

Tensor product in symmetric function spaces

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ABSTRACT

A concept of the multiplier of symmetric function space concerning to projective tensor product is introduced and studied. This allows us to obtain some concrete results. In particular, the well-known theorem of R. O’Neil about boundedness of tensor product in the Lorentz spaces L_{pq} is discussed.

0. Introduction

Let $x = x(s)$ and $y = y(t)$ be measurable functions on $I = [0, 1]$. We define the following bilinear operator:

$$B(x, y)(s, t) = (x \otimes y)(s, t) = x(s)y(t), (s, t) \in I \times I.$$

If X, Y, Z are symmetric function spaces, then the boundedness of B from $X \times Y$ into $Z(I \times I)$ is equivalent to the continuity of the embedding

$$X \otimes Y \subset Z,$$

where $X \otimes Y$ denotes the projective tensor product of spaces X and Y [1, p. 51].

The importance of operator B stems mainly from two facts: this operator arises naturally on study of geometric properties of symmetric spaces (see, for example, [2, p. 169–171], [3], [4]) and the problem of boundedness of tensor product is equivalent to the stability problem of integral operators ([1, p. 50], [5]).

The problem of boundedness of the operator B in concrete families of Banach spaces (for example, Lorentz, Marcinkiewicz and Orlicz spaces) has attracted the

attention of number of authors (see [1], [6–11]). In particular, R. O’Neil proved the following well-known theorem about boundedness of tensor product in the spaces L_{pq} [9].

Let us recall that the space L_{pq} ($1 < p < \infty, 1 \leq q \leq \infty$) consists of all measurable on I functions $x = x(t)$ for which

$$\|x\|_{pq} = \begin{cases} \left\{ \int_0^1 (x^*(t)t^{1/p})^q dt/t \right\}^{1/q}, & \text{if } 1 \leq q < \infty \\ \sup_{0 < t \leq 1} (x^*(t)t^{1/p}), & \text{if } q = \infty \end{cases}$$

is finite. Although the functional $\|x\|_{pq}$ is not subadditive, however it is equivalent to the norm $\|x\|'_{pq} = \|x^{**}\|_{pq}$, where $x^{**}(t) = 1/t \int_0^t x^*(s) ds$ and by $x^*(s)$ is denoted the right continuous non-increasing rearrangement of $|x(s)|$.

Let us notice that $L_{pr} \subset L_{pq}$ ($r \leq q$), $L_{pp} = L_p$ and usual norm of L_p $\|x\|_p = \|x\|_{pp}$. These spaces arise naturally in interpolation theory (see, for example, [12]).

Theorem (O’Neil)

Let $1 < p < \infty, 1 \leq q, r, s \leq \infty$. The boundedness of the operator B from $L_{pr} \times L_{pq}$ into $L_{ps}(I \times I)$ is equivalent to the conditions :

$$(1) \max(q, r) \leq s$$

and

$$(2) \frac{1}{p} + \frac{1}{s} \leq \frac{1}{q} + \frac{1}{r}.$$

We consider following problems connected with the exactness of this theorem:

a) Let us fix $1 < p < \infty, 1 \leq r \leq s \leq \infty$ and q^* is the maximum of all $q \geq 1$ for which the conditions of theorem are satisfied (if those exist). Is the space L_{pq^*} the largest among symmetric spaces E such that

$$B : L_{pr} \times E \rightarrow L_{ps}(I \times I) ?$$

b) Let us $1 < p < \infty, 1 \leq r \leq q \leq \infty$ and s^* is the minimum of all $s \geq 1$ for which the conditions of theorem are satisfied (if those exist). Is the space L_{ps^*} the smallest among symmetric spaces E such that

$$B : L_{pr} \times L_{pq} \rightarrow E(I \times I) ?$$

The answers to these questions are positive, of course, in case $r \leq p$, since B bounded from $L_{pr} \times L_{pr}$ into $L_{pr}(I \times I)$ ($r \leq p$) and from $L_p \times L_{pq}$ into L_{pq} ($p \leq q$) ([9], [11]).

In this article we shall study tensor product in general symmetric spaces. It will allow us to solve in part the problems a) and b).

The paper is organized in the following manner. At first, we introduce a concept of the multiplier $M(E)$ of a symmetric space E concerning to tensor product. The relations defining the fundamental function and the norm of dilation operators in $M(E)$ will be obtained. Moreover the upper and lower estimates of $M(E)$ allow to find it in some cases. In particular, we shall get the positive answer on the question a) in case $p < r = s$ (see the Corollary 1.12). Then we shall consider a quantity connected with a boundedness of tensor product in symmetric spaces.

