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A generalization of Eichler's trace formula

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Abstract

Eichler's trace formula for traces of the Brandt-Eichler matrices is proved for arbitrary totally definite orders in central simple algebras of prime index over global fields. A formula for type numbers of such orders is proved as an application.

Introduction

Let R be a Dedekind ring whose quotient field K is global. Recall that an R-order is a ring Λ containing R as a subring, finitely generated as an R-module and such that $A = K \otimes_R \Lambda$ is a finitely dimensional separable K-algebra. If K is a totally real field and A is a totally definite quaternion K-algebra, then Eichler's formula for traces of the Brandt-Eichler matrices relates the number of (left) principal ideals with given norm in orders representing different isomorphism classes in the genus of Λ to the class numbers of some maximal commutative suborders of Λ and the numbers of orbits for the action of unit groups in the completions of Λ on the completions of these suborders.

By a totally definite order, we mean an R-order Λ such that the quotient of the unit groups Λ^*/R^* is finite. We consider a generalization of Eichler's trace formula to totally definite orders in arbitrary central simple algebras of prime index over K. Even if this situation is more general than that considered in [5], the idea of the proof is essentially the same. Our main tool is a combinatorial class number formula proved in [1]. The proof is in principle similar to that given in [10], Thm. 4.12 for Brandt-Eichler matrices corresponding to definite quaternion orders over number fields. See also [7], where a different method is used in the case of quaternion orders. In the second section of the paper, we show how to prove a formula for type numbers of totally definite orders in algebras satisfying the above assumptions.

1. A proof of Eichler's trace formula

In this section, we give a proof of Eichler's formula for traces of the Brandt-Eichler matrices and a closely related "two-sided" variant of them in the case of totally definite orders in central simple algebras of prime index over global fields.

Let R be a Dedekind ring with quotient field K. Assume that K is a global field and $\dim_K A = n^2$. We denote by Tr the reduced trace, and by Nr the reduced norm from A to K.

Recall that I is a locally principal left Λ-ideal in A if $I_\mathfrak{p} = \Lambda_\mathfrak{p}\alpha_\mathfrak{p}$, where $I_\mathfrak{p}$ and $\Lambda_\mathfrak{p}$ are completions of I and Λ at non-zero prime ideals $\mathfrak{p}$ in R and $\alpha_\mathfrak{p} \in A_\mathfrak{p}$, where $A_\mathfrak{p}$ is the completion of A at $\mathfrak{p}$. The right order of I is the R-order $O_r(I) = \{\alpha \in A : I\alpha \subseteq I\}$. Let $I_1 = \Lambda, \ldots, I_h$ represent all isomorphism classes of locally principal left Λ-ideals with corresponding right orders $\Lambda_1 = \Lambda, \ldots, \Lambda_h$. $h = h(\Lambda)$ is called the class number of Λ. Notice that the class number could be defined by means of locally principal right Λ-ideals giving the same value of $h(\Lambda)$.

The number of mutually non-isomorphic orders between $\Lambda_1 = \Lambda, \ldots, \Lambda_h$ is called the type number of Λ. It will be denoted by $t(\Lambda)$ or t. $t(\Lambda)$ is also the number of mutually non-isomorphic orders in the genus of Λ, since the genus of Λ is exactly the set of all right orders of the locally principal left Λ-ideals in A. Recall that an R-order Λ' belongs to the genus of Λ if for each non-zero prime ideal $\mathfrak{p}$ in R the completions $\Lambda_\mathfrak{p}$ and $\Lambda'_\mathfrak{p}$ are isomorphic as orders over the completion $R_\mathfrak{p}$ of R at $\mathfrak{p}$.

Assume that $\Lambda_1 = \Lambda, \ldots, \Lambda_t$ represent all isomorphism classes in the genus of Λ. Let $H(\Lambda_k)$ be the class number of locally principal two-sided Λ_k-ideals modulo principal two-sided Λ_k-ideals. It is easy to prove that each Λ_k is isomorphic to $H(\Lambda_k)$ orders between $\Lambda_1 = \Lambda, \ldots, \Lambda_h$. Thus

$$h(\Lambda) = \sum_{k=1}^{t(\Lambda)} H(\Lambda_k) \tag{1.1}$$

(see e.g. [11], p. 88).

