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On the Bloch space and convolution of functions in the L^p-valued case

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Abstract

We introduce the convolution of functions in the vector valued spaces $H^1(L^p)$ and $H^1(L^q)$ by means of Young's Theorem, and we use this to show that Bloch functions taking values in certain space of operators define bilinear bounded maps in the product of those spaces for $1 \leq p, q \leq 2$. As a corollary, we get a Marcinkiewicz-Zygmund type result.

Preliminaries

In all what follows, we shall write L^p $(p \geq 1)$ for the space $L^p(\sigma)$, where σ is the normalized Lebesgue measure in the torus $\mathbb{T} = \{w \in \mathbb{C};\ |w| = 1\}$. As usual, the norm of a function in L^p (or $L^p(X)$ in the vector valued case) will be expressed $||f||_p$ (or $||f||_{p,X}$).

Given a Banach complex space X, the Hardy space $H^1(X)$, whose elements are functions on $\mathbb{T}$ with values in X, is the closure of the set of all analytic X-valued polynomials, denoted by $\mathcal{P}(X)$, in the Lebesgue-Bochner space $L^1(X)$, and

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coincides with the set of functions in $L^1(X)$ whose negative Fourier coefficients vanish. Observe that a function f in $H^1(X)$ can be regarded as an analytic function in D if we define $f(z) = \int_{\mathbb{T}} f(e^{it})P(z, e^{-it})d\sigma(e^{it})$, where P is the Poisson kernel $P(z,w) = \frac{1-|z|^2}{|1-wz|^2}$ and its derivative f' is another analytic function on D, but not necessarily defined on $\mathbb{T}$. For any $0 < r < 1$ we get another function $f_r \in H^1(X)$, given by $f_r(e^{it}) = f(re^{it})$ and following the usual notation we write $M_{1,X}(f,r)$ for $||f_r||_{1,X}$. This norm grows with r, and the limit as $r \to 1$ is precisely $||f||_{H^1(X)}$. This is just the same as in the scalar case, but the reader should be aware that there are many "scalar theorems" that depend strongly on the geometry of the Banach space in order to keep or not true in the vector valued case. We refer the reader to [3, 7] for details on vector valued Hardy spaces in this setting.

One of the most important results in the theory of Hardy spaces which is not always true in the vector valued case is that the dual space of H^1 is $BMOA$. For a complex Banach space X, $BMOA(X)$ is the space of all functions $f \in H^1(X)$ such that

$$||f||_{*,X} = \sup_I \frac{1}{|I|} \int_I ||f(e^{it}) - f_I|| d\sigma(e^{it}) < \infty,$$

where the supremum is taken over all intervals $I \subset \mathbb{T}$, $|I|$ stands for the normalized Lebesgue measure of I and $f_I = \frac{1}{|I|}\int_I f(e^{it})d\sigma(e^{it})$.

Note that $||f||_{*,X}$ is a seminorm, and if we define the norm by

$$||f||_{BMOA(X)} = \left\| \int_{\mathbb{T}} f(e^{it})d\sigma(e^{it}) \right\| + ||f||_{*,X}$$

then $BMOA(X)$ is a Banach space.

For any Banach space X one has that $BMOA(X^*)$ is continuously included in $\left(H^1(X)\right)^*$, in the following sense: if $f \in BMOA(X^*)$ and $g \in \mathcal{P}(X)$, then

$$\left| \int_{\mathbb{T}} \left\langle f(e^{it}), g(e^{-it}) \right\rangle d\sigma(e^{it}) \right| \leq ||f||_{BMOA(X^*)} ||g||_{1,X}.$$

In the case that X has the UMD property (which was introduced in terms of vector valued martingales, and is equivalent to the boundedness of the Hilbert transform on $L^p(X)$ for any $1 < p < \infty$), then the pairing given by $\langle f, g\rangle$ as the integral above gives the duality

$$\left(H^1(X)\right)^* = BMOA(X^*).$$

It is well known that L^p is a UMD space only for $1 < p < \infty$.

(The reader is referred to [3, 6] for information on the duality problem in the vector valued setting).

