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Left and right on locally compact groups

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Abstract

Let G be a locally compact, non–compact group and f a function defined on G; we prove that, if f is uniformly continuous with respect to the left (right) structure on G and with a power integrable with respect to the left (right) Haar measure on G, then f must vanish at infinity. We prove that left and right cannot be mixed.

0. Introduction

A function $f : \mathbb{R} \to \mathbb{R}$, Riemann–integrable on every bounded interval, is absolutely integrable on $\mathbb{R}$ if f tends to zero, for $|x| \to \infty$, sufficiently fast (convergence tests based on infinitesimal order). On the other hand, the condition "f infinitesimal" is not necessary for the absolute convergence of the integral: see, for instance, the function

$$f(x) = \begin{cases} \exp\left[-\frac{(x-n)^2}{\left(|x-n|-\frac{1}{2n^2}\right)^2}\right] & \text{if } \ n-\frac{1}{2n^2} < x < n+\frac{1}{2n^2} \quad n \in \mathbb{N}\backslash\{0\} \\ 0 & \text{otherwise.} \end{cases}$$

We notice that this function f is not uniformly continuous; indeed, it is not hard to prove that a uniformly continuous function f, with a power absolutely integrable, must be infinitesimal.

In the paper we prove this result for function defined in an arbitrary locally compact group G (Theorem 1); this gives a partial converse of the Cantor–Heine theorem "a continuous function of a compact group is uniformly continuous" ([2], 4.16). If G is not unimodular, we have to distinguish between left or right Haar measure on G and left or right uniform structure on G: "to vanish at infinity" is necessary condition for functions which are left uniformly continuous and with a power absolutely integrable w.r. to the left Haar measure (the same for right uniformly continuity and right Haar measure), but it is not possible to mix left and right (this is proved in theorem 2).

1. Uniformly continuous and integrable functions on locally compact groups

We recall some definitions, results and notations about locally compact group (the main reference is [2]).

If G is a topological locally compact group, there exists a nonnegative, extended real valued, regular measure μ defined on the family $\mathcal{B}$ of Borel sets of G, which is left invariant (i.e. $\mu(gB) = \mu(B) \ \ \forall B \in \mathcal{B}\ , \ \ \forall g \in G$); μ is unique up to a multiplicative constant and is called a left Haar measure. Similarly, there exists a right Haar measure on G (i.e. $\mu(Bg) = \mu(B)$), denoted by ν.

If G is a locally compact group, μ (ν) is a fixed left (right) Haar measure on G and $1 \le p < +\infty$, we consider the functional spaces $\mathcal{L}_p(G,\mu)$ ($\mathcal{L}_p(G,\nu)$) of the (equivalence classes of) Borel measurable complex functions on G, whose p-powers are absolutely integrable w.r. to μ (ν):

$$f \in \mathcal{L}_p(G,\mu) \quad \Longleftrightarrow \quad \|f\|_p = \left(\int_G |f(g)|^p d\mu(g)\right)^{1/p} < +\infty\,.$$

If G and H are topological groups and $\mathcal{U}$ and $\mathcal{V}$, respectively, open bases at the identities of G and H, then a mapping $\varphi : G \to H$ is called left/left uniformly continuous if, for every $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ s.t.

$$x^{-1}y \in U \quad \Rightarrow \quad (\varphi(x))^{-1}\varphi(y) \in V$$

similarly, we have definitions of right/right, left/right and right/left uniform continuity:

$$\begin{aligned} yx^{-1} \in U \quad &\Rightarrow \quad \varphi(y)(\varphi(x))^{-1} \in V \\ x^{-1}y \in U \quad &\Rightarrow \quad \varphi(y)(\varphi(x))^{-1} \in V \\ yx^{-1} \in U \quad &\Rightarrow \quad (\varphi(x))^{-1}\varphi(y) \in V\,. \end{aligned}$$

We denote by $\mathcal{C}^l(G)$ (respectively, $\mathcal{C}^r(G)$) the space of complex functions on G, left (resp., right) uniformly continuous (obviously, if H is abelian, left/left and left/right is the same).

