(x_2\chi_{T\setminus G_m})$. So, by (2) and (3), we get

$$\begin{aligned}(1+\varepsilon)\|x^*\|^0 &\ge x^*(x)=x^*(x_1\chi_{G_m})+x^*(x_2\chi_{T\setminus G_m})\\ &=\xi_v(x_1\chi_{G_m})+\xi_v(x_2\chi_{T\setminus G_m})+\varphi(x_2)\\ &\ge \xi_v(x_1)-\varepsilon-\varepsilon+\varphi(x_2)>\|v\|^0_{\Phi^*}+\|\varphi\|^0-4\varepsilon,\end{aligned}$$

which finishes the proof of the first formula.

To prove the second formula we can assume without loss of generality that $\|x^*\|=1$. Take $\gamma>0$ satisfying $I_{\Phi^*}(v/\gamma)+\|\varphi\|/\gamma\le 1$ and $x\in S(L^\Phi_0)$. Let $k>0$ be such that

$$\|x\|^0_\Phi=\frac{1}{k}\big(1+I_\Phi(kx)\big).$$

Since $I_\Phi(kx)<+\infty$, applying Lemma 1.3, we get

$$\begin{aligned}\frac{1}{\gamma}x^*(kx)&=\frac{1}{\gamma}\xi_v(kx)+\frac{1}{\gamma}\varphi(kx)\le I_\Phi(kx)+I_{\Phi^*}(v/\gamma)+\|\varphi\|/\gamma\\ &\le I_\Phi(kx)+1=k\|x\|^0_\Phi=k\,,\end{aligned}$$

whence $x^*(x)\le\gamma$. By the arbitrariness of x from $S(L^\Phi_0)$ and $\gamma>0$ satisfying $I_{\Phi^*}(v/\gamma)+\|\varphi\|/\gamma\le 1$, we get

$$\|x^*\|\le\inf\big\{\lambda>0: I_{\Phi^*}(v/\lambda)+\|\varphi\|/\lambda\le 1\big\}\,. \tag{5}$$

If the inequality (5) is sharp, there exists $\delta>0$ such that

$$I_{\Phi^*}(v)+\|\varphi\|>1+\delta. \tag{6}$$

In fact, if such a number $\delta>0$ does not exist, then $I_{\Phi^*}(v)+\|\varphi\|\le 1$, which yields

$$\inf\big\{\lambda>0: I_{\Phi^*}(v/\lambda)+\|\varphi\|/\lambda\le 1\big\}\le 1=\|x^*\|\,,$$

so we have then equality in (5). In virtue of (6), there is $x_1 \in L^\Phi$ such that

$$I_\Phi(x_1) \le 1 \quad \text{and} \quad I_{\Phi^*}(v) + \varphi(x_1) > 1 + \delta.$$

From the equalities

$$\|x^*|_{E^\Phi}\| = \|\xi_v|_{E^\Phi}\| = \|\xi_v\| = \|v\|_{\Phi^*},$$

we get $\|v\|_{\Phi^*} \le \|x^*\| = 1$, which yields $I_{\Phi^*}(v) \le 1$. Using the left derivative $(\Phi^*)'_-(t,\cdot)$ of $\Phi^*(t,\cdot)$, we have the following Young equality

$$I_{\Phi^*}(v) = \int_T \big[(\Phi^*)'_-(t,|v(t)|)|v(t)| - \Phi(t,(\Phi^*)'_-(t,|v(t)|))\big]d\mu.$$

Define the sets

$$F_n = \{t \in T_n : |x_1(t)| \le n\},$$
$$H_n = \big\{t \in F_n : (\Phi^*)'_-(t,|v(t)|) \le n\big\},\ n = 1,2,\ldots .$$