Next we recall Kahane inequalities, which in their trigonometric version (see [14]) state that, for any $0 < p < \infty$ and any finite family (x_n) in X, we have

$$\int_{\mathbb{T}} \left\| \sum_{k=0}^{n} x_k e^{i2^k t} \right\| d\sigma(e^{it}) \sim \left(\int_{\mathbb{T}} \left\| \sum_{k=0}^{n} x_k e^{i2^k t} \right\|^p d\sigma(e^{it}) \right)^{1/p}.$$

Here, and in the sequel, $A \sim B$ stands for $C_1 A \leq B \leq C_2 A$ for absolute constants C_1 and C_2. In this case the constants only depend on p (and not on X). If we substitute x_k by scalars α_k we get the so called Khintchine inequalities because, by Plancherel, we have that

$$\left(\int_{\mathbb{T}} \left| \sum_{k=0}^{n} \alpha_k e^{i2^k t} \right|^2 d\sigma(e^{it}) \right)^{1/2} = \left(\sum_{k=0}^{n} |\alpha_k|^2 \right)^{1/2}.$$

In the same way (though not so classical), it holds that

$$\int_{\mathbb{T}} \left\| \sum_{k=0}^{n} x_k e^{i2^k t} \right\| d\sigma(e^{it}) \sim \left\| \sum_{k=0}^{n} x_k z^{2^k} \right\|_{BMOA(X)}.$$

A simple proof of this can be seen in [4].

An analytic function on D with values in X, say $f(z) = \sum_{n=0}^{\infty} x_n z^n$, is called a Bloch function if

$$\sup_{|z|<1} (1 - |z|)||f'(z)|| < \infty.$$

The set $\mathcal{B}(X)$ of Bloch functions taking values in X, denoted by $\mathcal{B}$ when $X = \mathbb{C}$, is a Banach space if we endow it with the norm $\max\{||f(0)||, ||f||_{\mathcal{B}(X)}\}$, where $|| \cdot ||_{\mathcal{B}(X)}$ stands for the supremum above (which is a seminorm).

Let us mention that for the Bloch norm we have that

$$\left\| \sum_{n=1}^{\infty} x_n z^{2^n} \right\|_{\mathcal{B}(X)} \sim \sup_{n \in \mathbb{N}} ||x_n||.$$

This follows from the scalar case (see [1, 2]) using the easy fact that

$$||f||_{\mathcal{B}(X)} = \sup_{||x^*|| \leq 1} ||x^* f||_{\mathcal{B}}.$$

As usual, the constant C in the proofs may vary from line to line.

The result

Recalling Young's Theorem, which says that the convolution $f * g$ of two functions $f \in L^p$ and $g \in L^q$, when $\frac{1}{p} + \frac{1}{q} \geq 1$, makes sense and verifies $||f * g||_r \leq ||f||_p ||g||_q$ if $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, we can give the following definition.

DEFINITION 1. Let $1 \leq p, q, r \leq \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Then, for any $f \in L^1(L^p)$ and any $g \in L^1(L^q)$ we define their convolution $f * g$ as the L^r-valued function given by

$$f * g(e^{i\theta}) = \int_{\mathbb{T}} f\big(e^{i(\theta - t)}\big) * g(e^{it}) d\sigma(e^{it}).$$

To justify our definition simply mention that the function $f(e^{i(\theta-t)}) * g(e^{it})$ is easily seen to be measurable in $\mathbb{T} \times \mathbb{T}$, and using Fubini and Young theorems we get that the integral $\int_{\mathbb{T}} ||f(e^{i(\theta-t)}) * g(e^{it})||_r d\sigma(e^{it})$ is finite for almost every $e^{i\theta} \in \mathbb{T}$. Then $f * g$ is a measurable function.

Remark 1. It is easy to see that if $f(e^{it}) = \sum_{-N}^{N} \varphi_n e^{int}$ and $g(e^{it}) = \sum_{-N}^{N} \psi_n e^{int}$ then

$$f * g(e^{i\theta}) = \sum_{-N}^{N} \varphi_n * \psi_n e^{in\theta}.$$

Now we can state the following result whose elementary proof is left to the reader.