Finally we study a boundedness of operator B in the spaces L_{pq} , using the family of Lorentz-Zygmund spaces $L_{pq}(\log L)^\alpha$ that involve, in particular, spaces L_{pq} . The main result —Theorem 2.1— is similar with O’Neil Theorem. To begin with it allows to define more exactly values of B on $L_{pr} \times L_{pq}$ for $p < r \leq q$, $q^{-1} + r^{-1} - p^{-1} \geq 0$. Namely, we shall find (see Corollary 2.7) the symmetric space $E = E(p, r, q) \subset L_{ps}$, $E \neq L_{ps}$, $s^{-1} = q^{-1} + r^{-1} - p^{-1}$ such that

$$B : L_{pr} \times L_{pq} \rightarrow E(p, r, q) (I \times I).$$

The last means that the answer on the problem b) is negative in this case. Besides, O’Neil Theorem does not describe the set $B(L_{pr}, L_{pq})$, if $r^{-1} + q^{-1} - p^{-1} < 0$ (a family L_{pq} is too narrow for study of this problem). Consideration of spaces $L_{pq}(\log L)^\alpha$ allows to solve this problem to within any positive power of logarithm (see Theorems 2.1 and 2.4). The complex method of interpolation is used here (the definitions of interpolation theory see in [12] or [13]).

1. The multiplier of symmetric space concerning to tensor product

If $z = z(w)$ is a measurable function on I or $I \times I$, μ is the usual Lebesgue measure, then the distribution function of $|z(s)|$ is $n_z(\tau) = \mu\{w : |z(w)| > \tau\} (\tau > 0)$. The function $x(t)$ is equimeasurable with $y(t)$ if $n_x(\tau) = n_y(\tau) (\tau > 0)$.

Let us recall that the Banach space E of measurable functions defined on I is said to be symmetric if these conditions are satisfied: a) $y \in E$ and $|x(t)| \leq |y(t)|$ imply that $x \in E$ and $\|x\| \leq \|y\|$; b) if $y \in E$ and the function $x(t)$ is equimeasurable with $y(t)$, then $x \in E$ and $\|x\| = \|y\|$.

Let E be a symmetric space on $I = [0, 1]$. We shall denote by $M(E)$ a set of all measurable functions $x = x(s) (s \in I)$ for which $x \otimes y \in E(I \times I)$ with arbitrary $y \in E$. Then $M(E)$ is symmetric space on I concerning to the norm

$$\|x\| = \sup\{\|x \otimes y\|_{E(I \times I)} : \|y\|_E \leq 1\}.$$

Obviously, $M(E) \subset E$.

EXAMPLE 1.1: Let us find $M(E)$, if E is Lorentz space $\Lambda(\phi)$, where positive concave function $\phi(u)$ increases on $(0, 1]$. Recall that $\Lambda(\phi)$ consists of all measurable functions $x = x(s)$ for which

$$\|x\|_{\Lambda(\phi)} = \int_0^1 x^*(s) d\phi(s) < \infty.$$

In particular, $L_{p1} = \Lambda(t^{1/p})$ ($1 < p < \infty$).

For arbitrary $e \subset I$, $x \in \Lambda(\phi)$ functions $\chi_e \otimes x$ and $\sigma_{\mu(e)}x$ are equimeasurable, where $\chi_e(s) = 1$ ($s \in e$), $\chi_e(s) = 0$ ($s \notin e$) and the dilation operators $\sigma_t y(u) = y(u/t)\chi_{[0,1]}(u/t)$. As the rearrangements of equimeasurable functions are equal, then

$$\begin{aligned} \|\chi_e \otimes x\|_{\Lambda(\phi)} &= \int_0^1 (\sigma_{\mu(e)} x)^*(u) d\phi(u) = \int_0^{\mu(e)} x^*(u/\mu(e)) d\phi(u) \\ &= \int_0^1 x^*(v) d\phi(\mu(e)v). \end{aligned}$$

Since $\|\chi_e\|_{\Lambda(\phi)} = \phi(\mu(e))$, then we have in view of [12, p. 151]

$$\|x\|_{M(\Lambda(\phi))} \leq 2 \sup_{0 < t \leq 1} \int_0^1 x^*(v) d \left[\frac{\phi(tv)}{\phi(t)} \right].$$

Introduce the notation:

$$\mathcal{M}_\phi(v) = \sup \left\{ \frac{\phi(tv)}{\phi(t)}, 0 < t \leq \min(1, 1/v) \right\}.$$

Using last inequality, we obtain

$$\Lambda(\mathcal{M}_\phi) \subset M(\Lambda(\phi)).$$

Assume in addition that

$$\mathcal{M}_\phi(v) = \lim_{t \rightarrow t_0} \frac{\phi(tv)}{\phi(t)}, \tag{1}$$

where $t_0 \in [0, 1]$ does not depend on $v \in [0, 1]$. In this case

$$\|x\|_{\Lambda(\mathcal{M}_\phi)} = \lim_{t \rightarrow t_0} \int_0^1 x^*(v) d \left[\frac{\phi(tv)}{\phi(t)} \right] = \lim_{t \rightarrow t_0} \frac{\|\chi_{(0,t)} \otimes x\|_{\Lambda(\phi)}}{\phi(t)} \leq \|x\|_{M(\Lambda(\phi))}.$$

Consequently, if (1) holds, then

$$M(\Lambda(\phi)) = \Lambda(\mathcal{M}_\phi).$$

Let again E be arbitrary symmetric space on I . We shall obtain the relations connecting the fundamental functions and the norms of dilation operators of spaces E and $M(E)$.