The sequence (H_n) is increasing and in virtue of the assumption (∞_1), we have that $\Phi^*(t,\cdot)$ is finitely valued for μ-a.e. $t \in T$ and consequently $(\Phi^*)'_-(t,\cdot)$ is also finitely valued for μ-a.e. $t \in T$. Therefore, $\mu(T \setminus \bigcup_{n=1}^\infty H_n) = 0$. Moreover,

$$(\Phi^*)'_-(t,|v(t)|)\chi_{H_n} \in E^\Phi \quad \text{and} \quad x_1\chi_{H_n} \in E^\Phi,$$

for each $n \in \mathbb{N}$. By the Beppo-Levi theorem we have

$$\int_{H_n} \big[(\Phi^*)'_-(t,|v(t)|)|v(t)| - \Phi(t,(\Phi^*)'_-(t,|v(t)|))\big]d\mu \to$$
$$\int_T \big[(\Phi^*)'_-\big(t,|v(t)|\big)|v(t)| - \Phi(t,(\Phi^*)'_-(t,|v(t)|))\big]d\mu,$$
$$\int_{T\setminus H_n} |x_1(t)v(t)|d\mu \to 0 \quad \text{and} \quad \int_{T\setminus H_n} \Phi\big(t,x_1(t)\big)d\mu \to 0\,.$$

So, we can find n large enough such that the function

$$x_2(t) = (\Phi^*)'_-(t,|v(t)|)\chi_{H_n}(t)$$

satisfies

$$I_{\Phi^*}(v) - \delta/6 < \int_T \big[x_2(t)|v(t)| - \Phi(t,x_2(t))\big]d\mu, \tag{7}$$

$$(8) \qquad \int_{T\setminus H_n} |x_1(t)v(t)|d\mu < \delta/6 \quad \text{and} \quad \int_{T\setminus H_n} \Phi(t, x_1(t))d\mu < \delta/6.$$

Clearly, $x_2 \in E^\Phi$. Define

$$x(t) = \begin{cases} x_1(t)\ \mathrm{sgn}(v(t)) & \text{if } t \in T \setminus H_n \\ x_2(t)\ \mathrm{sgn}(v(t)) & \text{if } t \in H_n. \end{cases}$$

In virtue of (6), (7) and (8), we get

$$\begin{aligned} &\xi_v(x) + \varphi(x) - I_\varphi(x) \\ &\quad = \xi_v(x_1\chi_{T\setminus H_n}) + \xi_v(x_2\chi_{H_n}) + \varphi(x_1\chi_{T\setminus H_n}) - I_\Phi(x_1\chi_{T\setminus H_n}) - I_\Phi(x_2\chi_{H_n}) \\ &\quad > -\delta/6 + \xi_v(x_2) + \varphi(x_1) - \delta/6 - I_\Phi(x_2) \\ &\quad = \xi_v(x_2) - I_\Phi(x_2) + \varphi(x_1) - \delta/3 > I_{\Phi^*}(v) + \varphi(x_1) - \delta/2 \\ &\quad > 1 + \delta - \delta/2 = 1 + \delta/2 > 1, \end{aligned}$$

whence

$$\xi_v(x) + \varphi(x) > 1 + I_\Phi(x).$$

By this inequality and the inequality $\|x\|_\Phi^0 \leq 1 + I_\Phi(x)$, we get

$$\begin{aligned} 1 = \|x^*\| &\geq x^*(x/\|x\|_\Phi^0) = \xi_v(x/\|x\|_\Phi^0) + \varphi(x/\|x\|_\Phi^0) \\ &= (\xi_v(x) + \varphi(x))/\|x\|_\Phi^0 > \frac{1 + I_\Phi(x)}{\|x\|_\Phi^0} \geq 1, \end{aligned}$$

a contradiction, which finishes the proof that the non-sharp inequality in (5) is equality in fact. So, the theorem is proved. □

Remark 1.5. Note that the first formula from Lemma 1.4 holds also true without the assumption that the Musielak-Orlicz function Φ satisfies condition (∞_1), which is not used in the proof.

The second formula from Lemma 1.4 holds also true without condition (∞_1) if we assume that the set $A = \{t \in T : \alpha(t) := \sup[u > 0 : \Phi(t, u) < +\infty] < +\infty\}$ is measurable and $(\Phi^*)'_-(t, \alpha(t)) < +\infty$ for μ-a.e. $t \in A$. The proof can be proceeded without any change because $I_{\Phi^*}(v) \leq 1$ implies that $\Phi^*(t, v(t)) < +\infty$ for μ-a.e. $t \in T$ and so, in virtue of our assumption, we have $(\Phi^*)'_-(t, v(t)) < +\infty$ for μ-a.e. $t \in T$. Note that condition (∞_1) is equivalent to the fact that $\Phi^*(t, \cdot)$ is finitely valued for μ-a.e. $t \in T$.