Proposition 1

Given $f \in L^1(L^p)$ and $g \in L^1(L^q)$, the convolution $f * g$ is in $L^1(L^r)$ and verifies $||f * g||_{L^1(L^r)} \leq ||f||_{L^1(L^p)} ||g||_{L^1(L^q)}$.

Moreover, if $f \in H^1(L^p)$ and $g \in H^1(L^q)$ then $f * g \in H^1(L^r)$.

Our next result is the key point for the main theorem, this is the extension of a classical result of Hardy and Littlewood (see [11, 9]) to the L^p-valued case for certain values of p. We refer the reader to [4, 5] for a proof of a more general statement, and further information about the Banach spaces for which the same result holds. For instance, it is shown there that the result doesn't extend to the rest of values of p.

Theorem 1 (See [5], Lemma 1.1)

Let $1 \leq p \leq 2$. There is an absolute constant $C > 0$ such that

$$\left(\int_0^1 (1-s) M^2_{1,L_p}(f', s) ds \right)^{1/2} \leq C ||f||_{1,L_p}$$

for any $f \in H^1(L^p)$.

Proof. By a theorem of Rosenthal ([15]) we know that L^p is isometrically contained in L^1 for $1 < p \leq 2$. Therefore we only have to prove the result for $p = 1$.

Let f be an analytic polynomial with values in L^1, say $f(z) = \sum_{n=0}^{N} \varphi_n z^n$. Given $e^{it} \in \mathbb{T}$ we shall write f_t for the scalar polynomial with coefficients $(\varphi_n(e^{it}))$; note that then we have $(f_t)' = (f')_t$, and we shall write simply f'_t. We have

$$\begin{aligned}\int_0^1 (1-s) M^2_{1,L_1}(f', s) ds &= \int_0^1 (1-s) \left(\int_{\mathbb{T}} ||f'(se^{i\theta})||_1 d\sigma(e^{i\theta}) \right)^2 ds \\ &= \int_0^1 (1-s) \left(\int_{\mathbb{T}} \int_{\mathbb{T}} |f'_t(se^{i\theta})| d\sigma(e^{it}) d\sigma(e^{i\theta}) \right)^2 ds \\ &= \int_0^1 (1-s) \left(\int_{\mathbb{T}} \int_{\mathbb{T}} |f'_t(se^{i\theta})| d\sigma(e^{i\theta}) d\sigma(e^{it}) \right)^2 ds \\ &= \left(\int_0^1 (1-s) g(s) \int_{\mathbb{T}} \int_{\mathbb{T}} |f'_t(se^{i\theta})| d\sigma(e^{i\theta}) d\sigma(e^{it}) ds \right)^2\end{aligned}$$

(where g is a certain norm one function in $L^2((1-s)ds)$)

$$\begin{aligned}&= \left(\int_{\mathbb{T}} \Big(\int_0^1 (1-s) M_1(f'_t, s) g(s) ds \Big) d\sigma(e^{it}) \right)^2 \\ &\leq \left(\int_{\mathbb{T}} \Big(\int_0^1 (1-s) M_1^2(f'_t, s) ds \Big)^{1/2} d\sigma(e^{it}) \right)^2 ,\end{aligned}$$

and using the scalar inequality (see [11]) for each t we get

$$\begin{aligned}\int_0^1 (1-s) M^2_{1,L_1}(f', s) ds &\leq C \left(\int_{\mathbb{T}} ||f_t||_1 d\sigma(e^{it}) \right)^2 \\ &= C \left(\int_{\mathbb{T}} \int_{\mathbb{T}} |f_t(e^{i\theta})| d\sigma(e^{i\theta}) d\sigma(e^{it}) \right)^2 \\ &= C \left(\int_{\mathbb{T}} \int_{\mathbb{T}} |f_t(e^{i\theta})| d\sigma(e^{it}) d\sigma(e^{i\theta}) \right)^2 = C ||f||^2_{H^1(L^1)} . \ \square\end{aligned}$$

We shall prove now our main result:

