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Analytic properties of the spectrum in Banach Jordan systems

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Abstract

For Banach Jordan algebras and pairs the spectrum is proved to be related to the spectrum in a Banach algebra. Consequently, it is an analytic multifunction, upper semicontinuous with a dense G_δ-set of points of continuity, and the *scarcity theorem* holds.

0. Introduction

In this paper we consider Banach Jordan algebras and pairs, i.e. complex Jordan algebras resp. pairs which have complete norms such that quadratic operators and squares are continuous (and even analytic). For details see [5]. As for associative Banach algebras the spectrum is an important notion in this more general context. Thus it is an obvious question whether its properties, besides being a compact subset of $\mathbb{C}$, are also true in the Jordan case.

In [1, § III.4] Aupetit showed some important analytic and topological properties of spectra in unital Banach algebras by using subharmonic and analytic functions as well as analytic multifunctions. The aim of this paper is the generalization of Aupetit's results to Banach Jordan algebras and pairs. Astonishingly this can be done very easily by using a fact which is partially due to Loos [5]: For an element x of a unital Banach Jordan algebra J or an element (x, y) of a Banach Jordan pair V the spectrum is equal to the spectrum of

$$\begin{pmatrix} 0 & \mathrm{Id}_J \\ -U_x & V_x \end{pmatrix} \quad \text{resp.} \quad \begin{pmatrix} 0 & \mathrm{Id}_{V^+} \\ -Q(x)Q(y) & D(x,y) \end{pmatrix}$$

in the associative Banach algebras $\mathrm{Mat}_2(\mathcal{L}(J))$ resp. $\mathrm{Mat}_2(\mathcal{L}(V^+))$, where $\mathcal{L}$ denotes the set of bounded linear operators. Using the fact that the above matrices depend analytically on x resp. (x, y), it is not necessary to adapt Aupetit's proofs to the Jordan setting for most of the claims.

The main results are:

(1) For a unital Banach Jordan algebra J or a Banach Jordan pair V the mappings $x \mapsto \mathrm{Sp}_J(x)$ resp. $(x, y) \mapsto \mathrm{Sp}_V(x, y)$ are upper semicontinuous. If f is an analytic function from a domain D of $\mathbb{C}$ into J or $V^+ \times V^-$, we get analytic multifunctions $\lambda \mapsto \mathrm{Sp}_J(f(\lambda))$ resp. $\lambda \mapsto \mathrm{Sp}_V(f^+(\lambda), f^-(\lambda))$.

(2) The sets of points of continuity of $x \mapsto \mathrm{Sp}_J(x)$ resp. $(x, y) \mapsto \mathrm{Sp}_V(x, y)$ are dense G_δ-subsets of J resp. $V^+ \times V^-$ containing all elements with totally disconnected spectrum.

(3) *Scarcity of spectrum-finite elements*: For an analytic function f from a domain D of $\mathbb{C}$ into J or $V^+ \times V^-$ either "almost all" $f(\lambda)$ have infinite spectrum or the number of spectral values is equal to a fixed $n \in \mathbb{N}$ for "almost all" $\lambda \in D$ and smaller than n for the rest.

In the sequel R is an arbitrary field. We use the notation Sp' for the union of the spectrum with $\{0\}$, and ρ for the spectral radius. If A is an associative or Jordan algebra, the subset of invertible elements is denoted by $A^\times$. All Banach spaces occurring are over the complex numbers.