Lemma 1.6

A functional $x^* \in (L^\Phi)^*$ is singular if and only if $\|x^*\|^0 = \|x^*\|$.

Proof. The necessity follows by Lemma 1.3. To prove the sufficiency, assume that $\|x^*\|^0 = \|x^*\|$ for $x^* = \xi_v + \varphi$, where $v \in L^{\Phi^*}$, $v \neq 0$, and φ is a singular functional. Applying Lemma 1.1, Lemma 1.3 and the first formula from Lemma 1.4, we get

$$\begin{aligned}\|x^*\|^0 = \|x^*\| = \|\xi_v + \varphi\| \leq \|\xi_v\| + \|\varphi\| = \|v\|_{\Phi^*} + \|\varphi\|^0 \\ < \|v\|^0_{\Phi^*} + \|\varphi\|^0 = \|x^*\|^0,\end{aligned}$$

a contradiction, which finishes the proof. □

Lemma 1.7

There is no nonzero singular functional $\varphi \in (L^\Phi)^*$ attaining its norm on $S(L^\Phi_0)$.

Proof. Let $\varphi \in (L^\Phi)^*$ be singular and let $\varphi \neq 0$. Assume that there is $x \in S(L^\Phi_0)$ such that $\varphi(x) = \|\varphi\| \neq 0$. In virtue of Lemmas 1.1 and 1.3, we get

$$\|\varphi\|^0 = \|\varphi\| = \varphi(x) \leq \|\varphi\|^0 \|x\|_\Phi < \|\varphi\|^0 \|x\|^0_\Phi = \|\varphi\|^0\,,$$

a contradiction that proves the lemma. □

Lemma 1.8

Let $x^* = \xi_v + \varphi$ be a linear continuous functional over L^Φ_0 such that $\|x^*\| = 1$, where $v \in L^{\Phi^*}$ and φ is a singular functional. Then x^* attains its norm at $x \in S\ (L^\Phi_0)$ (or equivalently $x^* \in \mathrm{Grad}(x)$) if and only if for some (equivalently for any) $k \in [k^*, k^{**}]$ there hold:

1^0. $I_{\Phi^*}(v) + \|\varphi\| = 1$;
2^0. $\|\varphi\| = \varphi(kx)$;
3^0. $\langle kx, v\rangle = I_\Phi(kx) + I_{\Phi^*}(v)$, i.e. $v(t) \in \partial\Phi(t, kx(t))$ *for μ-a.e. $t \in T$.*

Proof. It can be proceeded in the same way as for Orlicz spaces in [5]. □

Let us denote $\operatorname{supp} x = \{t \in T : x(t) \neq 0\}$, for any $x \in L^0$.

Lemma 1.9

If $x \in L^\Phi$ and $\theta(x) > 0$, then there exist $h_1, h_2 \in L^\Phi$ with $\mu(\operatorname{supp} h_1 \cap \operatorname{supp} h_2) = 0$, $h_1 + h_2 = x$ and two singular functionals φ_1 and φ_2 in $S((L^\Phi)^*)$ such that $\varphi_1(x) = \varphi_2(x) = \theta(x)$, $\|\varphi_1\| = \|\varphi_2\| = 1$, $\varphi_1 \neq \varphi_2$ and $\varphi_1(h_2) = \varphi_2(h_1) = 0$.

Proof. Let (T_n) be the sequence of sets defined in the Introduction. We have $I_\Phi(k\chi_{T_n}) < +\infty$ for every $k > 0$ and $n \in \mathbb{N}$. Take a sequence $(\lambda_i)_{i=1}^\infty$ of positive numbers such that $\lambda_1 < \lambda_2 < \dots,\ \lambda_i < \theta(x)$ and $\lambda_i \to \theta(x)$ as $i \to +\infty$. Since $I_\Phi(x/\lambda_1) = +\infty$, we can find $n_1 \in \mathbb{N}$ such that for the set