In the first place,

$$\phi_{M(E)}(t) = \|\sigma_t\|_{E \rightarrow E}, \quad 0 < t \leq 1, \tag{2}$$

where $\phi_X(t)$ denotes the fundamental function of symmetric space X : $\phi_X(t) = \|\chi_{(0,t)}\|_X$. Really, since the functions $\chi_{(0,t)} \otimes y$ and $\sigma_t y$ ($t \in [0, 1]$) are equimeasurable, then

$$\phi_{M(E)}(t) = \sup_{\|y\| \leq 1} \|\chi_{(0,t)} \otimes y\|_E = \sup_{\|y\| \leq 1} \|\sigma_t y\|_E = \|\sigma_t\|_{E \rightarrow E}.$$

Theorem 1.2

The following relations hold for an arbitrary symmetric space E on I :

$$\|\sigma_t\|_{M(E) \rightarrow M(E)} = \|\sigma_t\|_{E \rightarrow E}, \quad 0 < t \leq 1 \tag{3}$$

$$\|\sigma_{1/t}\|_{E \rightarrow E}^{-1} \leq \|\sigma_t\|_{M(E) \rightarrow M(E)} \leq \|\sigma_t\|_{E \rightarrow E}, \quad t > 1. \tag{4}$$

Proof. First of all, if $x(u)$ and $y(v)$ are measurable functions on I and

$$\text{supp } x = \{u : x(u) \neq 0\} \subset [0, \min(1, t^{-1})],$$

then for $t, s > 0$

$$\begin{aligned} n_{\sigma_t x \otimes y}(s) &= \int_0^1 \mu \left\{ u : |\sigma_t x(u)| > \frac{s}{|y(v)|} \right\} dv \\ &= t \int_0^1 \mu \left\{ u : |x(u)| > \frac{s}{|y(v)|} \right\} dv = t n_{x \otimes y}(s). \end{aligned}$$

Therefore

$$(\sigma_t x \otimes y)^*(s) = \sigma_t(x \otimes y)^*(s) \quad 0 \leq s \leq 1,$$

and thus

$$\|\sigma_t x \otimes y\|_E = \|\sigma_t(x \otimes y)^*\|_E \leq \|\sigma_t\|_{E \rightarrow E} \|x \otimes y\|_E. \tag{5}$$

If $0 < t \leq 1$, then we get from (5) by definition of the norm in $M(E)$

$$\|\sigma_t\|_{M(E) \rightarrow M(E)} \leq \|\sigma_t\|_{E \rightarrow E}. \tag{6}$$

In case $t > 1$ we define the function $\tilde{x}(u) = x(u)\chi_{(0,1/t]}(u)$. For it (5) is satisfied and, besides, the functions $\sigma_t\tilde{x}$ and σ_tx are equimeasurable. Therefore

$$\|\sigma_tx \otimes y\|_E = \|\sigma_t\tilde{x} \otimes y\|_E \leq \|\sigma_t\|_{E \rightarrow E} \|\tilde{x} \otimes y\|_E \leq \|\sigma_t\|_{E \rightarrow E} \|x \otimes y\|_E,$$

and thus (6) is satisfied for $t > 1$.

If $0 < t \leq 1$, then in view (2) and by the quality $\|\chi_{(0,1)}\|_{M(E)} = 1$

$$\|\sigma_t\|_{E \rightarrow E} = \phi_{M(E)}(t) = \|\sigma_t\chi_{(0,1)}\|_{M(E)} \leq \|\sigma_t\|_{M(E) \rightarrow M(E)}.$$

Using the submultiplicativity of the function $f(t) = \|\sigma_t\|_{M(E) \rightarrow M(E)}$ ($t > 0$) and inequality (6), we get in case $t > 1$:

$$\|\sigma_t\|_{M(E) \rightarrow M(E)} \geq \|\sigma_{1/t}\|_{M(E) \rightarrow M(E)}^{-1} \geq \|\sigma_{1/t}\|_{E \rightarrow E}^{-1}.$$

The theorem is proved. \square

The Boyd indices of symmetric space E are defined as

$$\alpha_E = \lim_{t \rightarrow 0} \frac{\ln \|\sigma_t\|_{E \rightarrow E}}{\ln t}, \quad \beta_E = \lim_{t \rightarrow \infty} \frac{\ln \|\sigma_t\|_{E \rightarrow E}}{\ln t}.$$