1. The quadratic spectrum

DEFINITION 1.1. Let A be a unital associative R-algebra. For commuting elements $u, v \in A$ we define

$$\mathrm{Sp}^2_A(u, v) := \left\{\lambda \in R : \lambda^2 1_A - \lambda v + u \notin A^\times\right\},$$
$$m(u, v) := \begin{pmatrix} 0 & 1 \\ -u & v \end{pmatrix} \in \mathrm{Mat}_2(A) =: B\,.$$

Proposition 1.2

In the situation of 1.1 we have:

$$\lambda^2 1_A - \lambda v + u \in A^\times \Leftrightarrow \lambda 1_B - m(u, v) \in B^\times\,.$$

Proof. "$\Rightarrow$": Let $w := \big(\lambda^2 1_A - \lambda v + u\big)^{-1}$. Since $\lambda^2 1_A - \lambda v + u$ commutes with u and v the same is true for w. Therefore

$$\begin{pmatrix} \lambda 1_A - v & 1 \\ -u & \lambda 1_A \end{pmatrix} w = w \begin{pmatrix} \lambda 1_A - v & 1 \\ -u & \lambda 1_A \end{pmatrix} =: n\,,$$

whence we can easily show that $\big(\lambda 1_B - m(u,v)\big)n = 1_B = n\big(\lambda 1_B - m(u,v)\big)$ and get $\lambda 1_B - m(u,v) \in B^\times$.

"$\Leftarrow$": Let

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} := \big(\lambda 1_B - m(u,v)\big)^{-1}.$$

The equations defining the inverse yield

$$-a + \lambda b - bv = 0\,, \tag{1}$$
$$\lambda a + bu = 1_A\,, \tag{2}$$
$$\lambda b - d = 0\,, \tag{3}$$
$$ub + \lambda d - vd = 1_A\,, \tag{4}$$

and four additional equations. Using (1) and (2) we get

$$b\big(\lambda^2 1_A - \lambda v + u\big) = \lambda\big(\lambda b - bv\big) + bu = \lambda a + bu = 1_A\,,$$

whereas (3) and (4) imply

$$\big(\lambda^2 1_A - \lambda v + u\big)b = ub + \big(\lambda 1_A - v\big)\lambda b = ub + \big(\lambda 1_A - v\big)d = 1_A\,.$$

Thus $\big(\lambda^2 1_A - \lambda v + u\big)$ is invertible (with inverse b). $\square$

Corollary 1.3

$$\mathrm{Sp}_A^2(u,v) = \mathrm{Sp}_B\big(m(u,v)\big)\,.$$

2. Relating the Jordan case to the associative case

Proposition 2.1

Let J be a unital Jordan algebra over R. Consider the associative R-algebra $B := \mathrm{Mat}_2(\mathrm{End}_R(J))$ and for $x \in J$ define

$$m(x) := m(U_x, V_x) = \begin{pmatrix} 0 & \mathrm{Id}_J \\ -U_x & V_x \end{pmatrix} \in B\,.$$

Then $Sp_J(x) = \mathrm{Sp}_B(m(x))$.

Proof. Let $A := \mathrm{End}_R(J), v := V_x = V_{1_J,x} = V_{x,1_J} \in A$ and $u := U_x \in A$. By [4, 1.6.7] $uv = U_x V_{1_J,x} = V_{x,1_J} U_x = vu$, whence we can apply 1.3. Using [4, 3.1.2] we get

$$\begin{aligned}\mathrm{Sp}_J(x) &= \big\{\lambda \in R : U_{\lambda 1_J - x} \text{ is not invertible}\big\} \\ &= \big\{\lambda \in R : \lambda^2 1_A - \lambda v + u \text{ is not invertible}\big\} \\ &= \mathrm{Sp}^2_A(u,v) = \mathrm{Sp}_B(m(x))\,. \ \square\end{aligned}$$

Proposition 2.2

Let V be a Jordan pair over R. Consider the associative R-algebra $B := \mathrm{Mat}_2(\mathrm{End}_R(V^+))$ and for $(x,y) \in V$ define

$$m(x,y) := m\big(Q(x)Q(y),\, D(x,y)\big) = \begin{pmatrix} 0 & \mathrm{Id}_{V^+} \\ -Q(x)Q(y) & D(x,y) \end{pmatrix} \in B\,.$$

Then $\mathrm{Sp}_V(x,y) = \mathrm{Sp}'_B\big(m(x,y)\big)$.

