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Weak compactness criteria and convergences in $L^1_E(\mu)$

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Abstract

New characterizations of conditionally weakly compact (resp. relatively weakly compact) subsets in Banach space E and $L^1_E(\mu)$ are presented. We discuss also several types of convergence in $L^1_E(\mu)$, in particular we generalize Szlenk's theorem on Cesàro norm-convergence of weakly null sequences in $L^1_{\mathbb{R}}(\mu)$ to the norm-summability with respect to a class of regular method of summability of weakly null sequences in $L^1_H(\mu)$ where H is a Hilbert space.

Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space, E a Banach space, and $L^1_E(\mu)$ the Banach space of Bochner integrable functions equipped with its usual norm. We discuss here the characterizations of conditionally weakly compact, i.e. sequentially weakly precompact (resp. relative weakly compact) subsets in a Banach space E and in $L^1_E(\mu)$. We refer to [27] and [13] for recent results on the problem of characterizing relatively weakly compact (r.w.c.) subsets of E and $L^1_E(\mu)$. In section 1, new characterizations of conditionally weakly compact (c.w.c.) (resp. r.w.c.) subsets in Banach spaces via a class of regular method of summability (RMS) $a = (a_{pq})$

(cf. [15] p. 75) are presented. A subset $K \subset E$ is c.w.c. (resp. r.w.c.) iff for any sequence (x_n) in K, there exists a subsequence (x_{n_k}) such that the sequence (s_k) with $s_k = \sum_{q=0}^{\infty} a_{kq} x_{n_q}$ $(k \in \mathbb{N})$ is well-defined and weakly Cauchy (resp. weakly convergent). This characterization is equivalent to the following: for any sequence (x_n) in K, there exists a sequence $(\tilde{x}_n)$ with $\tilde{x}_n \in \operatorname{co}\{x_m : m \geq n\}$, such that $(\tilde{x}_n)$ is weakly Cauchy (resp. weakly convergent). In section 2, several criteria for c.w.c. and r.w.c. subsets in $L^1_E(\mu)$ are presented. In particular we show that a bounded uniformly integrable and ball-conditionally weakly compact-tight subset in $L^1_E(\mu)$ is c.w.c. This paper also contains several types of convergence in $L^1_E(\mu)$ with applications to Mathematical Economics and Minimization problems. In particular we discuss Banach-Saks property for weakly null sequences in $L^1_{\mathbb{R}}(\mu)$. This result is as follows: Let H be a Hilbert space, $a = (a_{pq})$ a RMS such that $\lim_{p\to\infty} \sum_{q=0}^{\infty} |a_{pq}|^2 = 0$ and (u_n) a weakly null sequence in $L^1_H(\mu)$. Then there exist $\psi \in Si(\mathbb{N})$ such that

$$\lim_{p\to\infty} \sup_{\varphi \in Si(\mathbb{N})} \left\| \sum_{q=0}^{\infty} a_{pq}\, u_{\psi\circ\varphi(q)} \right\|_1 = 0$$

($Si(\mathbb{N})$ denotes the set of strictly increasing mapping $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$).

Most of our proofs are detailed and easy, except for some of them which rely on deeper results due to Rosenthal [22] and Talagrand [26].

Notations and Preliminaries

We will use the following notions and notations. We denote by

– $(\Omega, \mathcal{F}, \mu)$ a complete probability space,

– E a Banach space,

– E' the topological dual of E, E'_s (resp. E'_b) the vector space E' equipped with the $\sigma(E', E)$ (resp. norm) topology.

– $\overline{B}_E$ (resp. $\overline{B}_{E'}$) is the closed ball of center 0 and radius 1 in E (resp. E').

– $\mathcal{R}_{wk}(E)$ (resp. $\mathcal{R}_{cwc}(E)$) the collection of borelian subsets of E such that its intersection with any ball of E is relatively weakly (conditionally weakly) compact.

– $\delta^*(., A)$ is the support function of a subset A of E.

– $L^1_E(\mu)$ is the space of Bochner integrable mapping $u : \Omega \longrightarrow E$ and $L^\infty_{E'}(\mu)$ is the topological dual of $L^1_E(\mu)$ (cf. A. and C. Ionescu Tulcea [18]).

– If X is a topological space, $\mathcal{B}(X)$ is the Borel tribe of X.

– A multifunction $\Gamma : \Omega \longrightarrow \mathcal{B}(X)$ is measurable if its graph $Gr(\Gamma)$ belongs to $\mathcal{F} \otimes \mathcal{B}(X)$.

– $Si(\mathbb{N})$ is the set of strictly increasing mappings from $\mathbb{N}$ to $\mathbb{N}$.

– A subset $\mathcal{H}$ of $L^1_E(\mu)$ is $\mathcal{R}_{wk}(E)$ (resp. $\mathcal{R}_{cwc}(E)$)-tight if for every $\varepsilon > 0$, there exists a measurable multifunction $\Gamma_\varepsilon : \Omega \longrightarrow \mathcal{R}_{wk}(E)$ (resp. $\mathcal{R}_{cwc}(E)$) such that $\forall u \in \mathcal{H}$, $\mu[\{\omega \in \Omega : u(\omega) \notin \Gamma_\varepsilon(\omega)\}] < \varepsilon$.

– If (x_n) is a sequence in E, $w - Ls\{x_n\}$ is defined by

$$w - Ls\{x_n\} := \bigcap_{n=1}^{\infty} \overline{\{x_k : k \geq n\}}^{\sigma}$$

where $\overline{\{.\}}^{\sigma}$ denotes the closure for the $\sigma(E, E')$ topology.

§1. Weak compactness and conditionally weak compactness in Banach spaces

An infinite matrix $(a_{pq})_{(p,q)\in\mathbb{N}\times\mathbb{N}}$ is called a *regular method of summability* (RMS) if

(1.1) $\sup_{p\in\mathbb{N}} \sum_{q=0}^{\infty} |a_{pq}| < +\infty$

(1.2) $\forall q \in \mathbb{N}$, $\lim_{p\to\infty} a_{pq} = 0$

(1.3) $\lim_{p\to\infty} \sum_{q=0}^{\infty} a_{pq} = 1$.

It is easy to check that $a = (a_{pq})$ is a RMS iff for any sequence (x_n) in a Banach space E, converging to $x \in E$, then the sequence (x'_n) given by $x'_n = \sum_{q=0}^{\infty} a_{nq}x_q$, converges to x. A sequence (x_n) in a Banach space is called *summable* with respect to a RMS $a = (a_{pq})$ if the sequence (x'_n) given by $x'_n = \sum_{q=0}^{\infty} a_{nq}x_q$ is well-defined and converges for the norm of E. A RMS $a = (a_{pq})$ is *positive* if, $\forall p, q$, $a_{pq} \geq 0$.

Let us mention first an easy lemma before we state the main results.

Lemma 1.1

Let (a_{pq}) be a positive RMS and let (x_n) be a sequence in $\mathbb{R}$ such that the series $\sum_{q=0}^{\infty} a_{pq}x_q$ are convergent. Then we have

(1) $\liminf_{p\to\infty} x_p \leq \liminf_{p\to\infty} \sum_{q=0}^{\infty} a_{pq}x_q$

In particular, if $\tilde{x}_n \in \mathrm{co}\{x_k : k \geq n\}$, $\forall n$, then we have

(2) $\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} \tilde{x}_n$.

Proof. (1) let $(x_n) \subset \mathbb{R}$ such that the series $u_p := \sum_{q=0}^{\infty} a_{pq} x_q$ are convergent in $\mathbb{R}$. Let $r < \liminf_{n\to\infty} x_n = \sup_n \inf_{k\geq n} x_k$.

Then there exists a positive integer n_0 such that $k \geq n_0$ implies $r < x_k$. Hence $\forall q \geq n_0$, $a_{pq} r \leq a_{pq} x_q$. Therefore $\big(\sum_{q=n_0}^{\infty} a_{pq}\big) r \leq \sum_{q=n_0}^{\infty} a_{pq} x_q$. Consequently we get

$(*)$ $\big(\sum_{q=0}^{\infty} a_{pq}\big) r - \big(\sum_{q=0}^{n_0-1} a_{pq}\big) r \leq u_p - \sum_{q=0}^{n_0-1} a_{pq} x_q$.

Since $\lim_{p\to\infty} \sum_{q=0}^{\infty} a_{pq} = 1$, and $\lim_{p\to\infty} \sum_{q=0}^{n_0-1} a_{pq} = 0$ by virtue of properties (1.2) and (1.3) of the RMS and since $\lim_{p\to\infty} \sum_{q=0}^{n_0-1} a_{pq} x_q = 0$, then by taking the lim inf in $(*)$, we obtain

$$r \leq \liminf_{p\to\infty} \Big(u_p - \sum_{q=0}^{n_0-1} a_{pq} x_q\Big) = \liminf_{p\to\infty} u_p \,.$$

It follows that $\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} \sum_{q=0}^{\infty} a_{nq} x_q$.

(2) is easy consequence of (1). □

Now we are able to produce the main results of this section.

Theorem 1.2

Let K be a subset of a Banach space E and let $a = (a_{pq})$ be a positive RMS. Then the following are equivalent:

(1) K is conditionally weakly compact.

(2) given any sequence $(x_n)_n \subset K$, there exists a subsequence $(x_{n_k})_k$ such that the sequence $(s_k)_k$ with $s_k = \sum_{q=0}^{\infty} a_{kq} x_{n_q}$ $(k \in \mathbb{N})$ is well-defined and weakly Cauchy.

(3) given any sequence $(x_n)_n \subset K$, there exists a sequence $(\tilde{x}_n)_n$ with $\tilde{x}_n \in \mathrm{co}\,\{x_m : m \geq n\}$ such that $(\tilde{x}_n)$ is weakly Cauchy.