$$A_1 = \big\{t \in T_{n_1} : |x(t)| \le n_1\big\}$$

the inequality $I_\Phi(x\chi_{A_1}/\lambda_1) \ge 2$ is satisfied. We have $I_\Phi(x\chi_{T\setminus A_1}/\lambda_2) = +\infty$ because $I_\Phi(x\chi_{A_1}/\lambda_2) < +\infty$. We can find $n_2 \in \mathbb{N}$ such that the set

$$A_2 = \big\{t \in (T \setminus A_1) \cap T_{n_2} : |x(t)| \le n_2\big\}$$

satisfies $A_1 \cap A_2 = \emptyset$ and $I_\Phi(x\chi_{A_2}/\lambda_2) \ge 2$. Clearly, $I_\Phi(x\chi_{T\setminus(A_1\cup A_2)}/\lambda_3) = +\infty$. Repeating this inductional procedure, we can find a sequence $(A_n)_{n=1}^\infty$ of pairwise disjoint sets such that

$$I_\Phi(x\chi_{A_n}/\lambda_n) \ge 2 \qquad (n = 1, 2, \dots)\,.$$

Divide any set A_n into two measurable disjoint sets A'_n and A''_n in such a way that

$$I_\Phi(x\chi_{A'_n}/\lambda_n) = I_\Phi(x\chi_{A''_n}/\lambda_n) = \frac{1}{2} I_\Phi(x\chi_{A_n}/\lambda_n) \ge 1.$$

Define

$$A = \bigcup_{n=1}^\infty A'_n, \quad B = T \setminus A, \quad h_1 = x\chi_A, \quad h_2 = x\chi_B.$$

Take any $\lambda \in (0, \theta(x))$. Let $m \in \mathbb{N}$ be such that $\lambda \le \lambda_m$. Then

$$\begin{aligned} I_\Phi(h_1/\lambda) &\ge I_\Phi(h_1/\lambda_m) = I_\Phi(x\chi_A/\lambda_m) \\ &\ge I_\Phi\Big(x\chi_{\bigcup_{n=m}^\infty A'_n}/\lambda_m\Big) \ge \sum_{n=m}^\infty I_\Phi\big(x\chi_{A'_n}/\lambda_n\big) = +\infty. \end{aligned}$$

So, by the arbitrariness of $\lambda \in (0, \theta(x))$, we have $\theta(x) = \theta(h_1)$. By Lemma 1.2, we obtain $d_0(h_1) = \theta(x)$. In an analogous way, we get $I_\Phi(h_2/\lambda) = +\infty$ for any $\lambda \in (0, \theta(x))$, whence $d_0(h_2) = \theta(x)$. Define

$$E^\Phi(h_i) = \text{ span}\,\{h_i, E^\Phi\} \quad (i = 1, 2).$$

Every element $u \in E^\Phi(h_i)$ is uniquely represented in the form $u = ah_i + w$, where $a \in \mathbb{R}$ and $w \in E^\Phi$. In fact, if we also have $u = bh_i + z$ with $b \in \mathbb{R}$ and $z \in E^\Phi$, then in the case when $a \neq b$, we get $0 = (a-b)h_i + (w-z)$, whence $h_i = (z-w)/(a-b) \in E^\Phi$, a contradiction. So, $a = b$ and consequently $w = z$. For any $u \in E^\Phi(h_1)$ of the form $u = ah_1 + w\chi_B$, we have $|(h_2 - u)(t)| \geq |(h_2 - w)(t)|$ for μ-a.e. $t \in T$, whence

$$\|h_2 - u\|_\Phi^0 \geq \|h_2 - w\chi_B\|_\Phi^0 \geq d_0(h_2) = \theta(x).$$

So,

$$d_0\big(h_2, E^\Phi(h_1)\big) = \inf\big\{\|h_2 - u\|_\Phi^0 : u \in E^\Phi(h_1)\big\} \geq \theta(x).$$

Moreover, by $E^\Phi \subset E^\Phi(h_1)$, we have $d_0(h_2, E^\Phi(h_1)) \leq d_0(h_2) = \theta(x)$, whence $d_0(h_2, E^\Phi(h_1)) = \theta(x)$. In an analogous way we can get

$$d_0\big(h_1, E^\Phi(h_2)\big) = \inf\big\{\|h_1 - u\|_\Phi^0 : u \in E^\Phi(h_2)\big\} = \theta(x).$$