If I is a Λ-ideal in A, then its norm $\mathrm{Nr}(I)$ is the R-ideal generated by the reduced norms of the elements of I. Let $\mathfrak{a}$ be an ideal in R, and let $\iota(\Lambda, \mathfrak{a})$ be the number of principal left ideals in Λ whose norm is equal to $\mathfrak{a}$. We write $\iota^*(\Lambda, \mathfrak{a})$

to denote the number of two-sided principal ideals in Λ with norm $\mathfrak{a}$. Both these numbers are finite, since K is a global field. They are, of course, equal to 0 when $\mathfrak{a}$ is not a principal ideal. Our objective in this section is to compute

$$\mathrm{Tr}_\Lambda(\mathfrak{a}) = \sum_{k=1}^{t(\Lambda)} H(\Lambda_k)\iota(\Lambda_k, \mathfrak{a}), \tag{1.2}$$

and

$$\mathrm{Tr}^*_\Lambda(\mathfrak{a}) = \sum_{k=1}^{t(\Lambda)} H(\Lambda_k)\iota^*(\Lambda_k, \mathfrak{a}). \tag{1.3}$$

The notations are motivated by the fact that the first of these sums is the trace of the Brandt-Eichler matrix corresponding to Λ and $\mathfrak{a}$. Let us recall that if $\mathfrak{a}$ is an arbitrary ideal in R, then the Brandt-Eichler matrix

$$M_\Lambda(\mathfrak{a}) = [\rho_{kl}(\mathfrak{a})]$$

has as its elements $\rho_{kl}(\mathfrak{a})$ the numbers of locally principal left ideals in Λ_k, which have norm $\mathfrak{a}$ and are isomorphic to $I_k I_l^{-1}$ for $k, l = 1, \ldots, h(\Lambda)$.

If $\mathfrak{a} = Ra$, we shall also use notations $\mathrm{Tr}_\Lambda(a)$ and $\iota(\Lambda, a)$ instead of $\mathrm{Tr}_\Lambda(Ra)$ and $\iota(\Lambda, Ra)$ with the same conventions for Tr^* and ι^*.

Eichler's trace formula depends on finiteness assumptions on A, which we impose from now on. A will be called totally definite if for each R-order Λ in A, the group Λ^*/R^* is finite, where Λ^* and R^* denote the groups of units in Λ and R. It is not difficult to prove that if there is one order in A having this finiteness property, then A is totally definite. In fact, A is totally definite if and only if A does not satisfy the Eichler condition (see [3], p. 718 for the definition of the Eichler condition, and [5], Satz 2, for a proof of the equivalence, which generalizes from the case of quaternion algebras considered there to the general case).

Let

$$\mathcal{N}(\Lambda, a) = \{\lambda \in \Lambda : \mathrm{Nr}(\lambda) = a\}$$

and

$$\mathcal{N}^*(\Lambda, a) = \{\lambda \in \Lambda : \mathrm{Nr}(\lambda) = a \text{ and } \lambda\Lambda = \Lambda\lambda\},$$

for $a \in R$. It is not difficult to prove that for a totally definite A these sets are finite.

Denoting by $|X|$ the number of elements in a finite set X, we have the following equality (see [5], p. 143 (52) and [11], p. 144):

Lemma 1.4

$u_n|\Lambda^*/R^*|\iota(\Lambda, a) = \sum_{l=1}^{m} |\mathcal{N}(\Lambda, a_l)|$, where $a_1 = a, \ldots, a_m$ represent all orbits for the action by multiplication of R^{*n} on the set R^*a and $u_n = |\{x \in R^* : x^n = 1\}|$. Similar equality holds when ι and $\mathcal{N}$ are replaced by ι^* and $\mathcal{N}^*$.