From Theorem 1.2 follows

Corollary 1.3

If E is any symmetric space on I then

$$0 \leq \alpha_E = \alpha_{M(E)} \leq \beta_{M(E)} \leq \beta_E \leq 1.$$

Remark 1.4. The left-hand side of the inequality (4) cannot be improved, in general. We shall show that there exists symmetric space E on I such that

$$\|\sigma_t\|_{M(E) \rightarrow M(E)} \|\sigma_{1/t}\|_{E \rightarrow E} = 1 \quad (t \geq 1) \quad (7)$$

and

$$\lim_{t \rightarrow \infty} \frac{\|\sigma_t\|_{M(E) \rightarrow M(E)}}{\|\sigma_t\|_{E \rightarrow E}} = 0. \quad (8)$$

Let $\phi_\alpha(s) = s^\alpha \ln^{-1}(C/s)$, $E = \Lambda(\phi_\alpha)$ ($0 < \alpha < 1$). If $C > \exp\{1/(1-\alpha)\}$, then the function ϕ_α is concave on $(0, 1]$. The function $f_t(s) = \ln(C/s) \ln^{-1}[C/(st)]$ ($0 <$

$s \leq \min(1, 1/t)$ decreases, if $0 < t \leq 1$, and increases, if $t > 1$. Therefore for $0 < t \leq 1$

$$\mathcal{M}_{\phi_\alpha}(t) = t^\alpha \lim_{s \rightarrow 0^+} f_t(s) = t^\alpha.$$

Hence, in particular, it follows that the condition (1) is fulfilled for $\phi_\alpha(s)$ and consequently as it was showed in Example 1.1

$$M(\Lambda(\phi_\alpha)) = \Lambda(\mathcal{M}_{\phi_\alpha}) = \Lambda(t^\alpha).$$

In the same time for $t > 1$

$$\mathcal{M}_{\phi_\alpha}(t) = t^\alpha f_t(1/t) = t^\alpha \ln(Ct) \ln^{-1} C.$$

Since $\|\sigma_t\|_{\Lambda(\phi) \rightarrow \Lambda(\phi)} = \mathcal{M}_\phi(t)$ [12, p. 134], then in our case

$$\|\sigma_t\|_{E \rightarrow E} = \mathcal{M}_{\phi_\alpha}(t), \quad \|\sigma_t\|_{M(E) \rightarrow M(E)} = t^\alpha (t > 0).$$

As a result we obtain the relations (7) and (8).

Continuing the study of multiplier of a symmetric space, we shall find its upper and lower estimates in general case.

Theorem 1.5 (Upper estimate)

Suppose that E is a symmetric space on I , $p = 1/\alpha_E$, where α_E is lower Boyd index of space E . Then

$$M(E) \subset L_p$$

and the constant of this embedding does not depend on a space E .

Proof. We consider the nonnegative simple functions of type:

$$x(s) = \sum_{i=1}^m \alpha_i \chi_{(\frac{i-1}{m}, \frac{i}{m}]}(s), \quad \alpha_i \geq 0, \quad m \in \mathbb{N}. \tag{9}$$

For any $y \in E$ function $x \otimes y$ is equimeasurable with the function

$$z_{x,y}(t) = \sum_{k=1}^m \alpha_k y_k(t),$$

where $y_k \in E$ are disjointly supported functions having all the same distribution function and

$$n_{y_k}(s) = \frac{1}{m} n_y(s), \quad s > 0, \quad k = 1, 2, \dots, m. \tag{10}$$

We shall show that for every $\epsilon > 0$ there exists $y \in E$ (or $y_k, k = 1, 2, \dots, m$) for which the equalities (10) hold and

$$\|x \otimes y\|_E = \|z_{x,y}\|_E \geq \frac{1 - \epsilon}{1 + \epsilon} \|x\|_p \|y\|_E, \tag{11}$$

where x is arbitrary function of type (9) and $p = 1/\alpha_E$.

Indeed, in view of [2, p. 141] (see also [14]) one can choose disjointly supported equimeasurable functions $z_k \in E (k = 1, 2, \dots, m)$ such that for $\alpha_k \geq 0$

$$(1 - \epsilon) \left(\sum_{k=1}^m \alpha_k^p \right)^{1/p} \leq \left\| \sum_{k=1}^m \alpha_k z_k \right\|_E \leq (1 + \epsilon) \left(\sum_{k=1}^m \alpha_k^p \right)^{1/p}. \tag{12}$$

Denote

$$y_k = \frac{z_k}{m} \quad (k = 1, 2, \dots, m), \quad y = \frac{1}{m} \sum_{k=1}^m z_k.$$

These functions satisfy all necessary assumptions, in particular, the equalities (10). We shall prove (11).