By the Hahn-Banach theorem there exist singular functionals $\varphi_i \in (L^\Phi)^*$ $(i = 1, 2)$ such that $\|\varphi_i\| = 1$ and

$$\begin{aligned}
&\varphi_1(u_2) = 0 \quad \text{for any} \quad u_2 \in E^\Phi(h_2),\\
&\varphi_2(u_1) = 0 \quad \text{for any} \quad u_1 \in E^\Phi(h_1),\\
&\varphi_1(h_1) = d_0\big(h_1, E^\Phi(h_2)\big) = \theta(x),\\
&\varphi_2(h_2) = d_0\big(h_2, E^\Phi(h_1)\big) = \theta(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varphi_1(x) &= \varphi_1(h_1 + h_2) = \varphi_1(h_1) + \varphi_1(h_2) = \varphi_1(h_1) = \theta(x),\\
\varphi_2(x) &= \varphi_2(h_1 + h_2) = \varphi_2(h_1) + \varphi_2(h_2) = \varphi_2(h_2) = \theta(x).
\end{aligned}$$

The proof is complete. □

2. Main results

We start with the following result.

Theorem 2.1

If $\Phi(t,\cdot)$ is smooth for μ-a.e. $t \in T$, then $x \in S(L_0^\Phi)$ is a smooth point if and only if $\mathrm{Grad}(x)$ *contains a regular functional.*

Proof. Sufficiency. By the assumption there is $v_0 \in S(L^{\Phi^*})$ such that $\|x\|_\Phi^0 = \xi_{v_0}(x) = \int_T x(t)v_0(t)d\mu$. In virtue of Lemma 1.8, we get $I_{\Phi^*}(v_0) = 1$. We will show that ξ_{v_0} is the only support functional at x. Assume that a functional $x^* = \xi_v + \varphi$, where $v \in L^{\Phi^*}$ and φ is a singular functional on L^Φ, is a support functional at x. Then $(x^* + \xi_{v_0})/2$ is also a support functional at x, i.e. $\|(x^* + \xi_{v_0})/2\| = 1$ and

$$\frac{1}{2}\big(x^* + \xi_{v_0}\big)(x) = \frac{1}{2}x^*(x) + \frac{1}{2}\xi_{v_0}(x) = \|x\|_\Phi^0 = 1.$$

By Lemma 1.8 and the convexity of Φ^*, we have

$$(9) \qquad 1 = I_{\Phi^*}\left(\frac{v_0 + v}{2}\right) + \left\|\frac{1}{2}\varphi\right\| \le \frac{1}{2}I_{\Phi^*}(v_0) + \frac{1}{2}I_{\Phi^*}(v) + \left\|\frac{1}{2}\varphi\right\| = 1,$$

whence

$$(10) \qquad I_{\Phi^*}\left(\frac{v_0 + v}{2}\right) = \frac{1}{2}I_{\Phi^*}(v_0) + \frac{1}{2}I_{\Phi^*}(v).$$

By smoothness of $\Phi(t,\cdot)$ for μ-a.e. $t \in T$ we have strict convexity of $\Phi^*(t,\cdot)$, for μ-a.e. $t \in T$. Therefore, equality (10) yields $v_0 = v$, whence $I_{\Phi^*}(v) = 1$ and, in view of (9), it follows that $\varphi = 0$. So, $x^* = \xi_{v_0}$, which means that x is a smooth point.

Necessity. Assume that there is no regular support functional at x. Then it must be $x \notin E^\Phi$, i.e. $\theta(x) > 0$. In view of Lemma 1.9 there are two orthogonal elements $h_1, h_2 \in L^\Phi$ such that $h_1 + h_2 = x$ and two different singular functionals φ_1 and φ_2 such that $\|\varphi_1\| = \|\varphi_2\| = 1$, $\varphi_1(h_2) = \varphi_2(h_1) = 0$ and $\varphi_1(x) = \varphi_2(x) = \theta(x)$. Let $x^* = \xi_v + \varphi \in \mathrm{Grad}(x)$, where $v \in L^{\Phi^*}$ and φ is the singular part of x^*. Define two new functionals by

$$x_i^* = \xi_v + \|\varphi\|\varphi_i \qquad (i = 1, 2).$$

Then $x_i^* \in (L^\Phi)^*$, $x_1^* \ne x_2^*$ and in view of Lemma 1.8

$$I_{\Phi^*}(v) + \|\,\|\varphi\|\varphi_i\| = I_{\Phi^*}(v) + \|\varphi\| = 1,$$

whence $\|x_i^*\| = 1$ for $i = 1, 2$. Moreover, by Lemma 1.3,

$$\varphi_i(x) = \varphi_i(h_i) = \theta(x) \ge \varphi(x)/\|\varphi\| \quad (i = 1, 2).$$