Proof. Let $A := \mathrm{End}_R(V^+), v := D(x,y) \in A$ and $u := Q(x)Q(y) \in A$. By [4, JP 1] $uv = Q(x)Q(y)D(x,y) = Q(x)D(y,x)Q(y) = D(x,y)Q(x)Q(y) = vu$, whence we can apply 1.3. Defining $\mathrm{Sp}_V(x,y)$ as in [5, 1.2.1] we obtain analogously to [5, 1.2.3]

$$\begin{aligned}\mathrm{Sp}_V(x,y) &= \{0\} \cup \big\{\lambda \in R : B(\lambda^{-1}x, y) \text{ is not invertible}\big\} \\ &= \{0\} \cup \big\{\lambda \in R : \lambda^2 B(\lambda^{-1}x, y) \text{ is not invertible}\big\} \\ &= \{0\} \cup \big\{\lambda \in R : \lambda^2 1_A - \lambda v + u \text{ is not invertible}\big\} \\ &= \{0\} \cup \mathrm{Sp}^2_A(u,v) = \{0\} \cup \mathrm{Sp}_B\big(m(x,y)\big) = \mathrm{Sp}'_B\big(m(x,y)\big)\,. \ \square\end{aligned}$$

Theorem 2.3

(a) Let J be a unital Banach Jordan algebra, $x \in J$ and $B := \mathrm{Mat}_2\big(\mathcal{L}(J)\big)$. Then $m(x) \in B$, and

$$\mathrm{Sp}_J(x) = \mathrm{Sp}_B\big(m(x)\big)\,.$$

Recall that $m : J \to B$ is analytic being a continuous polynomial of degree 2.
(b) For a Banach Jordan pair $V, (x,y) \in V$ and $B := \mathrm{Mat}_2\big(\mathcal{L}(V^+)\big)$ we have $m(x,y) \in B$ and

$$\mathrm{Sp}_V(x,y) = \mathrm{Sp}'_B\big(m(x,y)\big)\,.$$

Again, $m : V \to B$ is an analytic function.

Corollary 2.4

Let f be an analytic function from a domain D of $\mathbb{C}$ into J or $V^+ \times V^-$. Then the mappings $\lambda \mapsto \mathrm{Sp}_J\big(f(\lambda)\big)$ resp. $\lambda \mapsto \mathrm{Sp}_V\big(f^+(\lambda), f^-(\lambda)\big)$ are analytic multifunctions.

Remark. This result improves a theorem by Aupetit and Zraïbi [2, Th. 1]. With different arguments it was also proved by Maouche [6, 4.3.8].

Knowing that the spectrum is an analytic multifunction for Jordan pairs, a great number of results of Chapters III, V and VII in [1] can be adapted or used directly. In the following sections we use the above theorem and other techniques for generalizing some of Aupetit's results on analytic properties of the spectrum in Banach algebras to the Banach Jordan context.

3. Continuity of the spectrum

Proposition 3.1

Let J be a unital Banach Jordan algebra, and $x, y \in J$ such that $U_x, U_y, U_{x,y}, V_x$, and V_y commute pairwise. Then

$$\mathrm{Sp}_J(y) \subseteq \mathrm{Sp}_J(x) + \rho_J(x - y)\,.$$

Proof. Let $B := \mathbb{C}[x, y]$ the Jordan subalgebra of J generated by x and y. By [3, 3.2.4] the subalgebra of $\mathrm{End}(J)$ generated by U_B and V_B equals the subalgebra generated by the five commuting operators $U_x, U_y, U_{x,y}, V_x, V_y$ and is therefore commutative. Thus B is strongly associative, whence there exists a closed, full and strongly associative subalgebra $A \subseteq J$ with $A \supseteq B$ [7, I.3 and II.2]. In particular, A is an associative commutative unital Banach algebra and $x, y, x - y$ have the same spectra in A and in J. Applying [1, III.4.1] we get

$$\mathrm{Sp}_J(y) = \mathrm{Sp}_A(y) \subseteq \mathrm{Sp}_A(x) + \rho_A(x - y) = \mathrm{Sp}_J(x) + \rho_J(x - y)\,. \ \square$$