Lastly, we denote by $\mathcal{C}_0(G)$ the space of complex continuous functions on G, which vanish at infinity, i.e.

$$f \in \mathcal{C}_0(G) \quad \Longleftrightarrow \quad \forall \varepsilon > 0 \;\; \exists K \subset G\,, \quad K \text{ compact}, \quad \text{such that } \; |f(x)| < \varepsilon \;\; \forall x \notin K\,.$$

Our result is

Theorem 1

Let G be a locally compact group, μ (respectively, ν) a left (right) Haar measure on G and p such that $1 \le p < +\infty$. Then, with the previous notations,

$$\begin{aligned} f \in \mathcal{C}^l(G) \cap \mathcal{L}_p(G,\mu) \quad &\Rightarrow \quad f \in \mathcal{C}_0(G) \\ (\textit{resp.}, \quad f \in \mathcal{C}^r(G) \cap \mathcal{L}_p(G,\nu) \quad &\Rightarrow \quad f \in \mathcal{C}_0(G) \quad). \end{aligned}$$

Proof. Obviously, G is supposed non-compact. We prove the left case (the adjustments for the right case are straightforward). Without loss of generality, we assume $f \ge 0$.

Absurdly, we suppose that

$$\exists \varepsilon > 0 \;\; \text{s.t.} \;\; \forall K \subset G, K \text{ compact}, \; \exists x_K \notin K : f(x_K) \ge \varepsilon.$$

Since every topological group has an open basis at the identity, consisting of symmetric neighborhoods U (i.e. $U = U^{-1}$), from the hypothesis "G locally compact" and "f left uniformly continuous", we derive the existence of a compact symmetric neighborhood of the identity $K \subset G$, such that $\mu(K) > 0$ and $x^{-1}y \in K \;\; \Rightarrow \;\; |f(x) - f(y)| < \dfrac{\varepsilon}{2}$.

If A and B are compact subsets of a topological group, then the subset $AB = \{ab : a \in A, b \in B\}$ is also compact and so xK^2 is compact for every $x \in G$. Therefore, there exists a sequence $\{x_n\} \subset G$ such that

$$\begin{array}{lcl} x_1 \notin K & \wedge & f(x_1) \ge \varepsilon \\ x_2 \notin x_1 K^2 & \wedge & f(x_2) \ge \varepsilon \\ \dots & \dots & \dots \\ x_{n+1} \notin \bigcup_{i=1}^{n} x_i K^2 & \wedge & f(x_{n+1}) \ge \varepsilon \\ \dots & \dots & \dots \end{array}$$

The subsets x_nK are disjoint, since

$$x_nK \cap x_mK \neq \emptyset \ (n < m) \ \Rightarrow \ \exists k, k' \in K : \ x_nk = x_mk' \ \Rightarrow \ x_m = x_nkk'^{-1} \in x_nK^2$$

contrarily to the hypothesis $x_m \notin \bigcup_{j=1}^{m-1} x_jK$.

Now we consider the subsets $S_n \subset G$, defined by

$$S_n = \Big(f^{-1}\big([\frac{\varepsilon}{2}, +\infty)\big)\Big) \cap x_nK \, .$$

There exists at most a finite number of indices n such that $S_n = x_nK$: otherwise, from $\mu(x_nK) = \mu(K) > 0$ and $x_nK \cap x_mK = \emptyset$ if $n \neq m$, it follows

$$\begin{aligned} \int_G (f(x))^p d\mu(x) &\geq \sum_{n=1}^{+\infty} \int_{S_n} (f(x))^p d\mu(x) \geq \sum_{j=1}^{+\infty} \int_{S_{n_j} = x_{n_j}K} (f(x))^p d\mu(x) \\ &\geq \Big(\frac{\varepsilon}{2}\Big)^p \sum_{j=1}^{+\infty} \int_{S_{n_j} = x_{n_j}K} d\mu(x) = \Big(\frac{\varepsilon}{2}\Big)^p \sum_{j=1}^{+\infty} \mu(K) = +\infty \end{aligned}$$

contrarily to the hypothesis $f \in \mathcal{L}_p(G, \mu)$.

