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Barrelled spaces with Boolean rings of projections

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Abstract

The talk presented a survey of results most of which have been obtained over the last several years in collaboration with M. Florencio and P. J. Paúl (Seville). The results concern the question of barrelledness of locally convex spaces equipped with suitable Boolean algebras or rings of projections. They are particularly applicable to various spaces of measurable vector valued functions. Some of the results are provided with proofs that are much simpler than the original ones.

I. Introduction

If E and F are (Hausdorff) locally convex spaces, $\mathtt{T} \subset L(E,F)$ and

(eqc) = T is equicontinuous
(sb) = T is strongly bounded, i.e., (uniformly) bounded on bounded subsets of E
(pb) = T is pointwise bounded

then

$$\text{(eqc)} \Longrightarrow \text{(sb)} \Longrightarrow \text{(pb)}\,.$$

A locally convex space E is

1. *quasi-barrelled* if (sb) $\Longrightarrow$ (eqc)
2. *Banach-Mackey* if (pb) $\Longrightarrow$ (sb)
3. *barrelled* if (pb) $\Longrightarrow$ (eqc) (Banach-Steinhaus Property)

holds for all locally convex spaces F and all $\mathtt{T} \subset L(E,F)$, or equivalently, for all $\mathtt{T} \subset E'$.

In connection with 2. recall the Banach-Mackey Theorem: *Sequentially complete locally convex spaces are Banach-Mackey.* Trivially,

$$\text{barrelled} = \text{quasi-barrelled} + \text{Banach-Mackey}$$

Also,

$$E \text{ Banach (or Fréchet)} \Longrightarrow E \text{ Baire} \Longrightarrow E \text{ barrelled.}$$

Basic Facts. *Let E be a locally convex space.*

(F1) *E metrizable $\Longrightarrow$ E quasi-barrelled.*

(F2) *$M \subset E'$ is bounded on every null sequence in E $\Longrightarrow$ M is strongly bounded.*

(F3) *E is barrelled iff closed-graph operators from E to Banach spaces are continuous.*

(F4) *E is barrelled iff E is quasi-barrelled and E' is sequentially $\sigma(E', E)$-complete.*

Elementary examples of natural non-complete normed barrelled spaces

1. m_0 = the subspace in ℓ_∞ consisting of elements $x = (\xi_n)$ with finite range.

More generally: If $\mathtt{R}$ is a ring of subsets of a set S, and $m_0(S, \mathtt{R})$ is the space of $\mathtt{R}$-simple scalar functions on S equipped with the sup norm, then

$$m_0(S, \mathtt{R}) \text{ is barrelled} \quad \Longleftrightarrow \quad \mathtt{R} \text{ has the Nikodym Property.}$$

We recall that '$\mathtt{R}$ has the Nikodym Property' means that every pointwise bounded family M of bounded finitely additive scalar measures on $\mathtt{R}$ is uniformly bounded.

That is, if $\sup_{\mu \in M} |\mu(A)| < \infty$ for all $A \in \mathtt{R}$, then $\sup_{A \in \mathtt{R}} \sup_{\mu \in M} |\mu(A)| < \infty$.

2. $(\ell_p, \|\cdot\|_1)$ for $0 < p < 1$, where $\|\cdot\|_1$ denotes the standard ℓ_1-norm.

More generally: If E is an F-space with total dual E', then E equipped with its Mackey topology $\mu(E, E')$ is barrelled. (Recall that $\mu(E, E')$ is the strongest locally convex topology on E that is weaker than its original topology.)

3. $\mathtt{Z}(\ell_1) := \{x \in \ell_1 : \operatorname{supp} x \in \mathtt{Z}\}$, where $\mathtt{Z}$ denotes the ideal of sets A of density zero in $\mathbb{N}$, i.e. such that $\lim_n n^{-1}|A \cap \{1, \ldots, n\}| = 0$.

This result (due to Köthe) is easy using (**F4**) and amounts to verifying that a scalar sequence (η_n) is bounded provided all its zero-density subsequences $(\eta_n)_{n \in A}$ are bounded, which is trivial.