Theorem 3.2

(a) Let J be a unital Banach Jordan algebra. Then the function $x \mapsto \mathrm{Sp}_J(x)$ is upper semicontinuous, i.e. for any open set $W \supseteq \mathrm{Sp}_J(x)$ there exists an open neighborhood Z of x in J such that $\mathrm{Sp}_J(z) \subseteq W$ for all $z \in Z$.
(b) Let V be a Banach Jordan pair. Then the function $(x, y) \mapsto \mathrm{Sp}_V(x, y)$ is upper semicontinuous, i.e. for any open set $W \supseteq \mathrm{Sp}_V(x, y)$ there exists an open neighborhood Z of (x, y) in $V^+ \times V^-$ such that $Sp_V(z, w) \subseteq W$ for all $(z, w) \in Z$.

Proof. (a) Put $A := \mathrm{Mat}_2\big(\mathcal{L}(J)\big)$. By [1, III.4.2] there exists an open neighborhood $\tilde{Z}$ of $m(x)$ in A such that $\mathrm{Sp}_A(a) \subseteq W$ for all $a \in \tilde{Z}$. Since m is analytic by 2.3(a), it is continuous. Thus there exists an open neighborhood Z of x in J such that $m(z) \in \tilde{Z}$ for all $z \in Z$. In particular, an application of 2.3(a) yields

$$\mathrm{Sp}_J(z) = \mathrm{Sp}_A\big(m(z)\big) \subseteq W.$$

(b) The assertion can be shown analogously to (a) by using 2.3(b). □

Theorem 3.3

Let J be a unital Banach Jordan algebra and V a Banach Jordan pair. Then the sets of points of continuity of the functions $x \mapsto \mathrm{Sp}_J(x)$ and $(x,y) \mapsto \mathrm{Sp}_V(x,y)$ are dense G_δ-subsets of J resp. $V^+ \times V^-$.

Proof. We will show the claim for J by following the proof of [1, III.4.3]: Let (f_n) be a dense sequence of elements of the algebra of real continuous functions (together with the topology of uniform convergence on compact subsets). For $x \in J$ we define $F_n(x) := \sup\big(f_n(\mathrm{Sp}_J(x))\big)$. Then $F_n : J \to \mathbb{R}$ is upper semicontinuous as a consequence of 3.2(a). Thus $z \mapsto \mathrm{Sp}_J(z)$ is continuous at $x \in J$ if and only if all F_n are continuous at x (cf. proof of [1, III.4.3]). The rest of the proof is identical to the respective part of the proof of of [1, III.4.3] since this part works for an arbitrary Banach space and does not depend on any algebraic structure. The claim for V can be shown analogously by using 3.2(b). □

Proposition 3.4

Let U, W be disjoint open subsets of $\mathbb{C}$.

(a) For a unital Banach Jordan algebra J and $x \in J$ suppose that $\mathrm{Sp}_J(x) \subseteq U \cup W$ and $\mathrm{Sp}_J(x) \cap W \neq \emptyset$. Then there exists an open neighborhood Z of x in J such that $\mathrm{Sp}_J(z) \cap W \neq \emptyset$ for all $z \in Z$.

(b) For a Banach Jordan pair V and $(x,y) \in V$ suppose that $Sp_V(x,y) \subseteq U \cup W$ and $Sp_V(x,y) \cap W \neq \emptyset$. Then there exists an open neighborhood Z of (x,y) in $V^+ \times V^-$ such that $Sp_V(z,w) \cap W \neq \emptyset$ for all $(z,y) \in Z$.