From (12) (if we take $\alpha_k = 1$ for $k = 1, 2, \dots, m$) it follows that

$$\|y\|_E \leq (1 + \epsilon)m^{1/p-1},$$

and therefore

$$\begin{aligned} \left\| \sum_{k=1}^m \alpha_k y_k \right\|_E &= \frac{1}{m} \left\| \sum_{k=1}^m \alpha_k z_k \right\|_E \geq \frac{1 - \epsilon}{m} \left(\sum_{k=1}^m \alpha_k^p \right)^{1/p} \\ &\geq \frac{1 - \epsilon}{1 + \epsilon} m^{-1/p} \|y\|_E \left(\sum_{k=1}^m \alpha_k^p \right)^{1/p} = \frac{1 - \epsilon}{1 + \epsilon} \|x\|_p \|y\|_E. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, then the following inequality can be obtained for functions $x(s)$ of type (9):

$$\|x\|_{M(E)} \geq \|x\|_p. \tag{13}$$

Let us extend the last inequality on countably-value dyadic functions

$$x(s) = \sum_{k=1}^{\infty} \alpha_k \chi_{(2^{-k}, 2^{-k+1}]}(s), \quad \alpha_k \geq 0.$$

Functions $x_n(s) = x(s)\chi_{(2^{-n}, 1]}(s)$ belong to the class (9), $x_n(s) \leq x(s)$ ($n = 1, 2, \dots$). Consequently,

$$\|x\|_{M(E)} \geq \|x_n\|_{M(E)} \geq \|x_n\|_p \quad (n = 1, 2, \dots).$$

Passing to the limit as $n \rightarrow \infty$, we obtain (13).

If now $y \in E$ is arbitrary, then we define the function

$$x_y(s) = \sum_{k=1}^{\infty} y^*(2^{-k+1}) \chi_{(2^{-k}, 2^{-k+1}]}(s).$$

It is clear that $x_y \leq y^* \leq \sigma_2(x_y)$. Since for any symmetric space X $\|\sigma_t\|_{X \rightarrow X} \leq \max(1, t)$ [12, p. 133], then

$$\|y\|_{M(E)} = \|y^*\|_{M(E)} \geq \|x_y\|_{M(E)} \geq \|x_y\|_p \geq \frac{1}{2} \|y\|_p.$$

The theorem is proved. \square

Theorem 1.6 (Lower estimate)

For arbitrary symmetric space E on I

$$M(E) \supset \Lambda(\psi),$$

where $\psi(t) = \|\sigma_t\|_{E \rightarrow E}$ ($0 < t \leq 1$). Moreover the constant of embedding does not depend on the space E .

Proof. We fix $y \in E$ and consider the operator $T_y x = x \otimes y$. If $x = \chi_e$, $e \subset [0, 1]$, $\mu(e) = t$, then

$$\|T_y \chi_e\|_E = \|\sigma_t y\|_E \leq \|\sigma_t\|_{E \rightarrow E} \|y\|_E = \psi(t) \|y\|_E.$$

In view of [12, p. 151] the following inequality is true for every $x \in \Lambda(\psi)$:

$$\|x \otimes y\|_E = \|T_y x\|_E \leq 2 \|y\|_E \|x\|_{\Lambda(\psi)}.$$

The theorem is proved. \square

Consider some corollaries of obtained theorems.

Corollary 1.7

The multiplier $M(E) = L_\infty$ iff the lower Boyd index of symmetric space E $\alpha_E = 0$.

Proof. If $\alpha_E = 0$, then $M(E) = L_\infty$ in view of Theorem 1.5. If $\alpha_E > 0$, then by Theorem 1.6 $M(E) \supset \Lambda(\psi)$ and $\psi(t) = \|\sigma_t\|_{E \rightarrow E} \rightarrow 0 (t \rightarrow 0+)$. Therefore $\Lambda(\psi) \neq L_\infty$. Since $\Lambda(\psi) \supset L_\infty$ [12, p. 124] then, all the more, $M(E) \neq L_\infty$. \square

Corollary 1.8

If $B : E \times E \rightarrow E(I \times I)$, then $E \subset L_{1/\alpha_E}$.

If E is Banach space of measurable functions defined on I , then dual space E' consists of all measurable on I functions $y = y(t)$ for which

$$\|y\|_{E'} = \sup \left\{ \int_0^1 x(t)y(t)dt : \|x\|_E \leq 1 \right\} < \infty.$$

The norm of symmetric space E is called order subcontinuous if from monotone convergence $x_n \uparrow x$ μ -a.e. ($x_n, x \in E$) it follows: $\|x_n\|_E \rightarrow \|x\|_E$.

Corollary 1.9

Let E be symmetric space with an order subcontinuous norm. If $B : E \times E \rightarrow E(I \times I)$ and $B : E' \times E' \rightarrow E'(I \times I)$, then there exists $p \in [1, \infty]$ such that $E = L_p$.

Proof. By Corollary 1.8 $E \subset L_{1/\alpha_E}$. For dual spaces the inverse embedding is satisfied:

$$E' \supset (L_{1/\alpha_E})' = L_{1/1-\alpha_E} \supset L_{1/\beta_{E'}} \supset L_{1/\alpha_{E'}},$$

since $1 - \alpha_E \geq \beta_{E'} \geq \alpha_{E'}$ [12, p. 144].

On the other hand, now in view of Corollary 1.8 $E' \subset L_{1/\alpha_{E'}}$ and thus $E' = L_{1/\alpha_{E'}}$.