Proof. The implication (1) $\Longrightarrow$ (2) follows easily from properties of the RMS. Let us prove (2) $\Longrightarrow$ (3).

Since K satisfies (2), K is bounded. Indeed it is enough to check that, $\forall x' \in E'$, we have $\delta^*(x', K) = \sup_{x\in K} \langle x', x\rangle < +\infty$. Take a sequence $(u_n) \subset K$ such that $\lim_{n\to\infty} \langle x', u_n\rangle = \delta^*(x', K)$. By (2), there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$ such that the sequence $(v_p)_p$ with $v_p := \sum_{q=0}^{\infty} a_{pq} u_{n_q}$ is well-defined and weakly Cauchy. Hence the sequence $(\langle x', v_p\rangle)_p$ with $\langle x', v_p\rangle = \sum_{q=0}^{\infty} a_{pq} \langle x', u_{n_q}\rangle$ converge in $\mathbb{R}$ to a point $v_{x'}$. Clearly by obvious properties of the RMS, we have

$$\begin{aligned}\delta^*(x', K) &= \lim_{p\to\infty} \langle x', u_{n_p}\rangle \\ &= \liminf_{p\to\infty} \sum_{q=0}^{\infty} a_{pq}\langle x', u_{n_q}\rangle = v_{x'} < +\infty.\end{aligned}$$

Now set $M := \sup\{\|x\| : x \in K\}$ and let us prove that K satisfies (3).

Let $(x_n) \subset K$ and let $s_k = \sum_{q=0}^{\infty} a_{kq} x_{n_q}$ given by (2). For each $x' \in E'$, let $r_{x'} = \lim_{k\to\infty}\langle x', s_k\rangle$. According to properties (1.1) and (1.2) of the RMS, it is easy to construct two strictly increasing sequences of positive integers (N_p) and (p_k) such that

$$\forall p,\ \forall k \geq 1,\ \sum_{q>N_p} a_{pq} \leq 2^{-p} \text{ and } \sum_{q=0}^{k-1} a_{p_k q} \leq 2^{-k}. \tag{1.2.1}$$

For every $k \geq 1$, set $\lambda_k := \sum_{q=k}^{N_{p_k}} a_{p_k q}$. Then by (1.2.1), we obtain

$$0 \leq \sum_{q=0}^{\infty} a_{p_k q} - \lambda_k \leq 2^{-k} + 2^{-p_k}.$$

Consequently by property (1.3) of the RMS, we deduce that $\lim_{k\to\infty} \lambda_k = 1$. Set

$$\forall k, \lambda_q^k := \frac{1}{\lambda_k} a_{p_k q} \text{ and } \tilde{x}_k := \sum_{q=k}^{N_{p_k}} \lambda_q^k x_{n_q}.$$

Then it is clear that $\tilde{x}_k \in \mathrm{co}\,\{x_{n_q} : q \geq k\}$. Moreover, for every k, we have

$$\begin{aligned}|\langle x', \tilde{x}_k\rangle - s_{x'}| &= \Big|\frac{1}{\lambda_k}\Big[\langle x', s_{p_k}\rangle - \Big\langle x', \sum_{q=0}^{k-1} a_{p_k q}\, x_{n_q} + \sum_{q>N_{p_k}} a_{p_k q}\, x_{n_q}\Big\rangle\Big] - s_{x'}\Big| \\ &\leq \Big|\frac{1}{\lambda_k}\langle x', s_{p_k}\rangle - s_{x'}\Big| + \frac{M\|x'\|}{\lambda_k}(2^{-k} + 2^{-p_k}).\end{aligned}$$

Hence it follows that $\lim_{k\to\infty}\langle x', \tilde{x}_k\rangle = s_{x'}$. Whence $(\tilde{x}_k)$ is weakly Cauchy and satisfies $\tilde{x}_k \in \mathrm{co}\,\{x_m : m \geq k\}$, $\forall k$.

Now it remains to prove (3) $\Longrightarrow$ (1). By using lemma 1.1, we can show similarly as in the previous implication that K is bounded. Assume by contradiction that K is not conditionally weakly compact. Then according to a result of H. P. Rosenthal (see [22]), there exist $r \in \mathbb{R}$, $\delta > 0$ and a sequence $(x_n)_n \subset K$ such that the sequence $(A_n, B_n)_{n\in\mathbb{N}}$ defined by

$$A_n = \{x' \in \overline{B}_{E'} : \langle x', x_n\rangle \geq r + \delta\} \quad \text{and} \quad B_n = \{x' \in \overline{B}_{E'} : \langle x', x_n\rangle \leq r\}$$

is independent. By (3), there exists $\tilde{x}_n \in \operatorname{co}\{x_m : m \geq n\}$ $(n \in \mathbb{N})$ such that $(\tilde{x}_n)$ is weakly Cauchy. Each $\tilde{x}_n$ has the form $\tilde{x}_n = \sum_{i=n}^{m_n} \lambda_i^n x_i$ with $\lambda_i^n \geq 0$, $\sum_{i=n}^{m_n} \lambda_i^n = 1$, $m_n \geq n$. Let $n_0 = 0$, $n_1 = m_0 + 1, \ldots, n_{k+1} = m_{n_k} + 1$. Then (n_k) is a strictly increasing sequence such that for all $i \neq j$, $[n_i, m_{n_i}] \cap [n_j, m_{n_j}] = \emptyset$.

Now let us consider the following sets

$$\widetilde{A}_k := \bigcap_{i=n_k}^{m_{n_k}} A_i \quad \text{and} \quad \widetilde{B}_k := \bigcap_{i=n_k}^{m_{n_k}} B_i\,.$$

Then $(\widetilde{A}_k, \widetilde{B}_k)$ is a sequence of disjoint pairs of subsets in $\overline{B}_{E'}$ and is independent. Indeed, let I and J be two finite, nonempty, disjoint subsets of $\mathbb{N}$. Then we have

$$\Big(\bigcap_{k\in I} \widetilde{A}_k\Big) \cap \Big(\bigcap_{k\in J} \widetilde{B}_k\Big) = \Big(\bigcap_{i\in I'} A_i\Big) \cap \Big(\bigcap_{i\in J'} B_i\Big) \tag{1.2.2}$$

where $I' := \bigcup_{k\in I} [n_k, m_{n_k}]$ and $J' := \bigcup_{k\in J} [n_k, m_{n_k}]$ are disjoint. Consequently, the intersection in (1.2.2) is nonempty. On the other hand, for every k, we have

$$\begin{aligned} x' \in \widetilde{A}_k \Longrightarrow \langle x', \tilde{x}_{n_k}\rangle &= \sum_{i=n_k}^{m_{n_k}} \lambda_i^{n_k} \langle x', x_i\rangle \\ &\geq \sum_{i=n_k}^{m_{n_k}} \lambda_i^{n_k} (r + \delta) = r + \delta \end{aligned}$$

and $x' \in \widetilde{B}_k \Longrightarrow \langle x', \tilde{x}_{n_k}\rangle \leq \sum_{i=n_k}^{m_{n_k}} \lambda_i^{n_k} r = r$.

By invoking again Rosenthal [22], we conclude that $(\tilde{x}_{n_k})$ is equivalent to the unit vector basis of l^1. This contradicts the fact that $(\tilde{x}_n)$ is weakly Cauchy, thus proving the implication (3) $\Longrightarrow$ (1). □

Here is an analogous criterion for relative weakly compact subset in a Banach space where equivalence (1) $\Longleftrightarrow$ (3) was stated by Ülger [27] and Diestel-Ruess-Schachermayer [13] by different methods.

Theorem 1.3

Let K be a subset of a Banach space E and let $a = (a_{pq})$ be a positive RMS. Then the following are equivalent:
(1) K is relatively weakly compact.
(2) given any sequence $(x_n)_n$ in K, there exists a subsequence $(x_{n_k})_k$ such that the sequence $(s_k)_k$ with $s_k = \sum_{q=0}^{\infty} a_{kq} x_{n_q}$ $(k \in \mathbb{N})$ is well-defined and weakly convergent.
(3) given any sequence $(x_n)_n$ in K, there exists a sequence $(\tilde{x}_n)$ with $\tilde{x}_n \in \mathrm{co}\{x_m : m \geq n\}$ such that $(\tilde{x}_n)$ is weakly convergent.
(4) given any sequence $(x_n)_n$ in K there exists y such that, $\forall x' \in E'$,

$$\liminf_{n\to\infty} \langle x', x_n \rangle \leq \langle x', y \rangle.$$

Proof. The proofs of implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) follow from the arguments we used in the proof of theorem 1.2.

(3) $\Longrightarrow$ (4) is an immediate consequence of lemma 1.1 applied to the sequences $(\langle x', x_n \rangle)$ and $(\langle x', \tilde{x}_n \rangle)$.

(4) $\Longrightarrow$ (1) follows from a classical characterization of relatively sequentially weakly compact subset in normed spaces (see e.g. Holmes [17] § 18.A). □

Remark. It would be interesting to address the following question: what happens if one replace "weakly relatively compactness" by "norm relatively compactness" in the statement of Theorem 1.3.