Therefore, there exists an integer N such that $S_n \subset x_nK \ \ \forall n \geq N$; for every $n \geq N$, let y_n be such that $y_n \in x_nK \ \wedge \ y_n \notin S_n$.

We have

$$\begin{aligned} &y_n \in x_nK \ \Rightarrow \ x_n^{-1}y_n \in K \ \Rightarrow \ |f(x_n) - f(y_n)| < \frac{\varepsilon}{2} \\ &y_n \notin S_n \ \Rightarrow \ f(y_n) < \frac{\varepsilon}{2} \end{aligned}$$

and so

$$f(x_n) = f(x_n) - f(y_n) + f(y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{(absurd).} \ \square$$

2. A counterexample

In this section we prove that, in the hypothesis of Theorem 1, left and right cannot be mixed; if G is not unimodular (and so left and right Haar integrals are different), right uniform continuity and integrability w.r. to the left Haar measure does not imply vanishing at infinity.

Let G be the multiplicative group of all matrices

$$g = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \ , \quad x, y \in \mathbb{R}, \ x \neq 0 \ ;$$

writing $g = (x, y)$, we have

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, x_1 y_2 + y_1) \quad \text{and} \quad (x, y)^{-1} = \left(\frac{1}{x}, -\frac{y}{x}\right) \ ;$$

G is a non-commutative, topological locally compact, non-compact group (topologized as a subset of $\mathbb{R}^2$). G is not unimodular: left Haar measure is $d\mu = \frac{dxdy}{x^2}$, right Haar measure is $d\nu = \frac{dxdy}{|x|}$ ([2], 15.17).
As an open basis at the identity $e = (1, 0)$, we consider

$$U = U(\delta, \varrho) = \{g = (x, y) \in G : \ |1 - x| < \delta \ \wedge \ |y| < \varrho\} \ .$$

Theorem 2

With the previous notations, we have

$$\exists f : G \to \mathbb{R} \ : \quad f \in \mathcal{L}_p(G, \mu) \ \ \forall p, \ 1 \leq p < +\infty \ , \quad f \in \mathcal{C}^r(G) \ , \quad f \notin \mathcal{C}_0(G) \ .$$

Proof. We consider the function $f : G \to [0, 1)$ defined by $f(g) = f(x, y) = k(x) l(y)$, where

$$k : \mathbb{R} \backslash \{0\} \to (0, 1) \qquad k(x) = \exp\left(-\frac{1}{|x|}\right)$$

$$l : \mathbb{R} \to [0, 1] \qquad l(y) = \begin{cases} \exp\left(-\frac{y^2}{(1-|y|)^2}\right) & \text{if } |y| < 1 \\ 0 & \text{if } |y| \geq 1 \end{cases}$$

f has the following properties:

(i) $f \in \mathcal{L}_p(G, \mu)$; indeed,

$$\int_G (f(g))^p d\mu(g) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (f(x, y))^p \frac{dxdy}{x^2}$$
$$= \left[\int_{-\infty}^{+\infty} (l(y))^p dy\right] \cdot \left[\int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{p}{|x|}\right)}{x^2} dx\right] = \frac{2\|l\|_p^p}{p} < \frac{2}{p}$$

(ii) $f \notin \mathcal{L}_p(G,\nu)$, since

$$\int_G (f(g))^p d\nu(g) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (f(x,y))^p \frac{dxdy}{|x|}$$
$$= \left[\int_{-\infty}^{+\infty} (l(y))^p dy\right] \cdot \left[\int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{p}{|x|}\right)}{|x|} dx\right] = +\infty$$