It is less obvious that also for $1 < p < \infty$ the subspace $\mathtt{Z}(\ell_p) := \{x \in \ell_p : \operatorname{supp} x \in \mathtt{Z}\}$ is barrelled. However, one can verify this easily using (**F4**) and a result due to Auerbach (1930): *If* (η_n) *is a scalar sequence and* $\sum_{n\in A} |\eta_n| < \infty$ *for all* $A \in \mathtt{Z}$*, then* $\sum_{n=1}^{\infty} |\eta_n| < \infty$. (See [9] and [10] for more information.) The question of whether $\mathtt{Z}(\ell_\infty)$ is barrelled turned out to be much harder; it has been answered in the positive in [7], see Theorem 8 below.

Throughout

$$(S, \Sigma, \mu) \quad \text{is a positive measure space.}$$

II. Barrelledness of $L_p(\mu, X)(1 \le p < \infty)$ and $\mathtt{P}(\mu, X)$

Theorems 1 and 2 presented in this section are taken from [3]. The proof of Theorem 1 given below was our *original* proof of that result; it 'disappeared' in the course of analyzing and extending its ideas that culminated in Theorem 3 of Section III-a below.

Theorem 1

Let the measure space (S, Σ, μ) *be finite and nonatomic. Then, for every normed space* X *and* $1 \le p < \infty$*, the Bochner space* $E = L_p(\mu, X)$ *is barrelled.*

Proof. We make use of (**F3**). Consider a closed-graph linear map $T : E \to F$, where F is a Banach space. Observe that for every $f \in E$,

$$E_f := \big\{\varphi f \in E : \varphi \in L_0(\mu)\big\}$$

is a norm complete subspace of E. (In fact, $E_f \cong L_p(\nu)$, where $\nu := \int \|f(\cdot)\|^p \, d\mu$.) Hence, by the closed graph theorem, $T|E_f$ is continuous. In particular, since the norm of E is absolutely continuous (that is, $\lim \|f\chi_A\| = 0$ as $\mu(A) \to 0$), we have

$$(*) \qquad \lim_{\mu(A)\to 0} \|T(f\chi_A)\| = 0, \qquad \forall\, f \in E\,.$$

Now we show that every linear map $T : E \to F$ satisfying $(*)$, where now F can be any normed space, is continuous. If it is not, then there is a sequence (f_n) in E with $\|f_n\| \to 0$ and $\|Tf_n\|$ unbounded.

Step 1. Choose n_1 so that $\|Tf_{n_1}\| > 1$. By $(*)$, there is $\delta > 0$ such that $\|T(f_{n_1}\chi_C)\| < \|Tf_{n_1}\| - 1$ whenever $\mu(C) < \delta$. Since μ is atomless, we can write

S as a finite union of disjoint measurable sets of μ measure $< \delta$. For one of these sets, say S_1, the sequence $(\|T(f_n\chi_{S_1})\|)$ must be unbounded. Set $A_1 = S \setminus S_1$; then, clearly, $\|T(f_{n_1}\chi_{A_1})\| > 1$.

Step 2. Choose $n_2 > n_1$ so that $\|T(f_{n_2}\chi_{S_1})\| > 2$. Proceeding as above, we find a set $S_2 \subset S_1$ for which the sequence $(\|T(f_n\chi_{S_2})\|)$ is unbounded, and such that for $A_2 = S_1 \setminus S_2$ we have $\|T(f_{n_2}\chi_{A_2})\| > 2$.