The following example shows that, in general, the statement of Theorem 1.3 is not true if one replace "weakly" by "norm". Let $E = c_0$ and let $K = \{e_n : n \in \mathbb{N}\}$ be the unit vector basis of c_0. Then K is not relatively compact for the norm topology since for $n \neq m$, $\|e_n - e_m\|_\infty = 1$, although K satisfies the following property: given any sequence $(x_n)_n \subset K$, there exists a sequence $(\tilde{x}_n)$ with $\tilde{x}_n \in \mathrm{co}\{x_m : m \geq n\}$ $(n \in \mathbb{N})$ such that $(\tilde{x}_n)$ converges for the norm topology. Indeed set $X = \{x_n : n \in \mathbb{N}\}$. If X is finite, there exists $m \in \mathbb{N}$ and a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that, $\forall k$, $x_{n_k} = e_m$, so that in this case, we can take, $\forall k$, $\tilde{x}_k = x_{n_k} = e_m$. If X is infinite, there exist two subsequences $(x_{p_k})_k$ and $(e_{q_k})_k$ of $(x_n)_n$ and (e_n) respectively such that, $\forall k$, $x_{p_k} = e_{q_k}$. Set $\tilde{x}_k = \frac{1}{k+1} \sum_{i=k}^{2k} e_{q_i}$, $\forall k$, then $\tilde{x}_k \in \mathrm{co}\{e_{q_i} : i \geq k\} \subset \mathrm{co}\{x_n : n \geq k\}$ and $(\tilde{x}_k)_k$ converges to 0 for the norm topology.

§2 - Weak compactness and convergence results in $L^1_E(\mu)$

Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space and E a Banach space. We aim to present in this section some compactness and convergence results in $L^1_E(\mu)$.

We begin by recalling the following result due to Talagrand ([26], Theorem 1).

Theorem 2.1

Let (u_n) be a bounded sequence in $L^1_E(\mu)$. Then there exists a sequence $(\tilde{u}_n)$ with $\tilde{u}_n \in \mathrm{co}\,\{u_m : m \geq n\}$ and two sets A and B in $\mathcal{F}$ with $\mu(A \cup B) = 1$ such that
(a) for each ω in A, the sequence $(\tilde{u}_n(\omega))$ is weakly Cauchy,
(b) for each ω in B, there exists an integer k such that the sequence $(\tilde{u}_n(\omega))_{n\geq k}$ is equivalent to the vector unit basis of l^1.

Remark. Although the thesis is more general than the one given in ([26], Theorem 1), in which (u_n) is bounded in $L^\infty_E(\mu)$, Theorem 2.1 is an easy consequence of Theorem 1 in ([26]). In the same vein, Diestel-Ruess-Schachermayer obtained a refinement of Talagrand's theorem by another method (see [13], lemma 2.5). Indeed let $v_n = \|u_n(.)\|, \forall n$. Then (v_n) is a bounded sequence in $L^1_{\mathbb{R}}(\mathcal{F})$. By ([9], Théorème 3.1 et Remarques, p. 60-61), there is a sequence $(\tilde{v}_n)$ with $\tilde{v}_n \in \mathrm{co}\,\{v_m : m \geq n\}$ such that $(\tilde{v}_n)$ converges almost everywhere to some $v \in L^1_{\mathbb{R}}(\mathcal{F})$. Each $\tilde{v}_n$ has the form $\tilde{v}_n = \sum_{k=n}^{\nu_n} \lambda^n_k v_k$ with $0 \leq \lambda^n_k \leq 1$ and $\sum_{k=n}^{\nu_n} \lambda^n_k = 1$. Extracting a subsequence if necessary and modifying the v_k , $k \in \mathbb{N}$, on a negligible set we may suppose that $(\tilde{v}_n(\omega))_n$ converges to $v(\omega)$ for all $\omega \in \Omega$. Set

$$\forall \omega \in \Omega, M(\omega) := 1 + \sup_n \tilde{v}_n(\omega) \quad \text{and} \quad h_n(\omega) := \frac{1}{M(\omega)} \sum_{k=n}^{\nu_n} \lambda^n_k\, u_k(\omega).$$

Then we can apply Talagrand's theorem to (h_n). There is a sequence $(\tilde{h}_n)$ with $\tilde{h}_n \in \mathrm{co}\,\{h_m : m \geq n\}$ which satisfies conditions (a) and (b) in the thesis of Theorem 2.1. Now it is enough to set $\tilde{u}_n(\omega) = M(\omega)\tilde{h}_n(\omega)$, $\forall \omega \in \Omega$.

Now we state our first result which is a direct application of Theorem 1.2. and Talagrand's results [26].

Theorem 2.2

Let $\mathcal{H}$ be a bounded subset of $L^1_E(\mu)$. Then the following are equivalent:
(1) $\mathcal{H}$ is conditionally weakly compact.
(2) $\mathcal{H}$ is uniformly integrable and given any sequence $(f_n) \subset \mathcal{H}$, there exists a sequence $(\tilde{f}_n)$ with $\tilde{f}_n \in \mathrm{co}\,\{f_m : m \geq n\}$ such that $(\tilde{f}_n(\omega))_n$ is weakly Cauchy in E for a.e. $\omega \in \Omega$.

Proof. Let us prove (1) $\Longrightarrow$ (2). It is well-known that conditionally weakly compact subsets of $L^1_E(\mu)$ are uniformly integrable (see [14]). Now let (f_n) be any sequence in $\mathcal{H}$. Then by Theorem 2.1, there exists a sequence $(\tilde{f}_n)$, with $\tilde{f}_n \in \mathrm{co}\{f_m : m \geq n\}$, and two sets, A, B in $\mathcal{F}$ with $\mu(A \cup B) = 1$, such that

(a) for each ω in A, $(\tilde{f}_n(\omega))_n$ is weakly Cauchy in E,

(b) for each ω in B, there exists an integer k, such that the sequence $(\tilde{f}_n(\omega))_{n\geq k}$ is equivalent to the vector unit basis of l^1.

Suppose that the measure of subset B of Ω is strictly positive. Then by Talagrand's Lemma 4, [26], there exists k such that the sequence $(\tilde{f}_n)_{n\geq k}$ is equivalent to the vector unit basis of l^1. But this contradicts the fact that $(\tilde{f}_n)$ is c.w.c. since $(\tilde{f}_n)$ lies in the set $\mathrm{co}(\mathcal{H})$, which is c.w.c. (see [23] or [7] Theorem 5.E). Therefore $\mu(B) = 0$, and for a.e. $\omega \in \Omega$, the sequence $(\tilde{f}_n(\omega))$ is weakly Cauchy.

Let us prove now (2) $\Longrightarrow$ (1). By Theorem 1.2, it is enough to check that given $(f_n) \subset \mathcal{H}$ and $(\tilde{f}_n)$ as in (2), the sequence $(\tilde{f}_n)$ is weakly Cauchy in $L^1_E(\mu)$. Let $g \in L^\infty_{E'}[E]$. Since $(\tilde{f}_n(\omega))_n$ is weakly Cauchy in E for a.e. $\omega \in \Omega$, the sequence $(\langle g(\omega), \tilde{f}_n(\omega)\rangle)_n$ converges a.e. Let

$$\varphi(\omega) := \lim_{n\to\infty} \langle g(\omega), \tilde{f}_n(\omega)\rangle \quad \text{for } \omega \notin N$$

where N is a negligible set and $\varphi(\omega) = 0$ for $\omega \in N$. Then by Fatou's lemma, $\varphi \in L^1_{\mathbb{R}}(\mu)$ and since $(\langle g, \tilde{f}_n\rangle)_n$ is uniformly integrable, by Vitali's theorem, we get

$$\lim_{n\to\infty} \int_\Omega \langle g, \tilde{f}_n\rangle d\mu = \int_\Omega \varphi \, d\mu$$

thus proving that (2) $\Longrightarrow$ (1). □

Concerning r.w.c subsets in $L^1_E(\mu)$ we recall the following which is essentially due to Ülger [27] and relies on the equivalence (1) $\Longleftrightarrow$ (3) in Theorem 1.3.

Theorem 2.3 (Ülger-Diestel-Ruess-Schachermayer).

Let E be a Banach space and $\mathcal{H}$ be a subset of $L^1_E(\mu)$. Then the following are equivalent:

(a) $\mathcal{H}$ is relatively weakly compact.

(b) $\mathcal{H}$ is uniformly integrable and given any sequence (u_n) in $\mathcal{H}$, there is a sequence $(\tilde{u}_n)$ with $\tilde{u}_n \in \mathrm{co}\{u_m : m \geq n\}$, $\forall n$, such that $(\tilde{u}_n(\omega))$ is weakly convergent in E for almost all $\omega \in \Omega$.

The following result is mentioned in Diestel ([12], p. 237). We provide the proof here for the sake of completeness.

Proposition 2.4

Let E be an arbitrary Banach space, K a nonempty subset of E. Then the following are equivalent:
(1) K is conditionally weakly compact.
(2) For every $\varepsilon > 0$, there exists a conditionally weakly compact set K_ε such that

$$K \subset K_\varepsilon + \varepsilon\,\overline{B}_E.$$

Proof. (1) $\Longrightarrow$ (2) being obvious, let us prove (2) $\Longrightarrow$ (1). Let (ε_p) be a decreasing sequence of strictly positive numbers with $\lim_{p\to\infty}\varepsilon_p = 0$, and (K_p) be a sequence of conditionally weakly compact subsets in E such that

$$\forall p, \quad K \subset K_p + \varepsilon_p\ \overline{B}_E\,. \tag{2.3.1}$$

We have to show that, given any sequence $(x_n) \subset K$, there exists a weakly Cauchy subsequence. By (2.3.1), for every p, and every n, there exists $y_p^n \in K_p$ such that $\|x_n - y_n^p\| \le \varepsilon_p$.

