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On the structure of tensor norms related to (p, σ)-absolutely continuous operators[(*)]

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Abstract

We define an interpolation norm on tensor products of p-integrable function spaces and Banach spaces which satisfies intermediate properties between the Bochner norm and the injective norm. We obtain substitutes of the Chevet – Persson – Saphar inequalities for this case. We also use the calculus of traced tensor norms in order to obtain a tensor product description of the tensor norm associated to the interpolated ideal of (p, σ)-absolutely continuous operators defined by Jarchow and Matter. As an application we find the largest tensor norm less than or equal to our interpolation norm.

The operator ideals $\mathfrak{P}_{p,\sigma}$ of (p, σ)-absolutely continuous operators were defined by U. Matter in [5] just by applying an interpolative procedure to the ideals $\mathfrak{P}_p$ of p-absolutely summing operators (see the paper [3] of Jarchow and Matter for a description of the interpolation method). These ideals are larger than the ideals $\mathfrak{P}_p$ but they preserve some properties of $\mathfrak{P}_p$. In this paper we show that the tensor norms associated to these operator ideals – which we have characterized in [4] – can be constructed as in the classical case of $\mathfrak{P}_p$ using the calculus of traced tensor norms given in [1]. In order to do this we define a special "pointwise interpolation norm"

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on $L_{p/(1-\sigma)}(\mu)\otimes E$ – which we denote by $\Delta_{p,\sigma}$ – satisfying intermediate properties between the natural norm Δ_p and the injective norm ε. We adapt in this way the description of the Saphar tensor norm d_p [6] – which is related to the classical p-absolutely summing operators – to the interpolation case. More precisely, we obtain the formula $d_{p,\sigma} = \Delta_{\bar{p}} \otimes_{\ell_{\bar{p}}} \Delta^t_{p',\sigma}$ where $\bar{p} = \left(\frac{p'}{1-\sigma}\right)'$ and $1 \leq p \leq \infty$. This is the interpolation version of the formula $d_p = \Delta_p \otimes_{\ell_p} \varepsilon$.

In section 2 we give the main result. It is an adaptation of the Chevet-Persson-Saphar inequalities $d_p \leq \Delta_p \leq d'_{p'}$ on $L_p(\mu)\otimes E$ for the interpolated norm $\Delta_{p,\sigma}$. In particular, we obtain a tensor norm $b_{p,\sigma}$ satisfying

$$b_{p,\sigma} \leq \Delta_{p,\sigma} \leq b'_{p',\sigma} \quad \text{on} \quad L_{\frac{p}{1-\sigma}}(\mu)\otimes E\,.$$

As an application we show that $b_{p,\sigma}$ is the largest tensor norm less than or equal to $\Delta_{p,\sigma}$ on $L_{p/(1-\theta)}(\mu)\otimes E$ extending in this way a classical result of Gordon and Saphar about the relation between Δ_p and d_p ([2], [1]).

0. Background and notation

Throughout this paper we use standard Banach space notation. The class of all Banach spaces is denoted by BAN. If $E \in$BAN, B_E is the closed unit ball and $\overset{\circ}{B}_E$ the open unit ball of E. We denote by $id_{L_p(\mu)}$ the identity map on a Lebesgue function space $L_p(\mu)$. If A is a measurable set on the measure space (Ω,μ), χ_A denotes the characteristic function of A. If $1 \leq p \leq \infty, p'$ is the real number satisfying $\frac{1}{p}+\frac{1}{p'}=1$. We use the same notation for a tensor $z=\sum_{i=1}^n f_i\otimes x_i \in L_p(\mu)\otimes E$ and for the associated Bochner integrable function $z(\omega)=\sum_{i=1}^n f_i(\omega)x_i \in L_p(\mu,E)$. If $(x_i)\in E^{\mathbb{N}}$, we define

$$\pi_p\big((x_i)\big) := \left(\sum_{i=1}^{\infty}\|x_i\|^p\right)^{1/p}, \; w_p\big((x_i)\big) := \sup_{x'\in B_{E'}}\left(\sum_{i=1}^{\infty}\big|\langle x_i,x'\rangle\big|^p\right)^{1/p}$$

and

$$\delta_{p,\sigma}\big((x_i)\big) := \sup_{x'\in B_{E'}}\left(\sum_{i=1}^{\infty}\Big(\big|\langle x_i,x'\rangle\big|^{1-\sigma}\|x_i\|^{\sigma}\Big)^{p/(1-\sigma)}\right)^{(1-\sigma)/p}.$$

If α is a tensor norm, we denote by α^t the transposed tensor norm, by α' the dual tensor norm and by $\overleftarrow{\alpha}$ the cofinite hull of α. If $\mathfrak{U}$ is an operator ideal, we write $\mathfrak{U}^*$ for the adjoint ideal (see [1] for more details).