Continuing in this manner, we construct a sequence $n_1 < n_2 < \ldots$ in $\mathbb{N}$ and a disjoint sequence (A_k) in Σ such that, denoting $g_k = f_{n_k}\chi_{A_k}$, we have $\|Tg_k\| > k$ for every k. Now, if we take a subsequence (g_{k_j}) with $\sum_{j=1}^{\infty}\|g_{k_j}\| < \infty$ then, as easily seen, the pointwise sum $g = \sum_{j=1}^{\infty} g_{k_j}$ will be in E. Since $\|T(g\chi_{A_{k_j}})\| = \|T(g_{k_j})\| > k_j \to \infty$ as $j \to \infty$, we have arrived at a contradiction with $(*)$. $\square$

The same proof works for the more general spaces $L(X)$, where $L = (L, \|\cdot\|_L)$ is a solid Banach lattice contained in $L_0(\mu)$ and having absolutely continuous norm, and $L(X)$ consists of those f in $L_0(\mu, X)$ for which $\|f(\cdot)\| \in L$, and is equipped with the norm defined by $\|f\|_{L(X)} = \| \, \|f(\cdot)\| \, \|_L$. Somewhat modified, it also yields the following result.

Theorem 2

Let the measure space (S, Σ, μ) be finite and nonatomic. Then, for every Banach space X, the space $\mathtt{P}(\mu, X)$ of all Pettis μ-integrable functions $f : S \to X$ (with its usual norm) is barrelled.

Similar results can be proved for other spaces of weakly measurable functions.

Remark. For μ as above and X of infinite dimension, the space $\mathtt{P}(\mu, X)$ is never complete (Pettis; Thomas; Janicka and Kalton); nor is it Baire or has Property (K) [11].

III-a. Barrelledness and Boolean algebras of projections

Having proved Theorems 1 and 2, it was tempting to look for a general unified approach that would cover possibly wide range of spaces of measurable vector functions and would be independent of the type of measurability involved. On analyzing the arguments and, especially, the role played by the 'characteristic' projections $P_A(f) = f\chi_A$, we arrived in [3] at the idea of (S, Σ, μ)-Boolean algebras of projections. (Of course, the general concept of Boolean algebras of projections had been used long since before.)

An (S, Σ, μ)-*Boolean algebra of projections* in a topological vector space E is a map $\mathbb{P}_\Sigma$ that assigns to every $A \in \Sigma$ a continuous linear projection P_A in E so that

(a) $P_S = \mathrm{id}_E$,
$P_{A\cap B} = P_A \circ P_B$,
$P_{A\cup B} = P_A + P_B \quad \text{if} \quad A \cap B = \emptyset.$

(b) $P_A = 0 \quad \text{if} \quad \mu(A) = 0.$

It is said to be *countably additive* if

(c) for every $x \in E$ the vector measure $p_x : \Sigma \to E\,;\ A \to P_A(x)$ is countably additive,

and *disjointly* K-*complete* if

(d) whenever (A_n) is a disjoint sequence in Σ and (x_n) is a null sequence in E such that

$$P_{A_n}(x_n) = x_n, \qquad n = 1, 2, \ldots$$

then there exist $n_1 < n_2 < \ldots$ in $\mathbb{N}$ and $x \in E$ with

$$P_{A_{n_k}}(x) = x_{n_k}, \qquad k = 1, 2, \ldots$$

Remarks.

1. In view of (b), condition (c) is equivalent to: $\lim P_A(x) = 0$ as $\mu(A) \to 0$.

2. If $\mathbb{P}_\Sigma$ is countably additive, then the requirement in (d) can be expressed as follows: there exist $n_1 < n_2 \ldots$ for which the series $\sum_{k=1}^{\infty} x_{n_k}$ converges in E.

3. Let us recall that a space E is said to have *Property* (K) if every null sequence (x_n) in E has a subsequence (x_{n_k}) for which the series $\sum_k x_{n_k}$ is convergent. Thus condition (d) can indeed be viewed as a 'disjoint' version of Property (K). Other variants of this property have been considered in [5] for spaces with Schauder type decompositions.

Obviously, if a space E of (strongly or weakly) measurable vector functions over a measure space (S, Σ, μ) is such that $f\chi_A \in E$ whenever $f \in E$ and $A \in \Sigma$, then it will be considered with its natural (S, Σ, μ)-Boolean algebra of projection $\mathbb{P}_\Sigma = \{P_A : A \in \Sigma\}$, where $P_A(f) := f\chi_A$.