Proof. Let $\Lambda\alpha_1, \dots, \Lambda\alpha_r$ be all principal left ideals in Λ with norm equal to Ra. Let $\lambda_1, \dots, \lambda_s$ represent the elements of Λ^*/R^*. The number of $\lambda_i\alpha_j$ equals $|\Lambda^*/R^*|\iota(\Lambda, a)$. We shall express this number in a different way.

Consider all solutions $x \in \Lambda$ to $\mathrm{Nr}(x) = a_l$ for $l = 1, \dots, m$ (if such exist). If x is a solution, then $\mathrm{Nr}(\Lambda x) = Ra$, so $\Lambda x = \Lambda\alpha_j$, that is, $x = \lambda\alpha_j$, where $\lambda \in \Lambda^*$. Hence $x = \varepsilon\lambda_i\alpha_j$, where $\varepsilon \in R^*$. Conversely, for every pair (i, j) there is $\varepsilon \in R^*$ such that $\varepsilon\lambda_i\alpha_j$ is a solution to $\mathrm{Nr}(x) = a_l$ for a suitable l. In fact, $\mathrm{Nr}(\Lambda\lambda_i\alpha_j) = Ra$ implies that $\mathrm{Nr}(\lambda_i\alpha_j) = \varepsilon_0^n a_l$ for a unique l and some $\varepsilon_0 \in R^*$. Thus $\varepsilon_0^{-1}\lambda_i\alpha_j$ is a solution to $\mathrm{Nr}(x) = a_l$. The number of different solutions x to $\mathrm{Nr}(x) = a_l$ with fixed i and j is u_n. Now split all $\lambda_i\alpha_j$ into classes belonging to different a_l for $l = 1, \dots, m$. The number of $\lambda_i\alpha_j$ in the class of a_l is $(1/u_n)\iota(\Lambda, a_l)$. Summing over l, we get the required equality. The same arguments give the last statement when at the beginning of the proof the left principal ideals are replaced by the two-sided. □

Now let $\mathcal{P}(\Lambda, a)$ be the set of the minimal polynomials for all $\lambda \in \mathcal{N}(\Lambda, a)$. $\mathcal{P}^*(\Lambda, a)$ is defined similarly for $\mathcal{N}$ replaced by $\mathcal{N}^*$. If f is the minimal polynomial of $\lambda \in \mathcal{N}(\Lambda, a)$, let $S_f = R[X]/(f)$ and $L_f = S_f \otimes_R K$. Let x_f be the residue class of X in S_f. Let $S_f \subseteq S \subset L_f$, where S is an R-order, and let $E(S, \Lambda)$ be the number of optimal embeddings $\varphi : S \to \Lambda$, that is, R-algebra homomorphisms such that $\Lambda/\varphi(S)$ is R-projective. Let $E^*(S, \Lambda)$ be the number of all optimal embeddings $\varphi : S \to \Lambda$ such that $\varphi(x_f)\Lambda = \Lambda\varphi(x_f)$. $e(S, \Lambda)$ and $e^*(S, \Lambda)$ will denote the corresponding numbers of orbits for the action by conjugation of Λ^* on the optimal embeddings φ. We shall also need the following simple equality due to M. Eichler (see [5], p. 143 and [2], (4.1)):

Lemma 1.5

$|\mathcal{N}(\Lambda, a)| = \sum_S E(S, \Lambda) = \delta_a u_n + \sum_{S \neq R} E(S, \Lambda)$, *where the sums are over all* S *such that* $S_f \subseteq S \subset L_f$ *for* $f \in \mathcal{P}(\Lambda, a)$ *and* $\delta_a = 0$ *or* 1 *depending on whether* a *belongs to* R^n *or not. Similar equality holds when* $\mathcal{N}$, E *and* $\mathcal{P}$ *are replaced by* $\mathcal{N}^*$, E^* *and* $\mathcal{P}^*$.