As $L_\infty \subset E$ and E is a subspace of E'' [15, p. 255] we obtain, finally, for $p = 1/(1 - \alpha_{E'})$ the equality: $E = E'' = L_p$. \square

If $\phi_E(s)$ is the fundamental function of a symmetric space E , then $\mathcal{M}_{\phi_E}(t) \leq \|\sigma_t\|_{E \rightarrow E}$. A space E is called a space of fundamental type if

$$\|\sigma_t\|_{E \rightarrow E} \leq C \mathcal{M}_{\phi_E}(t),$$

where C does not depend on $t > 0$. Let us notice that all most important in applications symmetric spaces are spaces of fundamental type.

Corollary 1.10

If E is a symmetric space of fundamental type, then $M(E) \supset \Lambda(\mathcal{M}_{\phi_E})$.

Let us recall that Banach space E is interpolation space between the spaces E_0 and E_1 if $E_0 \cap E_1 \subset E \subset E_0 + E_1$ and from boundedness of arbitrary linear operator in E_0 and E_1 follows its boundedness in E . In this case there exists $C > 0$ such that

$$\|T\|_{E \rightarrow E} \leq C \max_{i=0,1} \|T\|_{E_i \rightarrow E_i}$$

where C does not depend on operator T [12].

Theorem 1.11

Let symmetric space E be an interpolation space between the spaces L_p and $L_{p\infty}$ by some $p \in (1, \infty)$. Then $M(E) = L_p$.

Proof. Since $L_p \subset E \subset L_{p\infty}$, then, obvious, there exist $C_1 > 0$ and $C_2 > 0$ for which

$$C_1 t^{1/p} \leq \phi_E(t) \leq C_2 t^{1/p} \quad (0 \leq t \leq 1).$$

Hence, in particular,

$$\|\sigma_t\|_{E \rightarrow E} \geq C_3 t^{1/p} \quad (t > 0).$$

On the other hand, by condition of theorem

$$\|\sigma_t\|_{E \rightarrow E} \leq C_4 \max(\|\sigma_t\|_{L_p \rightarrow L_p}, \|\sigma_t\|_{L_{p\infty} \rightarrow L_{p\infty}}) = C_4 t^{1/p}.$$

Using obtained inequalities, we get in view of definition of Boyd indices $\alpha_E = \beta_E = 1/p$, and therefore by the Theorem 1.2 $M(E) \subset L_p$.

For the proof of inverse embedding we fix $y \in L_p$ and consider the linear operator $T_y x = x \otimes y$. Then

$$\|T_y\|_{L_p \rightarrow L_p} = \|y\|_p, \quad \|T_y\|_{L_{p\infty} \rightarrow L_{p\infty}} \leq C_5 \|y\|_p,$$

where $C_5 > 0$ does not depend on y [11]. Consequently, in view of interpolation property of a space E

$$\|T_y\|_{E \rightarrow E} \leq C_6 \|y\|_p,$$

that is, $y \in M(E)$ and $\|y\|_{M(E)} \leq C_6 \|y\|_p$. The theorem is proved. \square

In particular, L_{pq} ($p \leq q \leq \infty$) is interpolation space between L_p and $L_{p\infty}$ [12, p. 142]. Therefore we obtain the following

Corollary 1.12

If $1 < p < \infty, p \leq q \leq \infty$, then

$$M(L_{pq}) = L_p.$$

Remark 1.13. In case $q = \infty$ the last statement was proved in [11] by “calculation” of operator T_y from proof of Theorem 1.11 on a function $x_0(s) = s^{-1/p}$, that has the greatest rearrangement in the space $L_{p\infty}$.

Let us introduce a quantity connected with a boundedness of tensor product in the symmetric spaces. We define for symmetric space E , $m \in \mathbb{N}$ and $\alpha = (\alpha_k)_{k=1}^m \in \mathbb{R}^m$

$$\mathcal{K}_E^m(\alpha) = \sup \left\| \sum_{k=1}^m \alpha_k y_k \right\|_E,$$

where the supremum is taken over all $y \geq 0$, $\|y\|_E = 1$ and all collections of disjointly supported functions y_k ,

$$n_{y_k}(s) = \frac{1}{m} n_y(s), s > 0.$$

Theorem 1.14

Let E be a symmetric space on I with an order subcontinuous norm. Then B is the bounded operator from $E \times E$ into $E(I \times I)$ if and only if there exists $C > 0$ such that for all $m \in \mathbb{N}$, $\alpha = (\alpha_i)_{i=1}^m \in \mathbb{R}^m$

$$\mathcal{K}_E^m(\alpha) \leq C \left\| \sum_{i=1}^m \alpha_i \chi_{(\frac{i-1}{m}, \frac{i}{m}]} \right\|_E. \quad (14)$$

Proof. If $y \in E, \|y\|_E = 1$ and $x = x(s)$ is such as in (9), then functions

$$B(x, y)(s, t) = \sum_{i=1}^m \alpha_i \chi_{(\frac{i-1}{m}, \frac{i}{m}]}(s) y(t) \quad (s, t \in [0, 1])$$

and

$$v(t) = \sum_{k=1}^m \alpha_k y_k(t) \quad (t \in [0, 1])$$

(y_k are mutually disjoint and $n_{y_k}(u) = \frac{1}{m} n_y(u)$, $u > 0$) are equimeasurable. Consequently, the inequality (14) is equivalent to the following:

$$\|x \otimes y\|_E \leq C \|x\|_E, \tag{15}$$

where $\|y\|_E = 1$ and the function x belongs to the class (9). Hence boundedness of the operator B from $E \times E$ into $E(I \times I)$ implies (14).