Theorem 3

Let the measure space (S, Σ, μ) be finite and nonatomic. Assume that E is a locally convex space with an equicontinuous and countably additive (S, Σ, μ)-Boolean algebra of projections $\mathbb{P}_\Sigma$. If $\mathbb{P}_\Sigma$ is disjointly K*-complete, then E is Banach-Mackey.*

Sketch of the Proof. Let $M \subset E'$ be pointwise bounded. We have to show that M is strongly bounded. By (**F2**) it is enough to show that M is (uniformly) bounded on null sequences in E. Suppose it is not so. Then there is a null sequence (x_n) in E on which M is not bounded. Using μ-continuity of the scalar measures $u \circ p_x$ ($u \in E'$, $x \in E$), and proceeding as in the proof of Theorem 1, we find $n_k \uparrow$ in $\mathbb{N}$, $u_k \in M$, and disjoint sets $A_k \in \Sigma$ so that

$$\big|\langle u_k, P_{A_k}(x_{n_k})\rangle\big| \to \infty \quad \text{as } k \to \infty\,.$$

Since $\mathbb{P}_\Sigma$ is equicontinuous, $y_k = P_{A_k}(x_{n_k}) \to 0$ as $k \to \infty$. Also, $P_{A_k}(y_k) = y_k$ for all k. Let $y \in E$ be provided by (d) for a suitable subsequence of (y_k), still denoted by (y_k).

Clearly, the family of scalar measures $\{u \circ p_y : u \in M\}$ is pointwise bounded on Σ. Therefore, by the Nikodym theorem, it is uniformly bounded on Σ. In particular, the sequence $\langle u_k, p_y(A_k)\rangle = \langle u_k, P_{A_k}(x_{n_k})\rangle$, $k = 1, 2, \ldots$, must be bounded. A contradiction. □

Remark. The analogy between Property (K) and the disjoint K-completeness is further supported by the following consequence of a result due to Antosik that has been noted in [5]: *A locally convex space with Property* (K) *is Banach-Mackey.*

It is easy to see that Theorem 3 remains valid for nonatomic σ-finite measures μ. (Simply replace μ by a finite measure ν having the same null sets.) In fact, by modifying slightly the proof of Theorem 3 so as to have $\mu(A_n) < \infty$, it is enough to assume that $\mathbb{P}_\Sigma$ is disjointly K-complete in a somewhat weaker sense, viz.,

(d) whenever (A_n) is a disjoint sequence in Σ of sets of <u>finite</u> μ measure and (x_n) is a null sequence in E such that

$$P_{A_n}(x_n) = x_n, \qquad n = 1, 2, \ldots$$

then there exist $n_1 < n_2 < \ldots$ in $\mathbb{N}$ and $x \in E$ with

$$P_{A_{n_k}}(x) = x_{n_k}, \qquad k = 1, 2, \ldots\,.$$

Let us also observe that, by (**F2**), *if every null sequence in a locally convex space E is contained in a Banach-Mackey subspace of E, then E itself is Banach-Mackey.*

In consequence, we quickly arrive at the following extension of Theorem 3 proved in [6].

Theorem 4

Let the measure space (S, Σ, μ) have no atoms of finite measure. Assume that E is a locally convex space with an equicontinuous and countably additive (S, Σ, μ)-Boolean algebra of projections $\mathbb{P}_\Sigma$. If $\mathbb{P}_\Sigma$ is disjointly K*-complete and*

(e) *$\mathbb{P}_\Sigma$ is σ-finite (relative to μ), that is, for every $x \in E$ there is a set $B \in \Sigma$ of σ-finite μ measure such that $P_B x = x$,*

then E is Banach-Mackey.

An even more general result holds in which condition (e) is replaced by the following:

(e′) For every $x \in E$ the measure p_x 'vanishes at infinity', that is, for every neighborhood U of zero in E there exists $A \in \Sigma$ with $\mu(A) < \infty$ such that $P_{S \setminus A}(x) \in U$.