Theorem 2

Let X be a complex Banach space, let $1 \leq p, q \leq 2$ and take $1 \leq r \leq \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Then there exists an absolute constant C such that

$$\Big\| \sum_{n \geq 1} T_n(\hat{f}(n) * \hat{g}(n)) \Big\|_X \leq C ||f||_{H^1(L^p)} ||g||_{H^1(L^q)} ||h||_{\mathcal{B}} ,$$

for any $f \in \mathcal{P}(L^p)$, $g \in \mathcal{P}(L^q)$ and $h(z) = \sum_{n \geq 0} T_n z^n \in \mathcal{B}\big(\mathcal{L}(L^r, X)\big)$, where $\mathcal{L}(L^r, X)$ stands for the space of bounded operators from L^r into X.

Proof. Let $\varphi_n = \hat{f}(n) \in L^p$ and $\psi_n = \hat{g}(n) \in L^q$ for each n. Check that

$$\int_0^1 (1-s^3)^2 s^{3n-1} ds = \frac{2/3}{n(n+1)(n+2)}\,,$$

so we can write

$$\sum_{n\geq 1} T_n(\varphi_n * \psi_n) = \frac{3}{2}\int_0^1 (1-s^3)^2 \sum_{n\geq 1} n s^{n-1} T_n((n+1)\varphi_n s^n * (n+2)\psi_n s^n) ds\,.$$

Let $u(z) = zf(z)$ and $v(z) = zg(z)$. Using that if $S_n \in \mathcal{L}(L^r, X)$ and $\phi_n \in L^r$ then

$$\int_{\mathbb{T}} \left(\sum_{n=0}^{\infty} S_n e^{int}\right)\left(\sum_{n=0}^{N} \phi_n e^{-int}\right) d\sigma(e^{it}) = \sum_{n=0}^{N} S_n(\phi_n)$$

and Remark 1 one can obtain that the sum in the above integral is the same as

$$\int_{\mathbb{T}} h'(se^{i\theta})\big[(u'_s * (v'_s + g_s))(e^{-i\theta})\big] e^{i\theta} d\sigma(e^{i\theta}) =: A(s)\,.$$

By Proposition 1 and the definition of $||\cdot||_{\mathcal{B}}$ we have

$$\begin{aligned}
||A(s)||_X &\leq \int_{\mathbb{T}} ||h'(se^{i\theta})||\, ||(u'_s * (v'_s + g_s))(e^{-i\theta})||_r d\sigma(e^{i\theta}) \\
&\leq \frac{1}{1-s}||h||_{\mathcal{B}} \int_{\mathbb{T}} ||u'(se^{i\theta})||_p d\sigma(e^{i\theta}) \int_{\mathbb{T}} (||v'(se^{i\theta})||_q + ||g(se^{i\theta})||_q) d\sigma(e^{i\theta}) \\
&= \frac{1}{1-s}||h||_{\mathcal{B}} M_{1,L^p}(u',s)\,(M_{1,L^q}(v',s) + M_{1,L^q}(g,s)) \\
&\leq \frac{1}{1-s}||h||_{\mathcal{B}} M_{1,L^p}(u',s)(M_{1,L^q}(v',s) + ||g||_{H^1(L^q)})\,,
\end{aligned}$$

and using now that $1 - s^3 \leq 3(1-s)$ along with Cauchy-Schwarz inequality and Theorem 1, we get

$$\begin{aligned}
\Big\|\sum_{n\geq 1} T_n(\varphi_n * \psi_n)\Big\|_X &\leq \frac{3}{2}\int_0^1 \big(1-s^3\big)^2 ||A(s)||_X ds \\
&\leq \frac{27}{2}||h||_{\mathcal{B}} \int_0^1 (1-s) M_{1,L^p}(u',s)\big(M_{1,L^q}(v',s) + ||g||_{H^1(L^q)}\big) ds \\
&\leq \frac{27}{2}||h||_{\mathcal{B}} \left(\int_0^1 (1-s) M^2_{1,L^p}(u',s) ds\right)^{1/2} \\
&\quad \left(\left(\int_0^1 (1-s) M^2_{1,L^q}(v',s) ds\right)^{1/2} + ||g||_{H^1(L^q)}\right) \\
&\leq \frac{27}{2}||h||_{\mathcal{B}}\, C||u||_{H^1(L^p)}\big(C||v||_{H^1(L^q)} + ||g||_{H^1(L^q)}\big) \\
&= \frac{27}{2} C(C+1)||h||_{\mathcal{B}} ||f||_{H^1(L^p)} ||g||_{H^1(L^q)}\,. \ \square
\end{aligned}$$