Proof. (a) Put $A := \mathrm{Mat}_2\big(\mathcal{L}(J)\big)$. Then by 2.3(a) we have $\mathrm{Sp}_A\big(m(x)\big) \subseteq U \cup W$ and $\mathrm{Sp}_A(m(x)) \cap W \neq \emptyset$ whence we may apply [1, III.4.4] and get an open neighborhood $\tilde{Z}$ of $m(x)$ in A such that $\mathrm{Sp}_A(a) \cap W \neq \emptyset$ for all $a \in \tilde{Z}$. By continuity of $m : J \to A$ we can find an open neighborhood Z of x such that $m(z) \in \tilde{Z}$ for all $z \in Z$. Now the claim is true by 2.3(a).

(b) The proof works analogously if we use 2.3(b). □

Theorem 3.5

(a) Let J be a unital Banach Jordan algebra and $x \in J$ with $\mathrm{Sp}_J(x)$ totally disconnected. Then the function $z \mapsto \mathrm{Sp}_J(z)$ is continuous at x.
(b) Let V be a Banach Jordan pair and $(x,y) \in V$ with $Sp_V(x,y)$ totally disconnected. Then the function $(z,w) \mapsto \mathrm{Sp}_V(z,w)$ is continuous at (x,y).

Proof. The first part can be shown analogously to [1, III.4.5] if we use 3.2(a) and 3.4(a), the second part with the help of 3.2(b) resp. 3.4(b). □

4. The scarcity theorem

Definition 4.1. For a compact set $K \subseteq \mathbb{C}$ and $n \in \mathbb{N}$ the *n-th diameter of K* is defined by

$$\delta_n[K] := \max\left(\prod_{1\le i<j\le n+1} |\lambda_i - \lambda_j|^{2/(n^2+n)} : \lambda_1, \ldots, \lambda_{n+1} \in K\right).$$

Note that $\delta_n[K] \geq \delta_{n+1}[K]$ and that $\delta_1[K] = \max\big(|\lambda_1 - \lambda_2| : \lambda_1, \lambda_2 \in K\big)$ is the classical diameter of K.

Definition 4.2. For $X \subset \mathbb{C}$ we define the *capacity of X* as in [1, § A.1]:

(a) If $X = K$ is a compact set, then $c(K) := \lim_{n\to\infty} \delta_n[K] < \infty$, which is well-defined by 4.1.

(b) If $X = U$ is an open set, then $c(U) := \sup\big(c(K) : K \subseteq U \text{ compact}\big) \leq \infty$.

(c) If X is arbitrary, then $c(U) := \inf\big(c(U) : U \supseteq X \text{ open}\big) \leq \infty$.

Recall that sets having zero capacity are totally disconnected [1, A.1.28].

Theorem 4.3

Let J be a unital Banach Jordan algebra, V a Banach Jordan pair, and f an analytic function from a domain D of $\mathbb{C}$ into J or $V^+ \times V^-$. Put $F := \big\{\lambda \in D : \sharp\big(\mathrm{Sp}(f(\lambda))\big) < \infty\big\}$. Then either F is a Borel set with $c(F) = 0$ or there exists $n \in \mathbb{N}$ and a closed discrete subset E of D such that

$$\sharp\big(\mathrm{Sp}(f(\lambda))\big) \begin{Bmatrix} = n & for & \lambda \in D\backslash E \\ < n & for & \lambda \in E \end{Bmatrix}.$$

In this case we get analytic functions $h_1, \ldots, h_n : D\backslash E \to \mathbb{C}$ with $\mathrm{Sp}\big(f(\lambda)\big) = \big\{h_1(\lambda), \ldots, h_n(\lambda)\big\}$.

Proof. The claim is true by 2.3 and [1, III.4.25]. □

Remark. Analogously to the above proof most of the results of [1, § III.4] can easily be adapted to the Banach Jordan context.

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