Since each K_p is c.w.c., the sequence $(y_p^n)_n$ admits a weakly Cauchy subsequence. Then by induction we find a sequence (φ_n) in $Si(\mathbb{N})$ such that

$$\forall p,\ (y^p_{\varphi_0\circ\ldots\circ\varphi_p(n)})_n \text{ is weakly Cauchy in } E. \tag{2.3.2}$$

Let us consider the diagonal sequence $\psi(n) := \varphi_0\circ\ldots\circ\varphi_n(n)$, $\forall n$, and let us prove that $(x_{\psi(n)})$ is weakly Cauchy. Let $\varepsilon > 0$ be fixed. Choose p such that $\varepsilon_p < \frac{\varepsilon}{4}$. Then for any $x' \in \overline{B}_{E'}$, and for $m > k > p$, we have

$$\begin{aligned}|\langle x', x_{\psi(m)} - x_{\psi(k)}\rangle| &\le |\langle x', x_{\psi(m)} - y^p_{\psi(m)}\rangle| + |\langle x', x_{\psi(k)} - y^p_{\psi(k)}\rangle| \\ &\qquad + |\langle x', y^p_{\psi(m)} - y^p_{\psi(k)}\rangle| \\ &\le 2\varepsilon_p + \left|\langle x', y^p_{\psi(m)} - y^p_{\psi(k)}\rangle\right|.\end{aligned}$$

Since by (2.3.2), $(y^p_{\varphi_0\circ\ldots\circ\varphi_p(n)})_n$ is weakly Cauchy, so is $(y^p_{\psi(n)})_n$. Therefore $\lim_{\substack{m\to\infty\\ k\to\infty}}\langle x', y^p_{\psi(m)} - y^p_{\psi(k)}\rangle = 0$. Hence there exists $p_\varepsilon > p$ such that $m > k > p_\varepsilon$ implies $|\langle x', x_{\psi(m)} - x_{\psi(k)}\rangle| \le 2.\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon$, proving that $(x_{\psi(n)})_n$ is weakly Cauchy. $\square$

We need a couple of notions which are inspired by ([1] and [26]) before we state our c.w.c. criteria in $L^1_E(\mu)$. Let us recall that $\mathcal{R}_{cwc}(E)$ (resp. $\mathcal{R}_{wk}(E)$) is the class of subsets $K \in \mathcal{B}(E)$ such that, their intersection with any ball is c.w.c. (resp. r.w.c) in E. An element $K \in \mathcal{R}_{cwc}(E)$ (resp. $\mathcal{R}_{wk}(E)$) is called ball-c.w.c. (resp. ball-r.w.c). It is clear that $\mathcal{R}_{cwc}(E)$ and $\mathcal{R}_{wk}(E)$ are stable under finite unions and that they contains the empty set $\emptyset$.

A subset $\mathcal{H} \subset L^1_E(\mu)$ is called $\mathcal{R}_{cwc}(E)$-tight (resp. $\mathcal{R}_{wk}(E)$-tight) if, for every $\varepsilon > 0$, there exists a measurable multifunction L_ε from Ω into $\mathcal{R}_{cwc}(E)$ (resp. $\mathcal{R}_{wk}(E)$) such that

$$\forall u \in \mathcal{H},\ \mu[\{\omega \in \Omega : u(\omega) \notin L_\varepsilon(\omega)\}] < \varepsilon\,.$$

A subset $\mathcal{H} \subset L^1_E(\mu)$ has the *conditionally weak Talagrand property*, shortly, conditionally WTP, (resp. *weak Talagrand property*, shortly, WTP) if, for any sequence $(f_n) \subset \mathcal{H}$, there exists a sequence (g_n) with $g_n \in \operatorname{co}\{f_m : m \geq n\}$, $\forall n$, such that, for a.e. $\omega \in \Omega$, $(g_n(\omega))_n$ is weakly Cauchy (resp. weakly convergent) in E.

There is a folklore Lemma which characterizes the above tightness notion.

Lemma 2.5

Let E be a separable Banach space. Let $\mathcal{R}$ be a class of borelian subsets of E such that: $\emptyset \in \mathcal{R}; A, B \in \mathcal{R} \Longrightarrow A \cup B \in \mathcal{R}$. Let $\mathcal{H}$ be a subset of $L^1_E(\mu)$. Then the following are equivalent:
(a) For any $\varepsilon > 0$, there exists a measurable multifunction $L_\varepsilon : \Omega \longrightarrow \mathcal{R}$ such that

$$\forall u \in \mathcal{H},\ \mu[\{\omega \in \Omega : u(\omega) \notin L_\varepsilon(\omega)\}] < \varepsilon\,.$$

(b) There exists a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand $\varphi : \Omega \times E \longrightarrow [0, +\infty]$ such that for all $\omega \in \Omega$ and all $r \geq 0$, $\{x \in E : \varphi(\omega, x) \leq r\} \in \mathcal{R}$ and that

$$\sup_{u \in \mathcal{H}} \int_\Omega \varphi(\omega, u(\omega))\mu(d\omega) < +\infty\,.$$

(c) There exists a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand $\varphi : \Omega \times E \longrightarrow [0, +\infty]$ such that for all $\omega \in \Omega$ and all $r \in \mathbb{R}^+$, $\{x \in E : \varphi(\omega, x) \leq r\} \in \mathcal{R}$ and that

$$\lim_{\lambda \to +\infty} \sup_{u \in \mathcal{H}} \mu[\{\omega \in \Omega : \varphi(\omega, u(\omega)) \geq \lambda\}] = 0.$$

Proof. (a) $\Longrightarrow$ (b). Let $\varepsilon_p = 3^{-p}$ ($p \in \mathbb{N}$). By (a) there exists a measurable multifunction $L_p : \Omega \longrightarrow \mathcal{R}$ such that

$$\forall u \in \mathcal{H},\ \mu[\{\omega \in \Omega : u(\omega) \notin L_p(\omega)\}] < \varepsilon_p\,.$$

Let us consider the multifunctions $K_n : \Omega \longrightarrow \mathcal{B}(E)$ $(n \in \mathbb{N} \cup \{\infty\})$ given by:

$\forall \omega \in \Omega, K_0(\omega) = L_0(\omega), K_n(\omega) = L_n(\omega) \backslash K_{n-1}(\omega), \forall n \geq 1$

and $K_\infty(\omega) = E \setminus \bigcup_{n\in\mathbb{N}} K_n(\omega) = E \setminus \bigcup_{n\in\mathbb{N}} L_n(\omega)$

Then it is obvious that each K_n $(n \in \mathbb{N} \cup \{\infty\})$ is measurable and the sequence $(Gr(K_n))_{n\in\mathbb{N}\cup\{\infty\}}$ is a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable partition of $\Omega \times E$. Set

$$\varphi(\omega, x) = \begin{cases} 2^n & \text{if } (\omega, x) \in Gr(K_n), n \in \mathbb{N} \\ +\infty & \text{if } (\omega, x) \in Gr(K_\infty)\,. \end{cases}$$

We claim that φ is $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand which satisfies condition (b). Indeed, let $r \geq 0$. If $r < 1$, $\{x \in E : \varphi(\omega, x) \leq r\}$ is empty ; if $r \geq 1$, let m be the unique integer such that $m \leq \frac{\log r}{\log 2} < m + 1$. Then

$$\{(\omega, x) \in \Omega \times E : \varphi(\omega, x) \leq r\} = \bigcup_{n=0}^{m} Gr(K_n) \in \mathcal{F} \otimes \mathcal{B}(E)\,.$$

Similarly for all $\omega \in \Omega$, we have

$$\{x \in E : \varphi(\omega, x) \leq r\} = \bigcup_{n=0}^{m} K_n(\omega) = \bigcup_{n=0}^{m} L_n(\omega) \in \mathcal{R}\,.$$

It remains to check that $\sup_{u\in\mathcal{H}} \int_\Omega \varphi(\omega, u(\omega))\mu(d\omega) < +\infty$.

For each $u \in \mathcal{H}$ and each $n \in \mathbb{N} \cup \{\infty\}$, set

$$\Omega_n^u = \{\omega \in \Omega : u(\omega) \in K_n(\omega)\}\,.$$

Then $(\Omega_n^u)_{n\in\mathbb{N}\cup\{\infty\}}$ is a $\mathcal{F}$-measurable partition of Ω with $\mu(\Omega_n^u) < \varepsilon_{n-1}$, $\forall n \in \mathbb{N}^*$ and $\mu(\Omega_\infty^u) = 0$. Consequently we have

$$\begin{aligned}\int_\Omega \varphi(\omega, u(\omega))\mu(d\omega) &= \sum_{n=0}^{\infty} \int_{\Omega_n^u} \varphi(\omega, u(\omega))\mu(d\omega) = \sum_{n=0}^{\infty} 2^n \mu(\Omega_n^u) \\ &\leq 1 + \sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}} < +\infty\end{aligned}$$

thus proving the implication (a) $\Longrightarrow$ (b).

(b) $\Longrightarrow$ (c) follows immediately from Tchebyschev's inequality. Let us prove (c) $\Longrightarrow$ (a). For every $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that $\sup_{u\in\mathcal{H}} \mu[\{\omega \in \Omega : \varphi(\omega, u(\omega)) >$

$\lambda_\varepsilon\}] < \varepsilon$. Since φ is $\mathcal{F} \otimes \mathcal{B}(E)$-measurable, the multifunction $L_\varepsilon(\omega) := \{x \in E : \varphi(\omega, x) \leq \lambda_\varepsilon\}$, $\forall \omega \in \Omega$, is measurable and takes it values in $\mathcal{R}$ by (c). Since we have

$$\forall u \in \mathcal{H}, \mu[\{\omega \in \Omega : u(\omega) \notin L_\varepsilon(\omega)\}] = \mu[\{\omega \in \Omega : \varphi(\omega, u(\omega)) > \lambda_\varepsilon\}] < \varepsilon$$

(c) $\Longrightarrow$ (a) is proved. □

Now we are able to present our second conditionally weakly compact criterion in $L^1_E(\mu)$.

Theorem 2.6

Let E be a separable Banach space. Assume that $\mathcal{H}$ is uniformly integrable and $\mathcal{R}_{cwc}(E)$-tight subset of $L^1_E(\mu)$. Then $\mathcal{H}$ is conditionally weakly compact in $L^1_E(\mu)$.