Remark 2. The theorem shows that any function $h \in \mathcal{B}(\mathcal{L}(L^r, X))$ defines a bilinear map in $\mathcal{P}(L^p) \times \mathcal{P}(L^q)$ which extends by density to a bounded bilinear map

$$U_h: H^1(L^p) \times H^1(L^q) \to X \ ,$$

taking the value $\frac{3}{2}\int_0^1 (1-s^3)^2 A(s) ds$, with $A(s)$ as in the proof for every pair of functions f and g. When f *or* g is a polynomial, it equals the finite sum $\sum_{n\geq 1} T_n(\hat{f}(n) * \hat{g}(n))$, but the convergence of the series in X is no granted in the general case, due to the fact that $\sum_{n=0}^N \hat{f}(n)e^{int}$ does not need to converge in norm to $f \in H^1$.

Nevertheless, the series above is always summable in the sense of Abel: for every $0 < s < 1$ the series $\sum_{n\geq 1} s^n T_n(\hat{f}(n) * \hat{g}(n))$ is convergent, and its sum $x(s)$ converges in norm to $U_h(f,g)$ as $s \to 1$.

To see this, note that $x(s) = U_h(f_s, g)$ with $f_s(e^{i\theta}) = f(se^{i\theta})$, because f_s is the limit of the functions $e^{i\theta} \to \sum_{n=0}^N \hat{f}_s(n)e^{in\theta} = \sum_{n=0}^N s^n \hat{f}(n)e^{in\theta}$, and recall that f_s converges to f in $H^1(L^p)$.

Remark 3. The Theorem 2 for the case $X = \mathbb{C}$ gives that if $f \in H^1(L^p)$ and $g \in H^1(L^q)$ then the convolution $f * g$ belongs to the predual of $\mathcal{B}(L^r)$ whenever $1 \leq p, q \leq 2$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$.

Let e_n denote the function in $\mathbb{T}$ given by $e_n(w) = \overline{w}^n$. Note then that, for $\varphi \in L^p$, $\psi \in L^q$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, we have $\langle e_n, \varphi * \psi \rangle = \hat{\varphi}(n)\hat{\psi}(n)$.

Corollary 1

Let $1 \leq p$, $q \leq 2$ such that $\frac{1}{p} + \frac{1}{q} \geq \frac{3}{2}$ and let $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$.

If (α_n) is a sequence of scalars such that $\sum_{n=1}^N n^r |\alpha_n|^r = O(N^r)$, then there exists a constant C that verifies

$$\sum_{n\geq 1} |\alpha_n \hat{\varphi}_n(n) \hat{\psi}_n(n)| \leq C ||f||_{H^1(L^p)} ||g||_{H^1(L^q)}$$

for any pair of polynomials $f(z) = \sum \varphi_n z^n \in \mathcal{P}(L^p)$ and $f(z) = \sum \psi_n z^n \in \mathcal{P}(L^q)$.

Proof. The result will follow directly from Theorem 2 as soon as we show that the function $\sum_{n\geq 1} \alpha_n e_n z^n$ is in $\mathcal{B}(L^{r'})$. We have to see that

$$\left\| \sum_n n\alpha_n z^n e_n \right\|_{L^{r'}} \leq C(1 - |z|)^{-1} \quad \text{if} \quad |z| < 1.$$

Note that the assumption gives that $1 \leq r \leq 2$ and then Hausdorff-Young's theorem implies that

$$\left\| \sum_{n\geq 1} n\alpha_n z^n e_n \right\|_{L^{r'}} \leq C\left(\sum_{n\geq 1} n^r |\alpha_n|^r |z|^{rn} \right)^{1/r}.$$

Observe that on one hand

$$\frac{1}{1-|z|^r} \sum_{n\geq 1} n^r |\alpha_n|^r |z|^{rn} = \sum_{N\geq 1} \left(\sum_{n\geq 1}^{N} n^r |\alpha_n|^r \right) |z|^{rN} \leq C \sum_{N\geq 1} N^r |z|^{rN}.$$

On the other hand, from Stirling's formula one easily gets the following estimate

$$\sum_{n\geq 1} n^\alpha s^n \leq C\Big(\frac{1}{1-s}\Big)^{\alpha+1}$$

for any $0 < s < 1$ and $\alpha > -1$.