Let $(\mathfrak{P},\Pi_p)$ be the ideal of p-absolutely summing operators and $1 \le p \le \infty$. The following definition is due to Matter ([5]):

Definition 0.1. Let $0 \le \sigma < 1$ and $E, F \in$ BAN. We say that $T \in \mathcal{L}(E,F)$ is a (p,σ)-absolutely continuous operator if there exist $G \in$ BAN and an operator $S \in \mathfrak{P}_p(E,G)$ such that

$$\|Tx\| \le \|x\|^{\sigma}\|Sx\|^{1-\sigma} \ \forall\, x \in E\,. \tag{1}$$

In such case, we put $\Pi_{p,\sigma}(T) = \inf \Pi_p(S)^{1-\sigma}$, taking the infimum over all G and $S \in \mathfrak{P}_p(E,G)$ such that (1) holds. We denote by $(\mathfrak{P}_{p,\sigma},\Pi_{p,\sigma})$ the injective normed ideal of (p,σ)-absolutely continuous operators in BAN.

Theorem 0.2 (Matter [5])

For every operator $T : E \longrightarrow F$, the following are equivalent:

(i) $T \in \mathfrak{P}_{p,\sigma}(E,F)$.

(ii) There is a constant $C > 0$ and a probability measure μ on $B_{E'}$, such that

$$\|Tx\| \le C\left(\int_{B_{E'}}\left(\left|\langle x,x'\rangle\right|^{1-\sigma}\|x\|^{\sigma}\right)^{p/(1-\sigma)}d\mu(x')\right)^{(1-\sigma)/p} \quad \forall\, x \in E\,.$$

(iii) There exists a constant $C > 0$ such that for every finite sequence $x_1,\dots,x_n$ in E, $\pi_{p/(1-\sigma)}\big((Tx_i)\big) \le C\,\delta_{p,\sigma}\big((x_i)\big)$.

In addition, $\pi_{p,\sigma}(T)$ is the smallest number C for which (ii) and (iii) hold.

The ideal $\mathfrak{P}_{p,\sigma}$ is a particular case of a family $\mathfrak{D}_{p,\sigma,q,v}$ of operator ideals introduced in [4] which generalizes the classical ideal $\mathfrak{D}_{p,q}$ of (p,q)-dominated operators. The following result is a characterization of the ideal $(\mathfrak{D}_{p,\sigma,q,\nu}, D_{p,\sigma,q,\nu})$ of (p,σ,q,ν)-dominated operators (see [4]).

Theorem 0.3

Let $E, F \in$ BAN, $T \in \mathcal{L}(E,F)$, $1 \le r,p,q \le \infty$ and $0 \le \sigma,\nu < 1$ such that $\frac{1}{r} + \frac{1-\sigma}{p} + \frac{1-\nu}{q} = 1$. The following assertions are equivalent.

1) $T \in \mathfrak{D}_{p,\sigma,q,\nu}(E,F)$.

2) There exist a constant $C > 0$ and regular probabilities μ and τ on $B_{E'}$ and $B_{F''}$ respectively, such that for every $x \in E$ and $y' \in F'$ the following inequality holds

$$\begin{aligned}\left|\langle Tx,y'\rangle\right| \le C&\left(\int_{B_{E'}}\left(\left|\langle x,x'\rangle\right|^{1-\sigma}\|x\|^{\sigma}\right)^{p/(1-\sigma)}d\mu(x')\right)^{(1-\sigma)/p}\\ &\times\left(\int_{B_{F''}}\left(\left|\langle y',y''\rangle\right|^{1-\nu}\|y'\|^{\nu}\right)^{q/(1-\nu)}d\tau(y'')\right)^{(1-\nu)/q}.\end{aligned}$$

3) There exists a constant $C > 0$ such that for every $(x_i)_{i=1}^n \subset E$ and $(y_i')_{i=1}^n \subset F'$ the inequality $\pi_{r'}\big((\langle Tx_i, y_i'\rangle)_{i=1}^n\big) \le C\,\delta_{p,\sigma}\big((x_i)_{i=1}^n\big)\delta_{q,\nu}\big((y_i')_{i=1}^n\big)$ holds.
4) There are a Banach space G and operators $A \in \mathfrak{P}_{p,\sigma}(E,G)$ and $B \in \mathfrak{P}_{q,\nu}^{\mathrm{dual}}G, F$ such that $T = BA$.