Proof. Let $\varepsilon > 0$ be fixed. Since $\mathcal{H}$ is uniformly integrable, there exists $\delta > 0$ and $\alpha > 0$ such that

$$\sup_{u \in \mathcal{H}} \int_{[|u| > \alpha]} |u| d\mu < \frac{\varepsilon}{2} \quad \text{and}$$

$$\forall B \in \mathcal{F}, \mu(B) \leq \delta \Longrightarrow \sup_{u \in \mathcal{H}} \int_B |u| d\mu < \frac{\varepsilon}{2}\,.$$

By our assumption there exists a measurable multifunction $L_\delta : \Omega \longrightarrow \mathcal{R}_{cwc}(E)$ such that

$$\forall u \in \mathcal{H}, \mu[\{\omega \in \Omega : u(\omega) \notin L_\delta(\omega)\}] < \delta\,.$$

For each $u \in \mathcal{H}$, set $A_u = [|u| \leq \alpha]$, $B_u = \{\omega \in \Omega : u(\omega) \in L_\delta(\omega)\}$.

Then we have

$$u = 1_{A_u \cap B_u} u + 1_{A_u^c \cap B_u} u + 1_{B_u^c} u$$

and

$$\|1_{A_u^c \cap B_u} u + 1_{B_u^c} u\|_1 \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Set $\mathcal{H}_\varepsilon = \{1_{A_u \cap B_u} u : u \in \mathcal{H}\}$. Then it is obvious that

$$\mathcal{H} \subset \mathcal{H}_\varepsilon + \varepsilon\, \overline{B}_{L^1_E(\mu)}.$$

Now we claim that $\mathcal{H}_\varepsilon$ is conditionally weakly compact in $L^1_E(\mu)$. Let $(u_n)_n \subset \mathcal{H}$ and $v_n := 1_{A_{u_n} \cap B_{u_n}} u_n$, $\forall n$. Then

$$v_n(\omega) \in (L_\delta(\omega) \cup \{0\}) \cap \eta\, \overline{B}_E$$

for all $\omega \in \Omega$. Moreover $G_\delta(\omega) := (L_\delta(\omega) \cup \{0\}) \cap \eta\, \overline{B}_E$ is conditionally weakly compact in E because $L_\delta(\omega) \in \mathcal{R}_{cwc}(E)$. By Talagrand's theorem ([26], Theorem 1), there exist $A \in \mathcal{F}$ and a sequence $(\tilde{v}_n)$ with $\tilde{v}_n \in \mathrm{co}\,\{v_m : m \geq n\}$, $\forall n$, such that

(a) $\forall \omega \in A$, $(\tilde{v}_n(\omega))_n$ is weakly Cauchy in E

(b) for a.e. $\omega \in A^c$, there exists k such that $(\tilde{v}_n(\omega))_{n\geq k}$ is equivalent to the unit vector basis of l^1.

Now, $\forall \omega \in \Omega$, $\tilde{v}_n(\omega) \in \mathrm{co}\, G_\delta(\omega)$ and $\mathrm{co}(G_\delta(\omega))$ is conditionally weakly compact (see [23], or [7] Theorem 5.E). Hence $\mu(A^c) = 0$. So we conclude that $(\tilde{v}_n(\omega))_n$ is weakly Cauchy for a.e. $\omega \in \Omega$. By virtue of Theorem 2.2., $\mathcal{H}_\varepsilon$ is conditionally weakly compact in $L^1_E(\mu)$. Since $\mathcal{H} \subset \mathcal{H}_\varepsilon + \varepsilon\, \overline{B}_{L^1_E(\mu)}$, then by Proposition 2.3, $\mathcal{H}$ is conditionally weakly compact too. This completes the proof of Theorem 2.4. □

Remark. Theorem 2.6 is a slight refinement of some results obtained by Pisier [21] and Bourgain [6].

Similarly we have the following criterion for relatively weakly compact subsets of $L^1_E(\mu)$ (see [1], Théorème 6, p. 174 for proof).

Theorem 2.7

Let E be a separable Banach space. Let $\mathcal{H}$ be a uniformly integrable and $\mathcal{R}_{wk}(E)$-tight subset of $L^1_E(\mu)$. Then $\mathcal{H}$ is relatively weakly compact in $L^1_E(\mu)$.

The following result provides the connections between "tightness notions" and "Talagrand's properties".

Theorem 2.8

Let E be a separable Banach space. If $\mathcal{H}$ is a bounded $\mathcal{R}_{cwc}(E)$ (resp. $\mathcal{R}_{wk}(E)$)-tight subset of $L^1_E(\mu)$, then $\mathcal{H}$ is conditionally WTP (resp. WTP) in $L^1_E(\mu)$.

Proof. We have only to prove the thesis for the $\mathcal{R}_{cwc}(E)$-tight case, since the proof of $\mathcal{R}_{wk}(E)$-tight case is similar by invoking Theorem 2.7.

Let $(u_n) \subset \mathcal{H}$. By Slaby's lemma [24], there exists an increasing sequence (A_k) in $\mathcal{F}$ with $\lim_{k\to\infty} \mu(A_k) = 1$ and a subsequence (u_{n_k}) such that $(1_{A_k} u_{n_k})$ is uniformly integrable in $L^1_E(\mu)$, and that $(1_{A_k^c} u_{n_k})$ converges to 0 a.e. Set $\mathcal{K} = \{1_{A_k} u_{n_k} : k \in \mathbb{N}\}$.

We claim that $\mathcal{K}$ is $\mathcal{R}_{cwc}(E)$-tight. Let $\varepsilon > 0$. By our assumption, there exists a measurable multifunction $L_\varepsilon : \Omega \longrightarrow \mathcal{R}_{cwc}(E)$ such that

$$\forall u \in \mathcal{H}, \mu[\{\omega \in \Omega : u(\omega) \notin L_\varepsilon(\omega)\}] < \varepsilon\,.$$

Set $G_\varepsilon(\omega) := L_\varepsilon(\omega) \cup \{0\}$, $\forall \omega \in \Omega$. Then G_ε is a measurable multifunction from Ω to $\mathcal{R}_{cwc}(E)$ such that

$$\forall k \in \mathbb{N}, \mu[\{\omega \in \Omega : (1_{A_k} u_{n_k})(\omega) \notin G_\varepsilon(\omega)\}] = \mu[\{\omega \in A_k : u_{n_k}(\omega) \notin L_\varepsilon(\omega)\}] < \varepsilon\,.$$

Hence $\mathcal{K}$ is $\mathcal{R}_{cwc}(E)$-tight as desired. Since $\mathcal{K}$ is uniformly integrable, by Theorem 2.6, $\mathcal{K}$ is c.w.c. in $L^1_E(\mu)$. By virtue of Theorem 2.2, there exists a sequence (v_p) with $v_p \in \mathrm{co}\,\{1_{A_k} u_{n_k} : k \geq p\}$, $\forall p$, such that, for a.e. $\omega \in \Omega$, $(v_p(\omega))_p$ is weakly Cauchy in E. Each v_p has the form $v_p = \sum_{k=p}^{\nu_p} \lambda_k^p 1_{A_k} u_{n_k}$, with $\lambda_k^p \geq 0$, $\sum_{k=p}^{\nu_p} \lambda_k^p = 1$. Set $\tilde{u}_p = \sum_{k=p}^{\nu_p} \lambda_k^p u_{n_k}$, $\forall p$. Then $\tilde{u}_p = v_p + w_p$, where $w_p := \sum_{k=p}^{\nu_p} \lambda_k^p 1_{A_k^c} u_{n_k}$ with $w_p \longrightarrow 0$ a.e. since $1_{A_k^c} u_{n_k} \underset{k}{\longrightarrow} 0$ a.e. We deduce that for a.e. $\omega \in \Omega$, the sequence $(\tilde{u}_p(\omega))$ is weakly Cauchy in E, thereby proving the Theorem. □

Theorem 2.9

Let $\mathcal{H}$ be a bounded subset of $L^1_E(\mu)$. Then the following are equivalent:
(a) $\mathcal{H}$ has the weak Talagrand property (WTP).
(b) given any sequence (u_n) in $\mathcal{H}$, there are an increasing sequence (A_k) in $\mathcal{F}$ with $\lim_{k\to\infty} \mu(A_k) = 1$ and a subsequence (u_{n_k}) of (u_n) such that $(1_{A_k} u_{n_k})_k$ is relatively weakly compact in $L^1_E(\mu)$ and that $(1_{A_k^c} u_{n_k})_k$ converges a.e. to zero.
(c) given any sequence (u_n) in $\mathcal{H}$, there exists a sequence $(\tilde{u}_n)$ with $\tilde{u}_n \in \mathrm{co}\,\{u_m : m \geq n\}$, $\forall n$, and $u_\infty \in L^1_E(\mu)$ such that $(\tilde{u}_n)$ converges a.e. to u_∞ for the norm topology of E.