Finally, we need a formula proved in [1], (2.2), and a "commutative" version of it:

Theorem 1.6

Let $\Lambda_1 = \Lambda, \dots, \Lambda_t$ *represent the isomorphisms classes in the genus of* Λ. *If* S *is a maximal commutative suborder of* Λ, *then*

$$\sum_{k=1}^{t} H(\Lambda_k) e(S, \Lambda_k) = h(S) e_{U(\Lambda)}(S, \Lambda),$$

where $h(S)$ is the class number of the locally principal ideals in S and $e_{U(\Lambda)}(S,\Lambda) = \prod_{\mathfrak{p}} e(S_{\mathfrak{p}},\Lambda_{\mathfrak{p}})$, $\mathfrak{p} \in SpecR$, $\mathfrak{p} \neq (0)$. The same equality holds when e is replaced by e^* for $S = R[s]$ with images of s generating two-sided ideals.

Proof. Only the last statement needs a proof, which is an easy modification of the arguments in the proof of the first statement in [1], (2.1). Let, as in [1], (2.1), $S = R[s]$ be a maximal commutative subring of Λ. The relation $\mathcal{R}$ in [1], (2.1), is modified in such a way that the optimal embeddings $\varphi_{\mathfrak{p}} : S_{\mathfrak{p}} \to \Lambda'_{\mathfrak{p}}$, where Λ' is an R-order in the genus of Λ satisfy $\varphi_{\mathfrak{p}}(s)\Lambda'_{\mathfrak{p}} = \Lambda'_{\mathfrak{p}}\varphi_{\mathfrak{p}}(s)$. It is easy to check, following the details in [1], p. 204, that this context leads to the formula above with e replaced by e^*. □

Now we are ready to prove Eichler's trace formula:

Theorem 1.7

Let Λ be an R-order in a central simple totally definite K-algebra A of prime index p. If $\Lambda_1 = \Lambda, \ldots, \Lambda_t$ represent all isomorphism classes in the genus of Λ and $a \in R$, then

$$\mathrm{Tr}_{\Lambda}(a) = \delta_a M_{\Lambda} + \frac{1}{u_p} \sum_{S \neq R} \frac{1}{|S^*/R^*|} h(S) e_{U(\Lambda)}(S,\Lambda),$$

where the sum is over all S such that $S_f \subseteq S \subset L_f$ for $f \in \bigcup_{k=1}^{m} \mathcal{P}(\Lambda, a_k)$ (the notations as in (1.4) and (1.5)), and

$$M_{\Lambda} = \sum_{k=1}^{t} \frac{H(\Lambda_k)}{|\Lambda_k^*/R^*|}.$$

Similar result holds when Tr, e and $\mathcal{P}$ are replaced by Tr^*, e^* and $\mathcal{P}^*$.

Proof. First of all, notice that

$$E(S,\Lambda) = \frac{[\Lambda^*/R^*]}{[S^*/R^*]} e(S,\Lambda). \tag{1.8}$$

In fact, if φ is any optimal embedding of S into Λ, then its orbit under the action of Λ^* contains exactly $[\Lambda^* : \varphi(S)^*]$ elements. This is clear, since the isotropy group of φ is equal to the intersection of Λ^* with $K\varphi(S)$ (this is the only place in the paper where we use the assumption about the index of A: $K\varphi(S)$ is a maximal commutative subring of A). We have $\Lambda^* \cap K\varphi(S) = \varphi(S)^*$, since the embedding is optimal. Finiteness of $|\Lambda^*/R^*|$ implies that the index $[\Lambda^* : \varphi(S)^*]$ is equal to $|\Lambda^*/R^*|/|S^*/R^*|$. Using (1.4) – (1.6) and (1.8), we get:

$$\begin{aligned}
\mathrm{Tr}_\Lambda(a) &= \sum_{k=1}^{t} H(\Lambda_k)\iota(\Lambda_k, a) = \sum_{k=1}^{t} H(\Lambda_k)\frac{1}{u_p|\Lambda_k^*/R^*|}\sum_{l=1}^{m}|\mathcal{N}(\Lambda_k, a_l)| \\
&= \delta_a \sum_{k=1}^{t}\frac{H(\Lambda_k)}{|\Lambda_k^*/R^*|} + \sum_{k=1}^{t} H(\Lambda_k)\frac{1}{u_p|\Lambda_k^*/R^*|}\sum_{S\neq R} E(S,\Lambda_k) \\
&= \delta_a M_\Lambda + \sum_{k=1}^{t} H(\Lambda_k)\frac{1}{u_p|\Lambda_k^*/R^*|}\sum_{S\neq R}\frac{[\Lambda_k^*/R^*]}{[S^*/R^*]}e(S,\Lambda_k) \\
&= \delta_a M_\Lambda + \frac{1}{u_p}\sum_{S\neq R}\frac{1}{|S^*/R^*|}\sum_{k=1}^{t} H(\Lambda_k)e(S,\Lambda_k) \\
&= \delta_a M_\Lambda + \frac{1}{u_p}\sum_{S\neq R}\frac{1}{|S^*/R^*|}h(S)e_{U(\Lambda)}(S,\Lambda).
\end{aligned}$$

Similar computations give the result for $\mathrm{Tr}^*_\Lambda(a)$ using the corresponding commutative instances of (1.4) – (1.6) and (1.8). □

Remark 1.9. (a) In many cases the traces (1.2) and (1.3) are known explicitly. The formulae were obtained in [5] for hereditary orders in quaternion algebras, and generalized afterwards to different classes of orders in such algebras in [6], [7], [10] and [11]. The embedding numbers are explicitly known for all quaternion orders whose Gorenstein closure is a Bass order as well as for arbitrary hereditary orders (see [2], [6] and [4] in the case of function fields).

(b) Eichler's trace formula applied for $a = 1$ gives an expression of the class number of Λ (see (1.1) and (1.2)). Of course, it is necessary to know both the weight of the genus M_Λ as well as all other terms on the right hand side in order to carry out the computations. An explicit formula in the case of all definite quaternion orders over the integers is given in [2].

2. Type numbers of totally definite orders

The purpose of this section is to prove a formula for type numbers of orders in definite division algebras of prime index over global fields. This formula is well-known (at least in the case of index 2) and was proved by Eichler in [5]. Some inaccuracies of the proof given in [5] were corrected by Peters [8] and Pizer [9]. The first proof of the formula for arbitrary quaternion orders over number fields was given by Körner [7], Thm. 3. Even if the method in [7] is completely different, the idea of the final expression for $t(\Lambda)$ comes from that paper. Our method closely follows the original proof by Eichler in combination with the "commutative" version of (1.7).

First we need a simple combinatorial result for which we fix the following notations. Let $h(R)$ be the class number of R and let $\mathfrak{a}_1 = R, \dots, \mathfrak{a}_{h(R)}$ represent all classes of ideals in R. Let $I_1, \dots, I_{H(\Lambda)}$ represent all classes of locally principal two-sided Λ-ideals modulo principal two-sided Λ-ideals. Let $J_1, \dots, J_{p(\Lambda)}$ represent all classes of locally principal two-sided Λ-ideals modulo $\mathfrak{a}\Lambda$ for R-ideals $\mathfrak{a}$. Notice that $p(\Lambda)$ only depends on the genus of Λ. Finally, let $\Lambda\alpha_1, \dots, \Lambda\alpha_{m(\Lambda)}$ represent all classes of the two-sided principal Λ-ideals modulo the ideals Λa, where $a \in K^*$.

Lemma 2.1

$$h(R)p(\Lambda) = H(\Lambda)m(\Lambda)$$

Proof. We compute in two different ways the number of classes of the two-sided Λ-ideals modulo the ideals $a\Lambda$, where $a \in K^*$.

If I is a two-sided Λ-ideal, then $I = J_k\mathfrak{a} = J_k\mathfrak{a}_l a$, where $a \in K^*$, with unique $k \in \{1, \dots, p(\Lambda)\}$ and $l \in \{1, \dots, h(R)\}$. Thus the number, we are looking for is $h(R)p(\Lambda)$.