Moreover, the norm on $\mathfrak{D}_{p,\sigma,q,\nu}$ is $D_{p,\sigma,q,\nu}(T) = \inf C$, where the infimum in calculated over all "C" on 2) and 3).

1. The tensor product description of the tensor norm associated to the ideal of (p,σ)-absolutely continuous operators

DEFINITION 1.1. *Let μ be a measure on Ω, the function space $L_{p/(1-\sigma)}(\Omega,\mu)$, $E \in$ BAN, $1 \le p < \infty$. and $0 \le \sigma \le 1$. For every $z \in L_{p/(1-\sigma)}(\mu) \otimes E$, we define*

$$\Delta_{p,\sigma}(z) := \inf\left\{\sum_{i=1}^n \sup_{x'\in B_{E'}} \left(\int_\Omega \left(\left|\langle z_i(\omega), x'\rangle\right|^{1-\sigma} \|z_i(\omega)\|^{\sigma}\right)^{p/(1-\sigma)} d\mu(\omega)\right)^{(1-\sigma)/p} : \right.$$
$$\left. \text{where} \quad z = \sum_{i=1}^n z_i,\ z_i \in L_{p/(1-\sigma)}(\mu) \otimes E \quad \forall\, 1 \le i \le n \right\}.$$

Proposition 1.2

For every $z \in L_{p/(1-\sigma)}(\mu) \otimes E,\ \varepsilon(z) \le \Delta_{p,\sigma}(z) \le \Delta_{p/(1-\sigma)}(z)$.

The proof is standard and left to the reader.

Corollary 1.3

$\Delta_{p,\sigma}$ is a norm on $L_{p/(1-\sigma)}(\mu) \otimes E$.

Proof. If $\Delta_{p,\sigma}(z) = 0$, then $\varepsilon(z) = 0$ and hence $z = 0$. The triangle inequality holds easily. □

The following proposition gives us some information about the intermediate properties of the tensor product $L_{p/(1-\sigma)}(\mu) \otimes_{\Delta_{p,\sigma}} E$ in relation to the couple $\big(L_{p/(1-\sigma)}(\mu) \otimes_{\Delta_{p/(1-\sigma)}} E,\ L_{p/(1-\sigma)}(\mu) \otimes_\varepsilon E\big)$.

Proposition 1.4

For every $z \in L_{p/(1-\sigma)}(\mu) \otimes E,\ \Delta_{p,\sigma}(z) \le \Delta_{p/(1-\sigma)}(z)^{\sigma} \varepsilon(z)^{1-\sigma}$.

Proof.

$$\begin{aligned}\Delta_{p,\sigma}(z) &\leq \sup_{x'\in B_{E'}} \left(\int_\Omega \left(\left|\langle z(\omega),x'\rangle\right|^{1-\sigma}\|z(\omega)\|^\sigma\right)^{p/(1-\sigma)} d\mu(\omega)\right)^{(1-\sigma)/p} \\ &\leq \sup_{x'\in B_{E'}}\left(\int_\Omega \left|\langle z(\omega),x'\rangle\right|^{p/(1-\sigma)} d\mu\right)^{(1-\sigma)^2/p}\left(\int_\Omega \|z(\omega)\|^{p/(1-\sigma)} d\mu\right)^{\sigma(1-\sigma)/p} \\ &\leq \Delta_{p/(1-\sigma)}(z)^\sigma \varepsilon(z)^{1-\sigma},\end{aligned}$$

where the second inequality is obtained just by applying Hölder's inequality with indexes $1/\sigma$ and $1/(1-\sigma)$. □

The proof of the following proposition is standard.