Proof. (a) $\Longrightarrow$ (b). By Slaby's decomposition [24], there exist an increasing sequence (A_k) in $\mathcal{F}$ with $\lim_{k\to\infty} \mu(A_k) = 1$ and a subsequence (u_{n_k}) of (u_n) such that $(1_{A_k} u_{n_k})_k$ is uniformly integrable in $L^1_E(\mu)$ and that $(1_{A_k^c} u_{n_k})_k$ converges to zero a.e. Now we claim that the set $\mathcal{K} := \{1_{A_k} u_{n_k} : k \in \mathbb{N}\}$ has the (WTP). Indeed, by (a) there exists a subsequence $(u_{n_{k_p}})_p$ of (u_{n_k}) and a sequence (v_p) with $v_p \in \mathrm{co}\,\{u_{n_{k_j}} : j \geq p\}$, $\forall p$, such that for a.e. $\omega \in \Omega$, $(v_p(\omega))_p$ converges weakly to $v(\omega)$ in E. Each v_p has the form $v_p = \sum_{j=p}^{\nu_p} \lambda_j^p u_{n_{k_j}}$ with $\lambda_j^p \geq 0$ and $\sum_{j=p}^{\nu_p} \lambda_j^p = 1$. Set $w_p = \sum_{j=p}^{\nu_p} \lambda_j^p 1_{A_{k_j}} u_{n_{k_j}}$, $\forall p$. Then it is easily seen that $w_p(\omega) \longrightarrow v(\omega)$ weakly a.e. in E. As $w_p \in \mathrm{co}\,\{1_{A_{k_j}} u_{n_{k_j}} : j \geq p\}$, $\forall p$, $\mathcal{K}$ has the (WTP). Since $\mathcal{K}$ is uniformly integrable, by Ülger-Diestel-Ruess-Schachermayer Theorem (Theorem 2.3), one conclude that $\mathcal{K}$ is r.w.c. in $L^1_E(\mu)$.

(b) $\Longrightarrow$ (c). Let (A_k) and (u_{n_k}) according to (b). By Mazur's Lemma, we may assume, by extracting a subsequence if necessary, that there exists a sequence (v_k) with $v_k \in \mathrm{co}\,\{1_{A_m} u_{n_m} : m \geq k\}$, $\forall k$, such that $(v_k)_k$ converges a.e. to an element

$v_\infty \in L^1_E(\mu)$. Each v_k has the form $v_k = \sum_{j=k}^{\nu_k} \lambda_j^k 1_{A_j} u_{n_j}$, with $0 \le \lambda_j^k \le 1$, $\sum_{j=k}^{\nu_k} \lambda_j^k = 1$. Let $\tilde{u}_k = \sum_{j=k}^{\nu_k} \lambda_j^k u_{n_j}$, $\forall k$. Then $(\tilde{u}_k)$ has the desired properties.

(c) $\Longrightarrow$ (a) is obvious. □

Corollary 2.10

Let K be a convex bounded WTP set in $L^1_E(\mu)$ which is closed for the topology of the convergence in measure. Let $J : K \longrightarrow [0, +\infty[$ be a convex lower semicontinuous function for the topology of convergence in measure. Then J reaches its minimum on K.

The preceding corollary generalizes a result due to Levin [19]. (See [9] for details and references).

Let us mention the following consequence of Theorem 2.9.

Proposition 2.11

Let $\mathcal{H}$ be a bounded WTP set in $L^1_E(\mu)$. Then the following are equivalent:
(a) $\forall v \in L^\infty_{E'_s}(\mu)$, $\{\langle v(.), u(.)\rangle : u \in \mathcal{H}\}$ is uniformly integrable in $L^1_{\mathbb{R}}(\mu)$.
(b) $\mathcal{H}$ is relatively weakly compact in $L^1_E(\mu)$.

Proof. (b) $\Longrightarrow$ (a) being obvious, it is enough to prove that (a) $\Longrightarrow$ (b). We may suppose that E is separable. Let (u_n) be a sequence in $\mathcal{H}$. By Theorem 2.9, there are $u_\infty \in L^1_E(\mu)$ and a sequence $(\tilde{u}_n)$ with $\tilde{u}_n \in \operatorname{co}\{u_m : m \ge n\}$, such that $(\tilde{u}_n)$ converges a.e. to u_∞ for the norm topology of E. By (a), $\forall v \in L^\infty_{E'_s}(\mu)$, the sequence $(\langle v, \tilde{u}_n\rangle)_n$ is uniformly integrable, then,by Vitali's theorem $\lim_{n\to\infty} \int_\Omega \langle v, \tilde{u}_n\rangle d\mu = \int_\Omega \langle v, u_\infty\rangle d\mu$. By virtue of Theorem 1.3, we conclude that $\mathcal{H}$ is relatively weakly compact in $L^1_E(\mu)$. □

Now we present some nice properties of bounded WTP sequences in $L^1_E(\mu)$.

Theorem 2.12

Let (u_n) be a bounded WTP sequence in $L^1_E(\mu)$. Then the following properties hold:
(a) There exist an increasing sequence (A_p) in $\mathcal{F}$ with $\lim_p \mu(A_p) = 1$, a subsequence (u_{n_k}) of (u_n), a sequence $(\tilde{u}_n)$ with $\tilde{u}_n \in \operatorname{co}\{u_{n_k} : k \ge n\}$ and $u_\infty \in L^1_E(\mu)$ such that, $\forall p$, $(u_{n_k|A_p})_k$ $\sigma(L^1, L^\infty)$ converges to $u_\infty|A_p$ and that $(\tilde{u}_n(\omega))$ converges in norm to $u_\infty(\omega)$ for a.e. $\omega \in \Omega$.

(b) If (v_n) is a bounded sequence in $L^\infty_{E'_s}(\mu)$ converging in measure to $v_\infty \in L^\infty_{E'_s}(\mu)$ for the norm topology of the strong dual of E and if the sequence $(\langle v_n, u_n\rangle^-)_n$ is uniformly integrable in $L^1_{\mathbb{R}}(\mu)$, then we have

$$\liminf_{n\to\infty} \int_\Omega \langle v_n, u_n\rangle d\mu \geq \int_\Omega \langle v_\infty, u_\infty\rangle d\mu\,.$$

(c) If $\varphi : \Omega \times E \longrightarrow [0,\infty[$ is an $\mathcal{F}\otimes\mathcal{B}(E)$-measurable integrand such that, $\forall\omega \in \Omega$, $\varphi(\omega,.)$ is convex lower semicontinuous on E, then we have

$$\liminf_{n\to\infty} \int_\Omega \varphi(\omega, u_n(\omega))\mu(d\omega) \geq \int_\Omega \varphi(\omega, u_\infty(\omega))\mu(d\omega)\,.$$

Proof. (a) Repeating the Biting lemma ([9], [24]), we find an increasing sequence (A_p) in $\mathcal{F}$ with $\lim_{p\to\infty}\mu(A_p) = 1$ and a subsequence (u'_n) of (u_n) such that, for each p, $(u'_{n|A_p})$ is uniformly integrable. Since (u_n) is WTP, then $(u'_{n|A_p})_n$ is uniformly integrable and WTP in $L^1_E(A_p)$. By virtue of Theorem 2.3, $\forall p$, $(u'_{n|A_p})_n$ is relatively weakly compact. Consequently, by a straightforward diagonal procedure, there are $u_\infty \in L^1_E(\mu)$ and a subsequence (u_{n_k}) such that, for every p, $(u_{n_k|A_p})_k\, \sigma(L^1, L^\infty)$ converges to $u_{\infty|A_p}$.

Since $(u_{n_k})_k$ is WTP, by Theorem 2.9 there exist $v_\infty \in L^1_E(\mu)$ and $(\tilde{u}_n)$ with $\tilde{u}_n \in \operatorname{co}\{u_{n_k} : k \geq n\}$ such that $(\tilde{u}_n)$ converges a.e. to v_∞ for the norm topology of E.

For any fixed p, and $B \in \mathcal{F}\cap A_p$ and any $x' \in E'$, we have

$$\begin{aligned}\int_B \langle x', v_\infty\rangle d\mu = \lim_{n\to\infty}\int_B \langle x', \tilde{u}_n\rangle &= \lim_{k\to\infty}\int_B \langle x', u_{n_k}\rangle d\mu\\ &= \int_B \langle x', u_\infty\rangle d\mu\,.\end{aligned}$$

Hence $\langle x', v_\infty\rangle = \langle x', u_\infty\rangle$ a.e. on A_p, so $(\tilde{u}_n(\omega))$ converges in norm to $u_\infty(\omega)$ for a.e. $\omega \in \Omega$. This proves Assertion (a).

Assertion (b) follows from the arguments given in [8]. Let us check (c). We may suppose that $a := \liminf_n \int_\Omega \varphi(\omega, u_n(\omega))\mu(d\omega)$ is finite and by extracting a subsequence that $a = \lim_{n\to\infty}\int_\Omega \varphi(\omega, u_n(\omega))\mu(d\omega)$. Let $(\tilde{u}_n)$ and $u_\infty \in L^1_E(\mu)$ given by Assertion (a). Each $\tilde{u}_n$ has the form $\tilde{u}_n(\omega) = \sum_{j=n}^{\nu_n}\lambda^n_j u_{n_j}(\omega)$ with $0 \leq \lambda^n_j \leq 1$ and $\sum_{j=n}^{\nu_n}\lambda^n_j = 1$. By convexity, we have

$$\forall\omega, \forall n, \varphi(\omega, \tilde{u}_n(\omega)) \leq \sum_{j=n}^{\nu_n}\lambda^n_j\ \varphi(\omega, u_{n_j}(\omega))\,.$$

Hence

$$\limsup_n \int_\Omega \varphi(\omega, \tilde{u}_n(\omega))\mu(d\omega) \le a\,.$$

By lower semicontinuity of $\varphi(\omega, .)$ and by Fatou's lemma, we get

$$\liminf_{n\to\infty} \int_\Omega \varphi(\omega, \tilde{u}_n(\omega))\mu(d\omega) \ge \int_\Omega \varphi(\omega, u_\infty(\omega))\mu(d\omega)\,.$$

Hence

$$\liminf_{n\to\infty} \int_\Omega \varphi(\omega, u_n(\omega))\mu(d\omega) \ge \int_\Omega \varphi(\omega, u_\infty(\omega))\mu(d\omega). \ \square$$

Remarks. (1) Properties (a) and (b) yield a version of Fatou's lemma in Mathematical Economics. See [9] for a complete bibliography of this subject.