Conversely, let us assume that the inequality (14) (or (15)) is satisfied. Since the norm of E is order subcontinuous, then the inequality (15) can be extended, at first, on countably-value and then on arbitrary functions from E as in proof of the Theorem 1.2. \square

2. The tensor product in the Lorentz spaces L_{pq}

Let $1 < p < \infty$, $1 \leq q \leq \infty$, $\alpha \in \mathbb{R}$, $\phi_{p,\alpha}(u) = u^{1/p} \ln^\alpha(e/u)$. The Lorentz-Zygmund space $L_{pq}(\log L)^\alpha$ consists of all measurable functions on I (or $I \times I$) for which

$$\|x\|_{p,\alpha,q} = \begin{cases} \left\{ \int_0^1 (x^*(u) \phi_{p,\alpha}(u))^q du/u \right\}^{1/q}, & \text{if } q < \infty \\ \sup_{0 < u \leq 1} (x^*(u) \phi_{p,\alpha}(u)), & \text{if } q = \infty \end{cases}$$

is finite. As in case L_{pq} $\|x\|_{p,\alpha,q}$ is not a norm. However using the Minkovski inequality (see, for example, [16, p. 38]), it can be verified that this functional is equivalent to the norm $\|x\|'_{p,\alpha,q} = \|x^{**}\|_{p,\alpha,q}$. Moreover the fundamental function of space $L_{pq}(\log L)^\alpha$ is equivalent to the function $\phi_{p,\alpha}(u)$. For $\alpha = 0$ the Lorentz-Zygmund spaces are the standard Lorentz spaces L_{pq} , while for $p = q < \infty$ they are the Orlicz spaces $L_p(\log L)^{\alpha p}$. For $p = q = \infty$, they produce the exponential classes of Zygmund (see [17] for further details). The Lorentz-Zygmund spaces were studied of a number of authors (except [17] see, for example, [18]).

If E is symmetric space, then we use E^0 to denote a norm closure of L_∞ in E .

Theorem 2.1

Let $1 < p < \infty$, $p \leq r \leq q \leq \infty$, $\alpha = r^{-1} - p^{-1}$. The operator B is bounded operator from $L_{pr} \times L_{pq}$ into the space $(L_{pq}(\log L)^\alpha)^0(I \times I)$ (if $p < r < q = \infty$) and into $L_{pq}(\log L)^\alpha(I \times I)$ (otherwise).

We need for a proof of this theorem two lemmas. The first of them is proved immediately, using easy calculations.

Lemma 2.2

Let $1 < p < \infty$, $\alpha \in \mathbb{R}$, $\psi_{p,\alpha} = 1/\phi_{p,\alpha}$.

- 1) If $C = C(p, \alpha) > 0$ is large enough, then for $v > C$ the distribution function of $\psi_{p,\alpha}$ is equivalent to the function $N_{p,\alpha}(v) = v^{-p} \ln^{-p\alpha} v$.
- 2) If $\alpha_i < 1/p$ ($i = 0, 1$), then the functions $\psi_{p,\alpha_0} \otimes \psi_{p,\alpha_1}$ and $\psi_{p,\alpha_0+\alpha_1-1/p}$ are equimeasurable.

The second lemma can be proved, using ordinary interpolation techniques (we shall denote by $[X_0, X_1]_\theta$ the complex interpolation spaces, as defined, for example, in [13] or [19]). In detail see [11].

Lemma 2.3

Let $1 < p_1 \leq p_0 < \infty$, $-\infty < \alpha_1 \leq \alpha_0 < \infty$, $1 \leq q_0 \leq q_1 \leq \infty$, $0 \leq \theta \leq 1$ and

$$\max(p_0 - p_1, \alpha_0 - \alpha_1, q_1 - q_0) > 0. \quad (16)$$

Then the space

$$[L_{p_0, q_0}(\log L)^{\alpha_0}, L_{p_1, q_1}(\log L)^{\alpha_1}]_\theta$$

is isomorphic to $(L_{pq}(\log L)^\alpha)^0$, if $q_0 = q_1 = \infty$ and $0 < \theta \leq 1$ or $q_1 = \infty$ and $\theta = 1$, and to $L_{pq}(\log L)^\alpha$, otherwise. The numbers p, q, α are defined by the following way:

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \alpha = (1-\theta)\alpha_0 + \theta\alpha_1.$$

Remark 2.4. The inequality (16) implies the embedding

$$L_{p_0, q_0}(\log L)^{\alpha_0} \subset L_{p_1, q_1}(\log L)^{\alpha_1}$$

that is used by proof of Lemma 2.3. However this embedding is true even when (16) is not satisfied [17].