Combining both estimates we have then

$$\sum_{n\geq 1} n^r |\alpha_n|^r |z|^{rn} \leq C\Big(\frac{1}{1-|z|}\Big)^{r},$$

resulting the required inequality. □

Corollary 2

Let $1 \leq p_1, p_2 \leq 2$ and $1 \leq p_3 \leq \infty$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} \geq 2$ and let $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} - 2$. Let $f_k \in L^{p_1}$, $g_k \in L^{p_2}$ and $h_k \in L^{p_3}$ for $1 \leq k \leq n$. Then there exists a constant C that verifies

$$\left\| \sum_{k=1}^{n} h_k * f_k * g_k \right\|_r \leq C||f||_{H^1(L^{p_1})}||g||_{H^1(L^{p_2})}||h||_{\mathcal{B}(L^{p_3})}$$

where $f(z) = \sum_{k=1}^n f_k z^k$, $g(z) = \sum_{k=1}^n g_k z^k$ and $h(z) = \sum_{k=1}^n h_k z^k$.

Proof. Take $1 \leq s \leq \infty$ such that $\frac{1}{s} = \frac{1}{p_1} + \frac{1}{p_2} - 1$ and look at the function $h(z) \in L^{p_3}$ as the operator from L^s into L^r given by the convolution $h(z)(\phi) = h(z) * \phi$ which, from Young's inequality, has norm bounded by $||h(z)||_{p_3}$. □

An application

In the case when $q > 1$ and $X = \mathbb{C}$, the bilinear map considered in Remark 1 can be regarded as a bounded operator $H^1(L^p) \to BMOA(L^{q'})$, with q' the conjugate exponent of q. By means of the results in the preliminaries, we derive the next application of Theorem 2:

Theorem 3

There exists an absolute constant C such that, for any $1 \leq p \leq 2$ and $\frac{2p}{3p-2} \leq q \leq \frac{p}{p-1}$ we have, if $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$, that

$$\left\| \Big(\sum_{n=1}^{\infty} |g_n * f_n|^2 \Big)^{1/2} \right\|_{L^r} \leq C \sup_n \left\{ ||g_n||_q \right\} \left\| \Big(\sum_{n=1}^{\infty} |f_n|^2 \Big)^{1/2} \right\|_{L^p}$$

for every two sequences $(f_n) \subset L^p$ and $(g_n) \subset L^q$.

Proof. Note that p and q are required so that $2 \leq r \leq \infty$. Assume first that $r < \infty$, which corresponds to $q < p/(p-1)$. By the monotone convergence theorem, we just have to show the result for finite sequences $(f_n)_{1 \leq n \leq N}$, provided that the constant does not depend on N. Given such a sequence, let G and F denote the polynomials which take respectively the values $\sum_{n \leq N} g_n z^{2^n} \in L^q$ and $\sum_{n \leq N} f_n z^{2^n} \in L^p$. Recall then that $||G||_{\mathcal{B}(L^q)} \sim \sup_n \{||g_n||_q\}$. On the other hand

$$\begin{aligned} ||F||_{H^1(L^p)} \sim ||F||_{H^p(L^p)} &= \left(\int_{\mathbb{T}} \Big\| \sum_n f_n e^{i 2^n t} \Big\|_p^p d\sigma(e^{it}) \right)^{1/p} \\ &= \left(\int_{\mathbb{T}} \int_{\mathbb{T}} \Big| \sum_n f_n(e^{i\theta}) e^{i 2^n t} \Big|^p d\sigma(e^{i\theta}) d\sigma(e^{it}) \right)^{1/p} \\ &= \left(\int_{\mathbb{T}} \int_{\mathbb{T}} \Big| \sum_n f_n(e^{i\theta}) e^{i 2^n t} \Big|^p d\sigma(e^{it}) d\sigma(e^{i\theta}) \right)^{1/p}, \end{aligned}$$

and using the Khintchine inequality we get

$$||F||_{H^1(L^p)} \sim \left(\int_{\mathbb{T}} \Big(\sum_n |f_n|^2 \Big)^{p/2} d\sigma \right)^{1/p}.$$