(iii) $f \notin \mathcal{C}_0(G)$, since $\lim\limits_{|x|\to+\infty} f(x,0) = \lim\limits_{|x|\to+\infty} e^{-\frac{1}{|x|}} = 1.$

(iv) $f \notin \mathcal{C}^l(G)$: obviously, this is a consequence of Theorem 1, but it is easy to give a direct proof.
For $g,h \in G$, we use these notations

$$g = (x,y) \qquad h = (\alpha x, y+t) \qquad (\alpha \in \mathbb{R}\backslash\{0\}, t \in \mathbb{R})$$

and then

$$g^{-1}h = \left(\frac{1}{x}, -\frac{y}{x}\right) \cdot (\alpha x, y+t) = \left(\alpha, \frac{t}{x}\right)$$
$$hg^{-1} = (\alpha x, y+t) \cdot \left(\frac{1}{x}, -\frac{y}{x}\right) = (\alpha, y(1-\alpha)+t)$$

We prove that, given a neighborhood $U(\delta,\varrho)$ (we can suppose $\varrho \le 1$), there exist $g,h \in G$ such that

$$g^{-1}h \in U(\delta,\varrho) \qquad \wedge \qquad |f(g) - f(h)| \ge \exp(-1)$$

and so $f \notin \mathcal{C}^l(G)$.
We consider $g,h \in G$ such that $|x| > \frac{1}{\varrho}$, $y = 0$, $\alpha = 1$, $t = 1$;
then $g^{-1}h = \left(1, \frac{1}{x}\right) \in U(\delta,\varrho)$, but

$$|f(g) - f(h)| = |f(x,0) - f(x,1)|$$
$$= k(x)|l(0) - l(1)| = \exp\left(-\frac{1}{|x|}\right) > \exp(-\varrho) \ge \exp(-1)$$

(v) $f \in \mathcal{C}^r(G)$; indeed, we prove that, given $\varepsilon > 0$, there exist $\delta,\varrho > 0$ (depending only on ε) such that $hg^{-1} \in U(\delta,\varrho) \Rightarrow |f(g) - f(h)| < 2\varepsilon$.
We have

$$|f(g) - f(h)| = |k(x)l(y) - k(\alpha x)l(y+t)|$$
$$\le |k(x)l(y) - k(\alpha x)l(y)| + |k(\alpha x)l(y) - k(\alpha x)l(y+t)|$$
$$\le |k(x) - k(\alpha x)| + |l(y) - l(y+t)| \ .$$

Let us consider, separately, $|k(x)-k(\alpha x)|$ and $|l(y)-l(y+t)|$:

(1) from the uniform continuity of l, there exists $\tilde{\varrho}=\tilde{\varrho}(\varepsilon)>0$ (that we can suppose <1) such that

$$|y_1-y_2|<\tilde{\varrho} \quad \Rightarrow \quad |l(y_1)-l(y_2)|<\varepsilon \;;$$

(2) if $|x|\geq 1$, we have

$$|k(x)-k(\alpha x)|=\exp\left(-\frac{1}{|x|}\right)\left|1-\exp\left(\frac{|\alpha|-1}{|\alpha x|}\right)\right|\leq\left|1-\exp\left(\frac{|\alpha|-1}{|\alpha x|}\right)\right|=(*)$$

(2a) if $|\alpha|\geq 1$, then

$$(*)=\exp\left(\frac{|\alpha|-1}{|\alpha x|}\right)-1\leq\exp\left(\frac{|\alpha|-1}{|\alpha|}\right)-1<\varepsilon \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad 1\leq|\alpha|<1+\frac{\log(1+\varepsilon)}{1-\log(1+\varepsilon)}=1+\delta_1$$