Finally, in quasi-barrelled spaces the assumption that $\mathbb{P}_\Sigma$ is equicontinuous can be omitted (see [3]):

Lemma

Every countably additive (S, Σ, μ)-Boolean algebra of projections in a quasi-barrelled space is equicontinuous.

Corollary

If the space E in Theorems 3 or 4 (or in Theorem 6 below) is quasi-barrelled, then it is barrelled.

From this corollary it is easy to deduce both Theorems 1 and 2, as well as similar results for other spaces of measurable vector functions (see [3] and [6]).

IV. Barrelledness of $L_\infty(\mu, X)$

It was essential in the proof of Theorem 1 that the norm in $L_p(\mu, X)$ was absolutely continuous and, in the proof of Theorem 3, that the vector measures p_x were countably additive. Thus the case of the spaces $L_\infty(\mu, X)$ was covered neither by Theorem 1 nor Theorem 3. Its treatment required a new idea, and was carried out in [1].

Theorem 5

Let the measure space (S, Σ, μ) be σ-finite and nonatomic. Then, for every normed space X, the space $E = L_\infty(\mu, X)$ is barrelled.

Proof. Without loss of generality we may assume that μ is finite. Let $M \subset E'$ be pointwise bounded but not norm bounded. Note that if, for some $A \in \Sigma$, the set M is not norm bounded on $P_A(E) = L_\infty(A, \mu, X)$, and if $A = A' \cup A''$, where A', $A'' \in \Sigma$, $A' \cap A'' = \emptyset$, then M is not norm bounded on at least one of the subspaces $P_{A'}(E)$ or $P_{A''}(E)$. Using this observation with $\mu(A') = \mu(A'') = \frac{1}{2}\mu(A)$, we easily construct a sequence $A_n \downarrow \emptyset$ in Σ such that M is not norm bounded on $P_{A_n}(E) = L_\infty(A_n, \mu, X)$ for each n. Thus we can choose $u_n \in M$ and $f_n \in L_\infty(\mu, X)$ with $\|f_n\| = 1$ (or even $\|f_n\| \to 0$) and $\operatorname{supp} f_n \subset A_n$ so that

$$\bigl|\langle u_n, f_n\rangle\bigr| > n\,.$$

Now we may define a continuous linear operator

$$T : \ell_1 \to L_\infty(\mu, X) \quad \text{by} \quad T(a_n) = \sum_{n=1}^{\infty} a_n f_n \qquad \text{(pointwise sum)}\,.$$

Then the functionals $u \circ T$, $u \in M$, are pointwise bounded on ℓ_1, hence norm bounded, by the Banach-Steinhaus theorem. But

$$\bigl|\langle u_n \circ T, e_n\rangle\bigr| = \bigl|\langle u_n, f_n\rangle\bigr| > n, \qquad \forall\, n \in \mathbb{N}\,;$$

a contradiction. □

III-b. Barrelledness and Boolean algebras of projections

As was the case with Theorem 1, an analysis of the proof of Theorem 5 lead to the following general result [2].

Theorem 6

Let the measure space (S, Σ, μ) be σ-finite and nonatomic. Assume that E is a locally convex space with an equicontinuous (S, Σ, μ)-Boolean algebra of projections $\mathbb{P}_\Sigma$ such that

(d′) *whenever $A_n \downarrow \emptyset$ in Σ and (x_n) is a null sequence in E with $P_{A_n}(x_n) = x_n$, then there is an operator $T : \ell_1 \to E$ such that $Te_n = x_n$.*

Then E is Banach-Mackey.

Proof. Follow the ideas of the proof of Theorem 4 modified in the spirit of the proof of Theorem 3. □

Thus, in consequence, if μ is σ-finite and nonatomic and $L = (L, \|\cdot\|_L)$ is a solid Banach lattice $\subset L_0(\mu)$, then for every normed space X the space $L(X)$ is barrelled.

Remark. Theorem 6 holds also for an arbitrary measure space (S, Σ, μ) without atoms of finite measure provided that $\mathbb{P}_\Sigma$ is σ-finite. In view of Theorem 7 below, it is unlikely that this latter assumption could be substantially weakened.