Now let $K(z)=\sum_{n=1}^{N}(g_n * f_n)z^{2^n}$, with values in L^r. The same as with F, we see that

$$||K||_{BMOA(L^r)} \sim \left(\int_{\mathbb{T}}\Big(\sum_n |g_n * f_n|^2\Big)^{r/2} d\sigma\right)^{1/r}.$$

But, since the conjugate exponent of r verifies $1 < r' \leq 2$, we can identify the space $BMOA(L^r)$ with the dual space of $H^1(L^{r'})$ with the pairing indicated in the preliminaries; therefore we have

$$||K||_{BMOA(L^r)} = \sup\left\{|\langle K,\Phi\rangle|; \Phi\in\mathcal{P}(L^{r'}), ||\Phi||_{H^1(L^{r'})}\leq 1\right\}.$$

When Φ has coefficients (φ_n), the integral $\langle K,\Phi\rangle$ takes the value

$$\sum_{n\leq N}\langle\, g_n * f_n, \varphi_{2^n}\rangle,$$

and this is the same as

$$\sum_{n\leq N}\langle\, g_n, f_n * \varphi_{2^n}\rangle.$$

Then since $\frac{1}{q'}=\frac{1}{p}+\frac{1}{r'}-1$ we can apply Theorem 2 for $h=G\in\mathcal{B}(\mathcal{L}(L^{q'},\mathbb{C}))$, $f=F\in H^1(L^p)$ and $g=\Phi\in H^1(L^{r'})$ and we get

$$||K||_{BMOA(L^r)} \leq C||G||_{\mathcal{B}(L^q)}||F||_{H^1(L^p)},$$

which gives the result in this case.

The remaining case $r=\infty$ is much easier: Now q is the conjugate exponent of p, so $||g_n * f_n||_\infty \leq ||g_n||_q||f_n||_p$ for each n. Then we have

$$\begin{aligned}\left\|\Big(\sum_{n=1}^{\infty}|g_n * f_n|^2\Big)^{1/2}\right\|_\infty &\leq \left(\sum_{n=1}^{\infty}||g_n||_q^2||f_n||_p^2\right)^{1/2}\\ &\leq \sup_n\left\{||g_n||_q\right\}\left(\sum_{n=1}^{\infty}||f_n||_p^2\right)^{1/2}\\ &\leq \sup_n\left\{||g_n||_q\right\}\left\|\Big(\sum_{n=1}^{\infty}|f_n|^2\Big)^{1/2}\right\|_{L^p}\end{aligned}$$

where the last inequality indicates the well-known fact that L^p is a 2-concave space (see [12]). □

Remark 3. If we set $g_n = g \in L^q$ in the statement of Corollary 1, we get—regarding L^q as a subspace of $\mathcal{L}(L^p, L^r)$, by convolution—a special case of a classical theorem of Marcinkiewicz and Zygmund (see [13, 10]), which reads as follows: If $T\colon L^p \to L^r$ is a bounded linear operator, where $0 < p, r < \infty$, then T has an ℓ^2-valued extension, in the sense that

$$\left\| \Big(\sum_n |Tf_n|^2 \Big)^{1/2} \right\|_r \leq C||T|| \left\| \Big(\sum_n |f_n|^2 \Big)^{1/2} \right\|_p$$

for any sequence of functions $f_n \in L^p$, where C depends only on p and r.

However, the standard proof of this statement, via Kahane inequalities, doesn't give a bound for $||(\sum_n |T_n f_n|^2)^{1/2}||_r$ when T_n are uniformly bounded.

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