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Each operator in $\mathcal{L}(l^p, l^r)$ $(1 \le r < p < \infty)$ is compact

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Abstract

It is known that each bounded operator from $l^p \to l^r$ is compact. The purpose of this paper is to present a very simple proof of this useful fact.

1. Introduction

Using construction of normalized block basis it can be proved that each bounded operator from l^p into l^r, $1 \le r < p < \infty$) is compact (see [1], Proposition 2.e.3, p. 76). This result can also be obtained using theory of norm ideals (see [2], 5.1.2).

The aim of this note is to present very elementary proof of this important and useful fact.

2. The main result

By l^p we denote the sequence l^p-space equipped with the standard norm. Let $1 \le r < p < \infty$. By $\mathcal{K}(l^p, l^r)$ we denote the set of all compact operators from l^p into l^r equipped with the operator norm. And by $\mathcal{F}(l^p, l^r)$ we denote the set of all finite rank operators from l^p into l^r. Clearly $\overline{\mathcal{F}(l^p, l^r)}^{\|\cdot\|} = \mathcal{K}(l^p, l^r)$. And $\mathcal{K}(l^p, l^r)$ forms a closed subspace of the space of all bounded operators $\mathcal{L}(l^p, l^r)$.

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Remark. Let E be a normed space. Let $\mathbf{0} \neq \mathbf{x} \in E$ and let $\xi \in E^*$ be such that $\|\xi\| = 1$ and $\xi(\mathbf{x}) = \|\mathbf{x}\|$. Then $\|\mathbf{y} + \lambda\mathbf{x}\| \geq \frac{1}{2}\|\mathbf{y}\|$ for all $\mathbf{y} \in E$ with $\xi(\mathbf{y}) = 0$.

Indeed, $\|\mathbf{y}+\lambda\mathbf{x}\| \geq \|\mathbf{y}\| - |\lambda|\|\mathbf{x}\|$. Additionally $|\lambda|\|\mathbf{x}\| = |\xi(\mathbf{y}+\lambda\mathbf{x})| \leq \|\mathbf{y}+\lambda\mathbf{x}\|$. Thus $\|\mathbf{y} + \lambda\mathbf{x}\| \geq \max\{|\lambda|\|\mathbf{x}\|, \|\mathbf{y}\| - |\lambda|\|\mathbf{x}\|\} = \frac{1}{2}\|\mathbf{y}\|$.

For a normed spaces E , F we denote be $\mathbf{y} \otimes \xi$ the one dimensional operator defined by $\mathbf{y} \otimes \xi(\mathbf{x}) = \mathbf{y}\xi(\mathbf{x})$, $\mathbf{x} \in E,\ \mathbf{y} \in F,\ \xi \in E^*$.

By $P_n : l^p \to l^p$ we denote a projection defined by

$$P_n\mathbf{e}_i = \begin{cases} 1 & \text{if } \ i \leq n \\ 0 & \text{if } \ i > n \end{cases} .$$

By $Q_n : l^r \to l^r$ we denote the analogous projection for l^r.

Theorem

If $1 \leq r < p < \infty$ *then* $\mathcal{L}(l^p, l^r) = \mathcal{K}(l^p, l^r)$.

Proof. Suppose that there exists non compact $T \in \mathcal{L}(l^p, l^r)$. Without loss of generality we can assume that $\|T\| = 1$. Put

$$a = \frac{1}{6} d(T, \mathcal{F}(l^p, l^r)) = \inf\{\|F - T\| : F \in \mathcal{F}(l^p, l^r)\}.$$

Clearly $a > 0$ (since $\mathcal{F}(l^p, l^r) \subset \mathcal{K}(l^p, l^r) \subset \overline{\mathcal{K}(l^p, l^r)}^{\|\cdot\|}$). Consider a function $f(t) = (1+t^p)^{r/p} - a^r t^r$. Choose $t_0 > 0$ such that $f(t_0) < 1$ (for instance $t_0 = a^{r/p-r}$). Let $\varepsilon > 0$ be such that $(1-\varepsilon)^r + a^r t_0^r > (1 + t_0^p)^{r/p}$.