It is also worth noting that Theorem 6 can be applied in all particular cases where Theorem 4 has been previously used. In general, however, these two results seem to be incomparable.

V. Barrelledness of $\ell_\infty(S, X)$

We recall that a set T, or its cardinal number $|T|$, is said to be (*Ulam*) *measurable* if there exists a countably additive measure $\mu : \mathtt{P}(T) \to \{0, 1\}$ such that $\mu(T) = 1$ and $\mu(\{t\}) = 0$ for all $t \in T$.

The following is a particular case of a result established in [8].

Theorem 7

Let S be a set and X a barrelled normed space. If $|S|$ or $|X|$ is non-measurable, then the space $\ell_\infty(S, X)$ is barrelled.

Sketch of the Proof. For the case where $|S|$ is non-measurable. Let M be a pointwise bounded set in the dual of $\ell_\infty(S, X)$, and define a submeasure $\eta : \mathtt{P}(S) \to \overline{\mathbb{R}}_+$ by

$$\eta(A) = \sup\left\{|u(f)| : u \in M,\ \|f\| \le 1,\ \operatorname{supp} f \subset A\right\}.$$

Suppose M is not norm bounded; in other words, $\eta(S) = \infty$. Using the fact that $\ell_\infty(S, X)$ satisfies condition (d′) from Theorem 6, it is shown that

(1) whenever $A_n \downarrow A$ and $\eta(A_n) = \infty$ for all n, then also $\eta(A) = \infty$.

Moreover, it is shown that

(2) every disjoint family of sets A with $\eta(A) > 0$ is countable.

From (1) and (2) it is deduced that there is a set $S' \subset S$ with $\eta(S') < \infty$ such that η assumes only the values 0 and ∞ on subsets of $S'' = S \backslash S'$. Using (1) it is then easy to show that there is $S_0 \subset S''$ such that for every $A \subset S_0$ one of the values $\eta(A)$ and $\eta(S_0 \backslash A)$ is 0 and the other is ∞. Appealing to (1) one more time it is clear that $\mathtt{U} := \{A \subset S_0 : \eta(A) = \infty\}$ is an ultrafilter on S_0 which is closed under countable intersections. Moreover, $\{s\} \notin \mathtt{U}$ for all $s \in S_0$; indeed, since E is barrelled, $\eta(\{s\}) = 0$. It follows that S_0 is measurable, and so is S. A contradiction. □

VI. Barrelledness of $\mathtt{Z}(\ell_\infty)$ and similar spaces

In a somewhat different way, Boolean rings of projections entered the scene also in our extensions of the results mentioned in group 3 of examples in the Introduction. We first present some results from [7].

Let $\mathtt{R}$ be a ring of subsets of a set S. An $\mathtt{R}$-ring of projections in a locally convex space E is a map $\mathbb{P}_{\mathtt{R}}$ that assigns to every $A \in \mathtt{R}$ a continuous linear projection P_A in E so that $P_{A\cap B} = P_A \circ P_B$ for all $A, B \in \mathtt{R}$, and $P_{A\cup B} = P_A + P_B$ for all disjoint $A, B \in \mathtt{R}$. Given $\mathbb{P}_{\mathtt{R}}$, we are interested in the following subspace of E:

$$\mathtt{R}(E) := \big\{x \in E : P_A(x) = x \quad \text{for some} \quad A \in \mathtt{R}\big\}.$$

Recall that $\mathtt{Z}$ denotes the ideal of sets of density zero in $\mathbb{N}$.

Theorem 8

Assume that E is a locally convex space with an equicontinuous ring of projections $\mathbb{P}_{\mathtt{Z}}$. If E is barrelled, so is $\mathtt{Z}(E)$. In particular, $\mathtt{Z}(\ell_\infty)$ is barrelled.

A crucial point in establishing Theorem 8 was proving the following.

Theorem 9

$\mathtt{Z}$ has the Nikodym Property.