Therefore,

$$\begin{aligned} x_i^*(x) &= \xi_v(x) + \|\varphi\|\varphi_i(x) \ge \xi_v(x) + \varphi(x) \\ &= x^*(x) = \|x\|_\Phi^0 \qquad (i = 1, 2), \end{aligned}$$

whence by the obvious inequality $x_i^*(x) \le \|x\|_\Phi^0$, we get $x_i^*(x) = \|x\|_\Phi^0$ for $i = 1, 2$. This means that x is not a smooth point and the proof is complete. □

Denote by E_0^Φ the space E^Φ equipped with the Orlicz norm.

Theorem 2.2

The space E_0^Φ is smooth if and only if $\Phi(t,\cdot)$ is smooth for μ-a.e. $t \in T$.

Proof. Sufficiency. Since $(E_0^\Phi)^* = L^{\Phi^*}$, for any $x \in S(E_0^\Phi)$, by the Hahn-Banach theorem, we have $\mathrm{R}\,\mathrm{Grad}(x) \neq \emptyset$. By Theorem 2.1 and smoothness of $\Phi(t,\cdot)$ for μ-a.e. $t \in T$, every point $x \in S(E_0^\Phi)$ is smooth, i.e. E_0^Φ is smooth.

Necessity. If $\Phi(t,\cdot)$ is not smooth for μ-a.e. $t \in T$, then the set

$$H = \big\{t \in T : \Phi'_-(t,u) < \Phi'_+(t,u) \quad \text{for some } \ u \in \mathbb{R}_+\big\},$$

which is measurable by the measurability of the derivatives $\Phi'_-(t,\cdot)$ and $\Phi'_+(t,\cdot)$, has a positive measure. Define on H the following multifunction

$$\Gamma(t) = \big\{u \in \mathbb{R}_+ : \Phi'_-(t,u) < \Phi'_+(t,u)\big\}\,.$$

It follows by the definition of the Musielak-Orlicz function Φ (the Carathéodory conditions!) that Φ is $\Sigma \times \mathcal{B}$-measurable, where $\mathcal{B}$ is the Σ-algebra of Borel sets in $\mathbb{R}$. This easily yields the $\Sigma \times \mathcal{B}$-measurability of Φ'_- and Φ'_+. Therefore, Φ'_- and Φ'_+ are also $\Sigma_H \times \mathcal{B}$-measurable, where $\Sigma_H = \{A \cap H : A \in \Sigma\}$, whence

$$\begin{aligned}\mathrm{Graph}\,\Gamma &= \{(t,u) \in H \times \mathbb{R}_+ : u \in \Gamma(t)\}\\ &= \big\{(t,u) \in H \times \mathbb{R}_+ : \Phi'_-(t,u) < \Phi'_+(t,u)\big\} \in \Sigma_H \times \mathcal{B}.\end{aligned}$$

By Theorem 2.5 from [10] there is a measurable selector $f : H \to \mathbb{R}_+$ such that $f(t) \in \Gamma(t)$ for μ-a.e. $t \in H$. Define

$$\Psi_+(t) = \Phi^*\big(t, \Phi'_+(t, f(t))\big), \quad \Psi_-(t) = \Phi^*\big(t, \Phi'_-(t, f(t))\big),$$

and $g(t) = \Psi_+(t) - \Psi_-(t)$. Let A' be a measurable subset of H such that

$$0 < \int_{A'} \Psi_+(t)\, d\mu \leq 1.$$

There is $m \in \mathbb{N}$ such that

$$0 < \int_{A' \cap T_m} \Psi_+(t)\, d\mu \leq 1,$$

where T_m is a set from the sequence of sets defined in the Introduction but for Φ^* in place of Φ. Define