On the other side, $I = I_r\Lambda\alpha = I_r\Lambda\alpha_s b$, where $b \in K^*$, with unique $r \in \{1, \dots, h(\Lambda)\}$ and $s \in \{1, \dots, m(\Lambda)\}$. Therefore, the second expression for the required number is $h(\Lambda)m(\Lambda)$. $\square$

The sum

$$\sum_{k=1}^{t(\Lambda)} H(\Lambda_k)m(\Lambda_k) = t(\Lambda)h(R)p(\Lambda)$$

can be computed using the trace formula. Each of the $m(\Lambda)$ classes can be represented by a unique ideal, which can be defined in the following way. Let $\Lambda\alpha$ be a two-sided ideal. Consider

$$\mathfrak{a}_0 = \{a \in K : a\Lambda\alpha \subseteq \Lambda\}.$$

It is clear that $\mathfrak{a}_0\Lambda\alpha$ only depends on the class of $\Lambda\alpha$ modulo the ideals Λa, where $a \in K^*$. Notice that $\mathfrak{a}_0\Lambda\alpha$ is a primitive ideal in Λ (not necessarily principal), when an ideal I in Λ is called primitive if $\mathfrak{a}^{-1}I \subseteq \Lambda$ for an ideal $\mathfrak{a}$ in R implies $\mathfrak{a} = R$. Now observe that $\Lambda\alpha = (\mathfrak{a}_0\Lambda\alpha)\mathfrak{a}_0^{-1} = (\mathfrak{a}_0\Lambda\alpha)\mathfrak{a}_l a$ with a unique $\mathfrak{a}_l$. Thus $\mathfrak{a}_0\Lambda\alpha\mathfrak{a}_l$

is a two-sided principal ideal in Λ, which is uniquely determined by its class. For all $m(\Lambda)$ classes, consider now the norms $\mathrm{Nr}(\mathfrak{a}_0\Lambda\alpha\mathfrak{a}_l)$ and choose between them all non-equal, say, $Rb_1, \ldots, Rb_w$. With these notations, we have

Theorem 2.2

Let Λ be an R-order in a central simple K-algebra A satisfying the assumptions of (1.7). Then

$$t(\Lambda) = \frac{1}{h(R)p(\Lambda)} \sum_{i=1}^{w} \mathrm{Tr}^*_\Lambda(b_i),$$

where $h(R)$ is the class number of R, $p(\Lambda)$ is the class number of the two-sided locally free Λ-ideals modulo $\Lambda\mathfrak{a}$ for R-ideals $\mathfrak{a}$, and $Rb_1, \ldots, Rb_w$ are all different products between $\mathrm{Nr}(I)\mathfrak{a}_l^p$, where I runs over all two-sided primitive Λ-ideals and $\mathfrak{a}_1 = R, \ldots, \mathfrak{a}_h$ represent all ideal classes of R. $\mathrm{Tr}^*_\Lambda(b_i)$ can be computed according to (1.7).

Proof. We have,

$$t(\Lambda) = \frac{1}{h(R)p(\Lambda)} \sum_{k=1}^{t(\Lambda)} m(\Lambda_k)H(\Lambda_k).$$

Since each of $m(\Lambda_k)$ classes of the two-sided principal ideals in Λ_k contains exactly one ideal $I\mathfrak{a}_l$, where I is primitive, and Rb_i are all non-equal norms of such products, we get

$$m(\Lambda_k) = \sum_{i=1}^{w} \iota^*(\Lambda_k, b_i).$$

Therefore,

$$t(\Lambda) = \frac{1}{h(R)p(\Lambda)} \sum_{k=1}^{t(\Lambda)} \sum_{i=1}^{w} \iota^*(\Lambda_k, b_i)H(\Lambda_k) = \frac{1}{h(R)p(\Lambda)} \sum_{i=1}^{w} \mathrm{Tr}^*_\Lambda(b_i). \square$$

Remark 2.3. Notice that in the case of hereditary orders over the integers (or more generally, over principal ideal rings), the trace Tr^* can be replaced by Tr. In fact, each class of principal two-sided Λ-ideals contains then a unique ideal in Λ whose norm is square-free as an ideal in R. Choosing such ideals as representants of classes, we get in 2.2 the traces corresponding to square-free ideals Rb_i. But a one-sided locally principal ideal with a square-free norm in a hereditary order must be two-sided, which gives the equality of Tr and Tr^* in this case.

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