Now choose $\mathbf{x} \in l^p$ and $n, m \in \mathbb{N}$ such that $\|\mathbf{x}\| = 1,\ (I - P_n)\mathbf{x} = \mathbf{0},\ \|Q_m T\mathbf{x}\| > 1 - \varepsilon,\ \|(I - Q_m)T\mathbf{x}\| < at_0$. We find $\eta \in (l^r)^*$ such that $\eta(Q_m T\mathbf{x}) = \|Q_m T\mathbf{x}\|$ and $\|\eta\| = 1$. Put

$$\xi = \frac{(I - P_n)^* T^* Q_m^* \eta}{\|(I - P_n)^* T^* Q_m^* \eta\|},$$

(we admit $\frac{0}{0} = 0$).

Note that if $\xi(\mathbf{z}) = 0$ and $P_n\mathbf{z} = \mathbf{0}$ then $\eta(Q_m T\mathbf{z}) = 0$ and

$$\|Q_m T(\mathbf{x}+\mathbf{z})\| \geq |\eta\big(Q_m T(\mathbf{x}+\mathbf{z})\big)| = \|Q_m T\mathbf{x}\| \geq 1 - \varepsilon\,. \tag{1}$$

We fix $\mathbf{w} \in l^p$ such that $\|\mathbf{w}\| = \xi(\mathbf{w}) = 1$ (if $\xi = 0$ we put $\mathbf{w} = 0$). Obviously $P_n\mathbf{w} = 0$.

Put $R = (I - Q_m)(T - T\mathbf{w} \otimes \xi)(I - P_n)$. We have

$$6a \le \|R\| = \sup\left\{\frac{\|R\mathbf{z}\|}{\|\mathbf{z}\|} : \ \mathbf{z} \ne \mathbf{0}, P_n\mathbf{z} = \mathbf{0}\right\}$$
$$= \sup\left\{\frac{\|R(\mathbf{z} + \lambda\mathbf{w})\|}{\|\mathbf{z} + \lambda\mathbf{w}\|} : \ \mathbf{z} \ne \mathbf{0}, P_n\mathbf{z} = \mathbf{0}, \xi(\mathbf{z}) = 0, \lambda \ \text{ is a scalar}\right\},$$

and by the remark,

$$6a \le 2\sup\left\{\frac{\|R(\mathbf{z} + \lambda\mathbf{w})\|}{\|\mathbf{z}\|} : \ \mathbf{z} \ne \mathbf{0}, P_n\mathbf{z} = \mathbf{0}, \xi(\mathbf{z}) = 0, \lambda \ \text{ is a scalar}\right\}$$
$$= 2\sup\left\{\frac{\|(I - Q_m)T\mathbf{z}\|}{\|\mathbf{z}\|} : \ \mathbf{z} \ne \mathbf{0}, P_n\mathbf{z} = \mathbf{0}, \xi(\mathbf{z}) = 0\right\}.$$

Hence

$$\sup\left\{\frac{\|(I - Q_m)T\mathbf{z}\|}{\|\mathbf{z}\|} : \ \mathbf{z} \ne \mathbf{0}, P_n\mathbf{z} = \mathbf{0}\xi(\mathbf{z}) = 0\right\} \ge 3a.$$

Now choose $\mathbf{u} \in l^p$ such that $\|\mathbf{u}\| = t_0$, $P_n\mathbf{u} = \mathbf{0}$, $\xi(\mathbf{u}) = 0$ and

$$\|(I - Q_m)T\mathbf{u}\| > 2at_0.$$

Note that

$$\|(I - Q_m)T(\mathbf{x} + \mathbf{u})\| \ge \|(I - Q_m)T\mathbf{u}\| - \|(I - Q_m)T\mathbf{x}\| > at_0\,. \tag{2}$$

We have

$$\|T(\mathbf{x} + \mathbf{u})\|_r^r \le \|\mathbf{x} + \mathbf{u}\|_p^r = (1 + t_0^p)^{r/p}\,. \tag{3}$$

And

$$\|T(\mathbf{x} + \mathbf{u})\|_r^r = \|Q_mT(\mathbf{x} + \mathbf{u})\|_r^r + \|(I - Q_m)T(\mathbf{x} + \mathbf{u})\|_r^r\,.$$

And by (1) and (2) we get

$$\ge (1 - \varepsilon)^r + a^r t_0^r > (1 + t_0^p)^{r/p}.$$

This contradicts to (3), and the proof is complete. $\square$

References

1. J. Lindenstrauss and L.Tzafriri, *Classical Banach spaces I, Sequence spaces*, Springer Verlag, 1977.
2. A. Pietsch, *Operator Ideals*, North Holland, 1980.