Proof of Theorem 2.1. Let us define the numbers

$$s = \frac{pq}{r}, \theta = 1 - \frac{p}{r}. \tag{17}$$

Then $p \leq s \leq \infty, 0 \leq \theta \leq 1, p \leq r \leq \infty, s \leq q \leq \infty$ and statement of theorem is true for extreme values r and q , that is,

$$B : L_{pp} \times L_{ps} \rightarrow L_{ps}(I \times I) \tag{18}$$

$$B : L_{p\infty} \times L_{p\infty} \rightarrow L_{p\infty}(\log L)^{-1/p}. \tag{19}$$

Really, (18) follows from the O'Neil Theorem (or from the Corollary 1.12 of present article). In view of the Closed Graph Theorem for a proof (19) it is enough to show that $B(x, y)$ belong to $L_{p\infty}(\log L)^{-1/p}(I \times I)$, if $x \in L_{p\infty}, y \in L_{p\infty}$. Lemma 2.2 and definition of $L_{p\infty}(\log L)^{-1/p}$ imply that this is true for the functions $x_0(u) = u^{-1/p}, y_0(v) = v^{-1/p}$. Since these functions have the greatest rearrangement in $L_{p\infty}$, then (19) is proved.

For θ from (17) we use now the complex method of interpolation. First, we recall the following property (see [19] or [13, p. 125–126]):

If $(X_0, X_1), (Y_0, Y_1)$ and (Z_0, Z_1) are arbitrary Banach pairs, T is a bilinear operator from $X_i \times Y_i$ into Z_i ($i = 0, 1$), then T is bounded operator from $[X_0, X_1]_\theta \times [Y_0, Y_1]_\theta$ into $[Z_0, Z_1]_\theta$ ($0 \leq \theta \leq 1$).

Theorem 2.1 follows now from the last remarks and Lemma 2.3. \square

Also there is following result about the exactness of the Theorem 2.1. For the proof it is needed to consider operator B on the pairs of the functions of type

$$\psi_{p\alpha}(u) = u^{-1/p} \ln^{-\alpha}(e/u)$$

and to use then the statement of the Lemma 2.2.

Theorem 2.5

If $1 < p < \infty$ and $p < r \leq q \leq \infty$, then the operator B does not act from $L_{pr} \times L_{pq}$ into the space $L_{pq}(\log L)^\beta(I \times I)$ for any $\beta > r^{-1} - p^{-1}$.

Remark 2.6. If $r = q = \infty$, then the following assertion is true: $L_{p\infty}(\log L)^{-1/p}$ is the smallest among symmetric spaces E such that B is the bounded operator from $L_{p\infty} \times L_{p\infty}$ into $E(I \times I)$ (see Lemma 2.2 and Proof of Theorem 2.1).

We get from O'Neil Theorem and Theorem 2.2 following

Corollary 2.7

Let $1 < p \leq r \leq q < \infty$, $r^{-1} + q^{-1} - p^{-1} \geq 0$. If $\alpha = r^{-1} - p^{-1}$, $s^{-1} = r^{-1} + q^{-1} - p^{-1}$, then B is the bounded operator from $L_{pr} \times L_{pq}$ into the space

$$E(p, r, q)(I \times I) = L_{ps} \bigcap L_{pq} (\log L)^\alpha (I \times I).$$

Remark 2.8. The space $E(p, q, r)$ is not equal neither L_{ps} nor $L_{pq}(\log L)^\alpha$, if $p < r$. Assume, for example, that $r = q = 2p$. Then B acts from $L_{p,2p} \times L_{p,2p}$ into $E(p, 2p, 2p)(I \times I) = L_{p\infty} \bigcap L_{p,2p}(\log L)^{-1/2p} (I \times I)$.

On the one hand, it is easy to check that the function $\psi_{p,0}(u) = u^{-1/p}$ belongs to $L_{p\infty}$ and it does not belong to $L_{p,2p}(\log L)^{-1/2p}$.

On the other hand, if we assume that, on the contrary,

$$L_{p,2p}(\log L)^{-1/2p} \subset L_{p\infty},$$

then from the inequality for fundamental functions of these spaces we obtain false inequality: $\ln(e/t) \leq C$ ($C > 0$ does not depend on $t \in (0, 1]$).

Remark 2.9. As we remarked in beginning of this article the case

$$\frac{1}{r} + \frac{1}{q} - \frac{1}{p} < 0$$

is not considered in the O'Neil Theorem. The Theorems 2.1 and 2.4 complete it, characterizing the image $B(L_{pr}, L_{pq})$ in this case to within any positive power of logarithm.

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