(2b) if $|\alpha|\leq 1$, then

$$(*)=1-\exp\left(\frac{|\alpha|-1}{|\alpha x|}\right)\leq 1-\exp\left(\frac{|\alpha|-1}{|\alpha|}\right)<\varepsilon \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad 1\geq|\alpha|>1-\frac{-\log(1-\varepsilon)}{1-\log(1-\varepsilon)}=1-\delta_2$$

(3) case $|x|\leq 1$: from the uniform continuity of k, there exists $\delta_3=\delta_3(\varepsilon)>0$ such that

$$r,s\in\mathbb{R}\,,\quad |r-s|<\delta_3 \quad\Rightarrow\quad |k(r)-k(s)|<\varepsilon\;;$$

therefore, if α satisfies $|1-\alpha|<\delta_3$, we have

$$|x-\alpha x|=|x||1-\alpha|\leq|1-\alpha|<\delta_3 \quad\Rightarrow\quad |k(x)-k(\alpha x)|<\varepsilon$$

Given $\varepsilon>0$, we have determined $\tilde{\delta}=\tilde{\delta}(\varepsilon)>0$ ($\tilde{\delta}=\min(\delta_1,\delta_2,\delta_3)$) such that

$$|1-\alpha|<\tilde{\delta} \quad\Rightarrow\quad |k(x)-k(\alpha x)|<\varepsilon \quad \forall x\neq 0$$

and so

$$|1-\alpha|<\tilde{\delta}\ \wedge\ |t|<\tilde{\varrho} \quad\Rightarrow\quad |f(g)-f(h)|<2\varepsilon\,.$$

Now, we have to find $\delta,\varrho>0$ such that

$$hg^{-1}\in U(\delta,\varrho) \quad\Rightarrow\quad |1-\alpha|<\tilde{\delta}\ \wedge\ \begin{cases}|t|<\tilde{\varrho}\\ \quad\text{or}\\ l(y)=l(y+t)=0\end{cases}$$

Let δ, ϱ be such that

$$\delta = \min\left(\tilde{\delta}, \frac{\tilde{\varrho}}{4}\right) , \quad \varrho : \ \varrho + 2\delta < \tilde{\varrho} \qquad \left(\text{it is sufficient } \varrho < \frac{\tilde{\varrho}}{2}\right)$$

If $hg^{-1} \in U(\delta, \varrho)$, then, obviously, $|1 - \alpha| < \delta \leq \tilde{\delta}$; for the second implication, we have

(4) if $|y| \leq 2$, then

$$|y(1-\alpha) + t| < \varrho \quad \Rightarrow \quad |t| \leq |y(1-\alpha) + t| + |y||1-\alpha| < \varrho + 2\delta < \tilde{\varrho}$$

(5) if $|y| > 2$, then $l(y) = 0$; we prove that it is also true that $l(y + t) = 0$:

$$|1-\alpha| < \delta \leq \frac{1}{4} \quad \Rightarrow \quad \frac{3}{4} < \alpha < \frac{5}{4} \qquad \text{and so}$$

$$|y(1-\alpha) + t| < \varrho \quad \Rightarrow \quad -\varrho + \alpha y < y + t < \varrho + \alpha y \quad \Rightarrow$$

$$\Rightarrow \quad \begin{cases} y + t > -\varrho + \frac{3}{2} > -\frac{1}{2} + \frac{3}{2} = 1 & \text{if } y > 0 \\ y + t < \varrho - \frac{3}{2} < \frac{1}{2} - \frac{3}{2} = -1 & \text{if } y < 0 \end{cases}$$

that prove $l(y + t) = 0$. □

Remark. Theorem 1 has an interesting consequence concerning the coefficient space of an irreducible unitary representation of a locally compact group: in [1] we prove that, if G is a connected algebraic group over a field of characteristic zero, or a connected exponential solvable Lie group, and π is an infinite–dimensional irreducible unitary representation of G, then, in the coefficient space of π, the sup–norm and the norm of duality with the $\mathcal{C}^*$–algebra of G are not equivalent.

References

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