$$A_n = \big\{t \in A' \cap T_m : f(t) \leq n\big\}.$$

The sequence $(A_n)_{n=1}^{\infty}$ is ascending and $\mu((A'\cap T_m)\setminus\bigcup_{n=1}^{\infty} A_n) = 0$. So, it is possible to choose $n_0 \in \mathbb{N}$ in such a way that for $A := A_{n_0}$, we have

$$0 < \int_A \Psi_+(t)\, d\mu \le 1,$$

which implies

$$0 < \int_A g(t)\, d\mu \le 1.$$

Define on $\Sigma \cap A$ a new measure ν by

$$\nu(E) = \int_E g(t)\, d\mu \qquad \big(\forall\ E \in \Sigma \cap A\big).$$

The measure ν is nonatomic, so there is $B \in \Sigma \cap A$ such that $\nu(B) = \nu(A \setminus B)$, i.e.

$$\int_B \big(\Psi_+(t) - \Psi_-(t)\big) d\mu = \int_{A\setminus B} \big(\Psi_+(t) - \Psi_-(t)\big) d\mu,$$

which is equivalent to the equality

$$\int_B \Psi_+(t) d\mu + \int_{A\setminus B} \Psi_-(t) d\mu = \int_{A\setminus B} \Psi_+(t) d\mu + \int_B \Psi_-(t) d\mu.$$

Denote this common value by κ. Clearly, $\kappa \le 1$. The assumption (∞_1) implies that $\Phi^*(t, \Phi'_+(t,u)) \to +\infty$ as $u \to +\infty$ and $\Phi^*(t,\cdot)$ is finitely valued for μ-a.e. $t \in T$. So, we can find $c > 0$ such that

$$\int_{T\setminus A} \Phi^*\big(t, \Phi'_+(t,c)\big)\, d\mu \ge 2.$$

Next, there is $n_0 \in \mathbb{N}$ satisfying

$$\int_{(T\setminus A)\cap T_{n_0}} \Phi^*\big(t, \Phi'_+(t,c)\big) d\mu \ge 3/2.$$

Define

$$D_n = \{t \in (T \setminus A) \cap T_{n_0} : \Phi'_+(t,c) \le n\}; \quad n = 1, 2, \ldots\ .$$

Clearly, $D_n \subset D_{n+1}$ for any $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} D_n = (T \setminus A) \cap T_{n_0}$ up to a set of measure zero. So, the Lebesgue dominated convergence theorem yields

$$\int_{D_n} \Phi^*\big(t, \Phi'_+(t,c)\big) d\mu \to \int_{(T\setminus A)\cap T_{n_0}} \Phi^*\big(t, \Phi'_+(t,c)\big) d\mu.$$

Moreover, the integrals on the left side are finite. Hence, there is $k \in \mathbb{N}$ such that $1 \leq \int_{D_k} \Phi^*(t, \Phi'_+(t,c))d\mu < +\infty$. Define on $\Sigma \cap D_k$ a new measure ν_1 by

$$\nu_1(E) = \int_E \Phi^*\big(t, \Phi'_+(t,c)\big)d\mu \qquad \big(\forall E \in \Sigma \cap D_k\big).$$

Since the measure ν_1 is nonatomic and $\nu_1(D_k) \geq 1$, there is $C \in \Sigma \cap D_k$ such that $\nu_1(C) = 1 - \kappa$. Define

$$v_1(t) = c(t)\chi_{A\setminus B}(t) + d(t)\chi_B(t) + e(t)\chi_C(t),$$
$$v_2(t) = d(t)\chi_{A\setminus B}(t) + c(t)\chi_B(t) + e(t)\chi_C(t),$$

where

$$c(t) = \Phi'_-\big(t, f(t)\big) \quad \text{for} \quad t \in A,$$
$$d(t) = \Phi'_+\big(t, f(t)\big) \quad \text{for} \quad t \in A,$$
$$e(t) = \Phi'_+(t, c) \quad \text{for} \quad t \in C.$$