(2) Property (c) is a lower semicontinuity result. It turns out that (c) allows to state a minimization problem as in the corollary of Theorem 2.9. The details are left to the reader.

(3) If E is separable and if (u_n) is bounded and $\mathcal{R}_{wk}(E)$-tight, then one can check that $u_\infty(\omega) \in \overline{\text{co}}\; w - Ls\{u_n(\omega)\}$ a.e. We refer the reader to Amrani-Castaing-Valadier ([1], Théorème 8) for details.

There is a variant of Theorem 2.12.

Theorem 2.13

Assume that E'_b is separable. Let (u_n) be a bounded sequence in $L^1_E(\mu)$ such that

(i) $\forall A \in \mathcal{F}$, $\mathcal{H}_A := \bigcup_n \left\{ \int_A u_n d\mu \right\}$ is relatively weakly compact.

(ii) Any vector measure $m : \mathcal{F} \longrightarrow E$ with bounded variation such that, $\forall A \in \mathcal{F}$, $m(A) \in \overline{\text{co}}(\mathcal{H}_A)$, admits a density in $L^1_E(\mu)$.

Then properties (a), (b), (c) in the thesis of Theorem 2.12 hold.

Proof. We sketch only the proof. It is enough to repeat the arguments of the proof of Theorem 2.12 by noting that, for each p, $(u'_{n|A_p})$ is relatively $\sigma(L^1, L^\infty)$ compact. See ([10], Theorem 3.1). $\square$

To end this paper we will discuss some Banach-Saks properties with respect to a RMS (a_{pq}).

Let E be a Banach space. Let $a = (a_{pq})$ be a RMS. The Banach space E has the *Banach-Saks* (resp. *weak Banach-Saks) property with respect to the* RMS (a_{pq}) if any bounded (resp. weakly null) sequence in E, has a summable subsequence with respect to (a_{pq}) (see [15], p. 75 for reference, cf. also [16], p. 232). Analyzing Theorem 1.2 and 1.3 reveals that theses properties characterize relative weakly compact and conditionally weakly compact subsets in E. Hence it is noteworthy to study theses properties and their implications on convergence problems for bounded sequences in $L^1_E(\mu)$.

We need first a lemma.

Lemma 2.14

Let H be a Hilbert space and (a_{pq}) be a RMS such that

$$(*) \qquad \lim_{p\to\infty} \sum_{q=0}^{\infty} |a_{pq}|^2 = 0 .$$

If (x_n) is a weakly null sequence in H, then there exists $\varphi \in \mathrm{Si}(\mathbb{N})$ such that

$$\lim_{p\to\infty} \sup_{\psi\in \mathrm{Si}(\mathbb{N})} \Big\| \sum_{q=0}^{\infty} a_{pq} x_{\varphi\circ\psi(q)} \Big\| = 0 .$$

Proof. W.l.o.g., we may suppose that $\|x_n\| \leq 1$ for all n. Let $(\varepsilon_n)_{n\geq 1}$ be a decreasing sequence in $\mathbb{R}^{+*}$ such that $\sum_{n=1}^{\infty} \varepsilon_n^2 < +\infty$. Set $M = \sup_p \sum_{q=0}^{\infty} |a_{pq}| < +\infty$ and $n_0 = 0$. Choose $n_1 > n_0$ such that

$$|\langle x_{n_0}, x_{n_1}\rangle| < \frac{\varepsilon_1}{M} .$$

Take $n_2 > n_1$ such that

$$|\langle x_{n_0}, x_{n_2}\rangle| < \frac{\varepsilon_2}{M} \text{ and } |\langle x_{n_1}, x_{n_2}\rangle| < \frac{\varepsilon_2}{M} .$$

Then by induction, there exists a finite sequence with $n_k > n_{k-1} > \ldots > n_0$ such that

$$\forall j < k, \quad |\langle x_{n_j}, x_{n_k}\rangle| < \frac{\varepsilon_k}{M} .$$

Take $\varphi(k) := n_k$, $\forall k$. We shall show that φ has the desired property. Let $\psi \in \mathrm{Si}\,(\mathbb{N})$. For every $k \in \mathbb{N}$, we have

$$\begin{aligned}
\Big\| \sum_{i=0}^{\infty} a_{ki} x_{\varphi\circ\psi(i)} \Big\|^2 &= \sum_{i=0}^{\infty} |a_{ki}|^2 \|x_{\varphi\circ\psi(i)}\|^2 \\
&\quad + 2 \sum_{j<l} a_{kj} a_{kl} \langle x_{\varphi\circ\psi(j)}, x_{\varphi\circ\psi(l)}\rangle \\
&\leq \sum_{i=0}^{\infty} |a_{ki}|^2 + 2 \sum_{j<l} |a_{kj} a_{kl}| \frac{\varepsilon_{\varphi\circ\psi(l)}}{M} \\
&\leq \sum_{i=0}^{\infty} |a_{ki}|^2 + \frac{2}{M} \sum_{l=1}^{\infty} \sum_{j=0}^{l-1} |a_{kj} a_{kl}| . \varepsilon_l
\end{aligned}$$

since $\varepsilon_{\varphi\circ\psi(l)} \leq \varepsilon_l$, $\forall l$. On the other hand by Hölder inequality

$$\begin{aligned}\sum_{l=1}^{\infty}\sum_{j=0}^{l-1}|a_{kj}a_{kl}|\varepsilon_l &= \sum_{l=1}^{\infty}|a_{kl}|\varepsilon_l\Big(\sum_{j=0}^{l-1}|a_{kj}|\Big)\\ &\leq \sum_{l=1}^{\infty}|a_{kl}|\varepsilon_l M\\ &\leq M\sqrt{\sum_{l=1}^{\infty}\varepsilon_l^2}\sqrt{\sum_{l=1}^{\infty}|a_{kl}|^2}\,.\end{aligned}$$

Set $L := \sum_{l=1}^{\infty}\varepsilon_l^2$. Then we obtain

$$\Big\|\sum_{i=0}^{\infty}a_{ki}x_{\varphi\circ\psi(i)}\Big\|^2 \leq \sum_{i=0}^{\infty}|a_{ki}|^2 + 2\sqrt{L}\sqrt{\sum_{l=1}^{\infty}|a_{kl}|^2}\,.$$

Since by our assumption $\lim_{p\to\infty}\sum_{q=0}^{\infty}|a_{pq}|^2 = 0$, the assertion we are after follows from the preceding inequality. □

Remark 2.15. Let us consider the two following (RMS):

$$a_{pq} = \begin{cases}\frac{1}{p+1} & \text{if } 0 \leq q \leq p\\ 0 & \text{if } q > p\end{cases}$$

$$b_{pq} = \begin{cases}2^{p-q} & \text{if } q > p\\ 0 & \text{if } q \leq p\,.\end{cases}$$

It is easy to check that (a_{pq}) and (b_{pq}) are (RMS). Moreover, for all p, we have

$$\sum_{q=0}^{\infty}|a_{pq}|^2 = \frac{1}{p+1}$$

$$\sum_{q=0}^{\infty}|b_{pq}|^2 = \sum_{q>p}4^{p-q} = \frac{1}{3} > 0\,.$$

Then (a_{pq}) satisfies the condition

$$(*) \qquad \lim_{p\to\infty}\sum_{q=0}^{\infty}|a_{pq}|^2 = 0$$

whereas (b_{pq}) does not satisfy $(*)$.

Now it is worth to observe that the (RMS) which satisfy the condition $(*)$ are those for which the spaces $L^1(S,\Sigma,\nu)$, where (S,Σ,ν) is a Probability space, have the weak Banach-Saks property. Indeed let $a=(a_{pq})$ be a RMS that does not satisfy $(*)$.

Let $\Omega=[0,1]$ and μ be the Lebesgue measure on Ω. Let us consider the sequence (r_n) of Rademacher functions on $[0,1]$. It is well-known that (r_n) is an orthonormal system in the Hilbert space $L^2([0,1])$ and $r_n \longrightarrow 0$ for $\sigma(L^1,L^\infty)$ topology. Suppose by contradiction that there exists a subsequence (r_{n_k}) of (r_n) which is summable with respect to the RMS (a_{pq}) in $L^1([0,1])$. Then the sequence (s_p) with $s_p:=\sum_{q=0}^\infty a_{pq}r_{n_q}$ converges to 0 for the norm of L^1, hence converges to 0 in measure. Since (s_p) is uniformly integrable in $L^2([0,1])$, $s_p\longrightarrow 0$ for the norm of $L^2([0,1])$. As (r_n) is an orthonormal system in $L^2([0,1])$, we deduce that

$$\|s_p\|_2^2=\Big\|\sum_{q=0}^\infty a_{pq}r_{n_q}\Big\|_2^2=\sum_{q=0}^\infty |a_{pq}|^2\,.$$

This contradicts the fact that $a=(a_{pq})$ does not satisfy $(*)$. Hence $L^1([0,1])$ does not satisfy the weak Banach-Saks property with respect to the RMS (a_{pq}).

Now we are able to produce the following result which generalizes the Szlenk's one to (a_{pq})-summability in L^1_H where H is a Hilbert space.

Theorem 2.16

Let H be a Hilbert space. Let $a=(a_{pq})$ be a RMS.

(1) If (a_{pq}) satisfies the property

$$(*)\qquad \lim_{p\to\infty}\sum_{q=0}^\infty |a_{pq}|^2=0\,,$$

then, for any weakly null sequence (u_n) in $L^1_H(\mu)$, there exists $\psi\in \mathrm{Si}(\mathbb{N})$ such that

$$\lim_{p\to\infty}\sup_{\varphi\in\mathrm{Si}(\mathbb{N})}\Big\|\sum_{q=0}^\infty a_{pq}u_{\psi\circ\varphi(q)}\Big\|_1=0\,.$$

(2) Conversely, if all the spaces $L^1_{\mathbb{R}}(S,\Sigma,\nu)$ have the weak Banach-Saks property with respect to the RMS $a=(a_{pq})$, then a satisfies

$$(*)\qquad \lim_{p\to\infty}\sum_{q=0}^\infty |a_{pq}|^2=0\,.$$

The assertion (2) follows from the above Remark 2.15.