In fact, a result more general than Theorem 8 was shown to hold.

Theorem 10

Assume that E is a locally convex space with an equicontinuous ring of projections $\mathbb{P}_{\mathtt{R}}$.
(a) If E is quasi-barrelled, so is $\mathtt{R}(E)$.
(b) If $\mathtt{R}$ has the Nikodym Property and E is Banach-Mackey or barrelled, so is $\mathtt{R}(E)$.

Motivated by the above results, we investigated in [9] conditions assuring that a ring of sets has the Nikodym Property. In particular, we showed that the Nikodym Property of $\mathtt{Z}$ is an easy consequence of another remarkable property of $\mathtt{Z}$.

Theorem 11

Every finite submeasure η on $\mathtt{Z}$ is bounded: $\sup\{\eta(A) : A \in \mathtt{Z}\} < \infty$.

In [9], an application of Theorem 11 to results of the type of Theorem 8 was also indicated.

Theorem 12

For every solid Banach sequence space E, the subspace

$$\mathtt{Z}(E) := \{x \in E : \operatorname{supp} x \in \mathtt{Z}\}$$

of E is barrelled.

Proof. Consider a closed-graph linear operator $T : \mathtt{Z}(E) \to F$, where F is a Banach space. Note that, for each $A \in \mathtt{Z}$, the subspace

$$E_A := \{x \in E : \operatorname{supp} x \subset A\} \subset \mathtt{Z}(E)$$

is complete, and $T_A := T|E_A$ has a closed graph. By the closed graph theorem, T_A is continuous, so $\tau(A) := \|T_A\| < \infty$. Clearly, τ is a submeasure on $\mathtt{Z}$. By Theorem 11, τ is bounded on $\mathtt{Z}$. Thus, $\|T\| = \sup_{A\in\mathtt{Z}} \tau(A) < \infty$. By (**F3**), $\mathtt{Z}(E)$ is barrelled. □

The assumption in Theorem 10(b) that $\mathtt{R}$ has the Nikodym Property is essential:

Theorem 13

Let $\mathtt{R}$ be an ideal of subsets of $\mathbb{N}$ such that $\mathtt{R}(\ell_\infty)$ is barrelled. Then $\mathtt{R}$ has the Nikodym Property.

Proof. In view of [9, Prop. 2.1], it is enough to show that every finitely additive measure $\mu : \mathtt{R} \to \mathbb{R}_+$ is bounded. To this aim, define for every $A \in \mathtt{R}$ a measure μ_A on $\mathtt{P}(\mathbb{N})$ by $\mu_A(E) = \mu(A \cap E)$, and let x_A^* be the continuous linear functional on $\mathtt{R}(\ell_\infty)$ determined by μ_A. Thus $x_A^*(x) = \int_{\mathbb{N}} x\, d\mu_A$ for every $x \in \mathtt{R}(\ell_\infty)$ and, as easily seen, $\|x_A^*\| = \mu(A)$.

Now, if $x \in \mathtt{R}(\ell_\infty)$, then $B := \operatorname{supp} x \in \mathtt{R}$ and, for every $A \in \mathtt{R}$,

$$|x_A^*(x)| = \left|\int_{\mathbb{N}} x\, d\mu_A\right| = \left|\int_{\mathbb{N}} x\chi_B\, d\mu_A\right| = \left|\int_{\mathbb{N}} x\, d\mu_{A\cap B}\right| \\ \leq \mu_{A\cap B}(\mathbb{N}) \cdot \|x\|_\infty \leq \mu(B)\, \|x\|_\infty .$$

In consequence, the family $\{x_A^* : A \in \mathtt{R}\}$ is pointwise bounded on $\mathtt{R}(\ell_\infty)$. Since $\mathtt{R}(\ell_\infty)$ is barrelled, it follows that

$$\sup_{A\in\mathtt{R}} \mu(A) = \sup_{A\in\mathtt{R}} \|x_A^*\| < \infty,$$

i.e., μ is bounded. $\square$

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