We have $I_{\Phi^*}(v_1) = I_{\Phi^*}(v_2) = 1$, whence $\|\xi_{v_1}\| = \|v_1\|_{\Phi^*} = 1$ and $\|\xi_{v_2}\| = \|v_2\|_{\Phi^*} = 1$. Take $x \in E_0^\Phi$ defined by

$$x(t) = f(t)\chi_A(t) + c\chi_C(t).$$

Since $v_i(t) \in \partial\Phi(t, x(t))$ for μ-a.e. $t \in T$ $(i = 1, 2)$, we get

$$\|x\|_\Phi^0 \leq I_\Phi(x) + 1 = I_\Phi(x) + I_{\Phi^*}(v_i) = \int_T x(t)v_i(t)d\mu = \xi_{v_i}(x)$$

for $i = 1, 2$. On the other hand

$$\xi_{v_i}(x) \leq \|\xi_{v_i}\| \, \|x\|_\Phi^0 = \|x\|_\Phi^0,$$

so $\xi_{v_i}(x) = \|x\|_\Phi^0$ for $i = 1, 2$. Since $v_1 \neq v_2$, we have $\xi_{v_1} \neq \xi_{v_2}$, which means that x is not a smooth point. Consequently, the space E_0^Φ is not smooth, which completes the proof. □

Theorem 2.3

The space L_0^Φ is smooth if and only if $\Phi(t, \cdot)$ is smooth for μ-a.e. $t \in T$ and Φ satisfies the Δ_2-condition.

Proof. Sufficiency. Since $\Phi \in \Delta_2$ implies that $L_0^\Phi = E_0^\Phi$, by Theorem 2.2, smoothness of Φ yields smoothness of L_0^Φ.

Necessity. Smoothness of L_0^Φ implies smoothness of E_0^Φ, so by Theorem 2.2 also smoothness of Φ. To finish the proof, we only need to show that smoothness of L_0^Φ implies that $\Phi \in \Delta_2$. Since the necessity of smoothness of $\Phi(t,\cdot)$ for μ-a.e. $t \in T$ has been already proved we have to show that if $\Phi(t,\cdot)$ is smooth for μ-a.e. $t \in T$ and $\Phi \notin \Delta_2$, then L_0^Φ is not smooth. If $\Phi \notin \Delta_2$, then $(L_0^\Phi)^* \supsetneq L^{\Phi^*}$. The Bishop-Phelps theorem says that the closure of the linear span of the set of these $x^* \in (L_0^\Phi)^*$ which attain their norms on $S(L_0^\Phi)$ is equal to the whole $(L_0^\Phi)^*$. Since L^{Φ^*} is a closed subspace of $(L_0^\Phi)^*$ (by identifying functions $v \in L^{\Phi^*}$ with functionals ξ_v), there is a functional $x^* = \xi_v + \varphi \in (L_0^\Phi)^*$, where $v \in L^{\Phi^*}$ and φ is a nonzero singular functional, such that $\|x^*\| = 1$ and x^* attains its norm at some point $x \in S(L_0^\Phi)$. This implies that L_0^Φ is not smooth. In fact, if L_0^Φ is smooth then by Theorem 2.1 there is $v_0 \in L^{\Phi^*}$ such that $\xi_{v_0} \in \mathrm{Grad}(x)$. But this means that both x^* and ξ_{v_0} belong to $\mathrm{Grad}(x)$. Since $x^* \neq \xi_{v_0}$, x is not a smooth point, whence L_0^Φ is not a smooth space, a contradiction which finishes the proof. □

Note also that after proving the necessity of smoothness of $\Phi(t,\cdot)$ for μ-a.e. $t \in T$, the necessity of $\Phi \in \Delta_2$ in order that L_0^Φ be smooth can be proved in a different way. Namely, $\Phi \notin \Delta_2$ implies that there is $x \in S(L_0^\Phi)$ with $\theta(x) > 0$. Now, it is enough to apply Theorem 2.1.

Theorem 2.4

Let $x \in S(L_0^\Phi)$. Then x is a smooth point if and only if one of the following conditions is satisfied:

(i) $I_{\Phi^*}(\Phi'_- \circ k^*|x|) = 1$,

(ii) $\theta(k^*x) < 1$ *and* $I_{\Phi^*}(\Phi'_- \circ k^*|x|) = 1$.

Proof. This theorem can be proved in the same way as Theorem 2.1 in [5]. □

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