Proof. We shall divide the proof in two steps.

Step 1. Claim: For any $\varepsilon > 0$, there exists $\psi \in \mathrm{Si}(\mathbb{N})$ such that

$$\limsup_{p\to\infty} \sup_{\varphi\in\mathrm{Si}(\mathbb{N})} \left\| \sum_{q=0}^{\infty} a_{pq} u_{\psi\circ\varphi(q)} \right\|_1 \leq \varepsilon\,.$$

W.l.o.g. we may suppose that $\|u_n\|_1 \leq 1$ for all n. Let $M > \max(1, \sup_p \sum_{q=0}^{\infty} |a_{pq}|)$ and let $\varepsilon > 0$. As (u_n) is uniformly integrable, there is $\alpha > 0$ such that

$$\sup_n \int_{[\|u_n\|\geq\alpha]} \|u_n\| d\mu \leq \frac{\varepsilon}{3M}\,.$$

Set $A_n := [\|u_n\| \geq \alpha]$, $u'_n := 1_{A_n} u_n$ and $u''_n := 1_{A_n^c} u_n$. Since $\|u''_n\| \leq \alpha$ a.e., there exists $v \in L^\infty_H(\mu)$ such that $\|v\| \leq \alpha$ a.e. and a subsequence $(u''_{\psi(k)})$, $\psi \in \mathrm{Si}(\mathbb{N})$, such that $(u''_{\psi(k)})\sigma(L^\infty_H, L^1_H)$ converges to v. Hence $u'_{\psi(k)} = u_{\psi(k)} - u''_{\psi(k)}$ $\sigma(L^1_H, L^\infty_H)$ converges to $-v$. Moreover it is obvious that, $\forall k$, $\|u'_{\psi(k)}\|_1 \leq \frac{\varepsilon}{3M}$, hence $\|v\|_1 \leq \frac{\varepsilon}{3M}$. As $(u''_{\psi(k)} - v)$ $\sigma(L^2_H, L^2_H)$ converges to 0, then in view of lemma 2.14, we may suppose that

$$\lim_{p\to\infty} \sup_{\sigma\in\mathrm{Si}(\mathbb{N})} \left\| \sum_{q=0}^{\infty} a_{pq} (u''_{\psi\circ\sigma(q)} - v) \right\|_2 = 0\,.$$

There is $p_\varepsilon \in \mathbb{N}$ such that $p \geq p_\varepsilon$ implies

$$\sup_{\sigma\in\mathrm{Si}(\mathbb{N})} \left\| \sum_{q=0}^{\infty} a_{pq} (u''_{\psi\circ\sigma(q)} - v) \right\|_2 \leq \frac{\varepsilon}{3}\,.$$

Then for all $p \geq p_\varepsilon$ and $\varphi \in \mathrm{Si}(\mathbb{N})$, we have

$$\begin{aligned}
\left\| \sum_{q=0}^{\infty} a_{pq} u_{\psi\circ\varphi(q)} \right\|_1 &\leq \left\| \sum_{q=0}^{\infty} a_{pq} u'_{\psi\circ\varphi(q)} \right\|_1 + \left\| \sum_{q=0}^{\infty} a_{pq} (u''_{\psi\circ\varphi(q)} - v) \right\|_1 \\
&\quad + \left\| (\sum_{q=0}^{\infty} a_{pq}) v \right\|_1 \\
&\leq \sum_{q=0}^{\infty} |a_{pq}| . \frac{\varepsilon}{3M} + |\sum_{q=0}^{\infty} a_{pq}| . \|v\|_1 + \left\| \sum_{q=0}^{\infty} a_{pq} (u''_{\psi\circ\varphi(q)} - v) \right\|_2 \\
&\leq M . \frac{\varepsilon}{3M} + M . \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} = \varepsilon
\end{aligned}$$

thus proving our claim.

Second Step. Let (u_n) be a weakly null sequence in $L^1_H(\mu)$ with $\|u_n\|_1 \le 1$, $\forall n$. According to the first step, we find, by induction, $\varphi_0, \ldots, \varphi_k$ in $\mathrm{Si}(\mathbb{N})$ such that

$$\limsup_{p\to\infty} \sup_{\sigma\in \mathrm{Si}(\mathbb{N})} \Big\| \sum_{q=0}^{\infty} a_{pq} u_{\psi_k\circ\sigma(q)} \Big\|_1 \le 2^{-k} \tag{2.16.1}$$

with $\psi_k := \varphi_0 \circ \ldots \circ \varphi_k$. Let us consider the diagonal sequence $\psi(k) := \psi_k(k)$, $\forall k$ and let us show that $\psi(.)$ has the required property of the thesis of Theorem 2.16. Let $\theta \in \mathrm{Si}(\mathbb{N})$ and $k \in \mathbb{N}$ be fixed. Define

$$\varphi(n) := \begin{cases} n & \text{if } n \le k \\ \varphi_{k+1} \circ \ldots \circ \varphi_{\theta(n)}(\theta(n)) & \text{if } n \ge k+1\,. \end{cases}$$

Then $\varphi \in \mathrm{Si}(\mathbb{N})$ and, $\forall q \ge k+1$, $\psi \circ \theta(q) = \psi_k \circ \varphi(q)$. Moreover we have

$$\begin{aligned} \Big\| \sum_{q=0}^{\infty} a_{pq} u_{\psi\circ\theta(q)} \Big\|_1 &\le \Big\| \sum_{q=0}^{k} a_{pq} u_{\psi(\theta(q))} \Big\|_1 + \Big\| \sum_{q=k+1}^{\infty} a_{pq} u_{\psi_k\circ\theta(q)} \Big\|_1 \\ &\le \sum_{q=0}^{k} |a_{pq}| + \Big\| \sum_{q=0}^{\infty} a_{pq} u_{\psi_k\circ\varphi(q)} - \sum_{q=0}^{k} a_{pq} u_{\psi_k\circ\varphi(q)} \Big\|_1 \\ &\le 2\sum_{q=0}^{k} |a_{pq}| + \sup_{\sigma\in \mathrm{Si}(\mathbb{N})} \Big\| \sum_{q=0}^{\infty} a_{pq} u_{\psi_k\circ\sigma(q)} \Big\|_1 . \end{aligned} \tag{2.16.2}$$

By (2.16.1) and (2.16.2), it follows that

$$\limsup_{p\to\infty} \sup_{\theta\in \mathrm{Si}(\mathbb{N})} \Big\| \sum_{q=0}^{\infty} a_{pq} u_{\psi\circ\theta(q)} \Big\|_1 \le 2 \lim_{p\to\infty} \sum_{q=0}^{k} |a_{pq}| + 2^{-k} = 2^{-k}\,. \tag{2.16.3}$$

Since k is arbitrary, assertion (1) follows immediately from (2.16.3). $\square$

Corollary 2.17

Let H be a Hilbert space and $a = (a_{pq})$ be a RMS which satisfies property $(*)$. Let (u_n) be a bounded sequence in $L^1_H(\mu)$. Then there exist $\psi \in \mathrm{Si}(\mathbb{N})$ and $u \in L^1_H(\mu)$ such that, for all $\varphi \in \mathrm{Si}(\mathbb{N})$, the sequence $\big(\sum_{q=0}^{\infty} a_{pq} u_{\psi\circ\varphi(q)}\big)_p$ converges in measure to u.

Proof. By Theorem 2.9, we may suppose that there exists an increasing sequence (A_n) in $\mathcal{F}$ with $\lim_{n\to\infty}\mu(A_n^c)=0$ such that $(1_{A_n}u_n)$ $\sigma(L^1,L^\infty)$-converges to $u\in L^1_H(\mu)$ and $(1_{A_n^c}u_n)$ converges μ-a.e. to 0. Now we apply Theorem 2.16 to the weakly null sequence $v_n=1_{A_n}u_n-u$. Then there exists $\psi\in\mathrm{Si}(\mathbb{N})$ such that, $\forall\varphi\in\mathrm{Si}(\mathbb{N})$, the sequence $\big(\sum_{q=0}^{\infty}a_{pq}v_{\psi\circ\varphi(q)}\big)_p$ converges in $L^1_H(\mu)$ to 0. Let $\varphi\in\mathrm{Si}(\mathbb{N})$ be fixed and set $\theta=\psi\circ\varphi$. Then

$$\begin{aligned}\sum_{q=0}^{\infty}a_{pq}u_{\theta(q)}&=\sum_{q=0}^{\infty}a_{pq}1_{A_{\theta(q)}}u_{\theta(q)}+\sum_{q=0}^{\infty}a_{pq}1_{A^c_{\theta(q)}}u_{\theta(q)}\\&=\Big(\sum_{q=0}^{\infty}a_{pq}\Big)u+\sum_{q=0}^{\infty}a_{pq}v_{\theta(q)}+\sum_{q=0}^{\infty}a_{pq}1_{A^c_{\theta(q)}}u_{\theta(q)}\,.\end{aligned}$$

As (a_{pq}) is a RMS, the sequence $\big(\big(\sum_{q=0}^{\infty}a_{pq}\big)u\big)_p$ pointwisely converges to u and the sequence $\big(\sum_{q=0}^{\infty}a_{pq}1_{A^c_{\theta(q)}}u_{\theta(q)}\big)_p$ converges μ-a.e. to 0. Hence $\big(\sum_{q=0}^{\infty}a_{pq}u_{\theta(q)}\big)_p$ converges in measure to u. □

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