Proposition 1.5

Let $E,F\in$ BAN and $T\in\mathcal{L}(E,F)$. Then for each $L_{p/(1-\sigma)}(\mu)$ the map $id_{L_{p/(1-\sigma)}(\mu)}\otimes T : L_{p/(1-\sigma)}(\mu)\otimes_{\Delta_{p,\sigma}} E \longrightarrow L_{p/(1-\sigma)}(\mu)\otimes_{\Delta_{p,\sigma}} F$ is continuous and $\|id_{L_{p/(1-\sigma)}}\otimes T\|\leq\|T\|$.

Remark 1.6. Following the definition given at [1], proposition 1.2 and 1.5 mean that $\Delta_{p,\sigma}$ is a right tensor norm, just like Δ_p. Moreover, $\Delta_{p,\sigma}$ is by definition a finitely generated right tensor norm.

Remark 1.7. Let $\Omega=\mathbb{N}$ and ν the measure satisfying $L_{p/(1-\sigma)}(\Omega,\nu)=\ell_{p/(1-\sigma)}$. Then for every (λ_i) the associated function $f:\mathbb{N}\longrightarrow\mathbb{K}$ is given by $f(i)=\lambda_i$. If $x\in E$, consider the usual representation $f(n)\otimes x=(\lambda_i)\otimes x=(\lambda_i x)\in E^{\mathbb{N}}$. Then for every $\sum_{j=1}^n f_j(n)\otimes x_j=\left(\sum_{j=1}^n \lambda_i^j x_j\right)\in\ell_{p/(1-\sigma)}\otimes E$ the following equalities hold.

$$\begin{aligned}&\sup_{x'\in B_{E'}}\left(\int_{\mathbb{N}}\left(\left|\langle\sum_{j=1}^n f_j(n)x_j,x'\rangle\right|^{1-\sigma}\left\|\sum_{j=1}^n f_j(n)x_j)\right\|^\sigma\right)^{p/(1-\sigma)} d\nu(n)\right)^{(1-\sigma)/p} \\ &=\sup_{x'\in B_{E'}}\left(\sum_{i=1}^\infty\left(\left|\langle\sum_{j=1}^n\lambda_i^j x_j,x'\rangle\right|^{1-\sigma}\left\|\sum_{j=1}^n\lambda_i^j x_j)\right\|^\sigma\right)^{p/(1-\sigma)}\right)^{(1-\sigma)/p} = \delta_{p,\sigma}\Big(\big(\sum_{j=1}^n\lambda_i^j x_j\big)\Big).\end{aligned}$$

Thus, if $z\in\ell_{p/(1-\sigma)}\otimes E$ then $\Delta_{p,\sigma}(z)=\inf\left\{\sum_{j=1}^n\delta_{p,\sigma}\big((x_i^j)\big)\,|\,z=\sum_{j=1}^n(x_i^j)\right\}$.

Remark 1.8. As in the case of the injective tensor norm ε, the completion $\ell_{p/(1-\sigma)}\,\hat{\otimes}_{\Delta_{p,\sigma}} E$ can be represented as the set $\left\{(x_i)\in E^{\mathbb{N}}\,|\,\Delta_{p,\sigma}\big((x_i)_{n=\mathbb{N}}^\infty\big)\longrightarrow 0\right\}$. The proof is standard and left to the reader.

DEFINITION 1.9. Let $1 \le p, q, r \le \infty$ and $0 \le \sigma, \nu < 1$ verifying $\frac{1}{r} + \frac{1-\sigma}{p'} + \frac{1-\nu}{q'} = 1$, and $E, F \in$ BAN. We define on $E \otimes F$ the function

$$\alpha_{p,\sigma,q,\nu}(z) := \inf \left\{ \pi_r\big((\lambda_i)\big)\delta_{q',\nu}\big((x_i)\big)\delta_{p',\sigma}\big((y_i)\big) \,\Big|\, z = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i \right\}.$$

We have proved on [4] that this expression defines a tensor norm and that $\alpha'_{p,\sigma,q,\nu}$ is the associated tensor norm to the maximal operator ideal $\mathfrak{D}_{q',\nu,p',\sigma'}$ of (q', ν, p', σ)-dominated operators.

DEFINITION 1.10. Let $1 \le p \le \infty$, $0 \le \sigma < 1, E, F \in$ BAN and denote $\bar{p} = \big(\frac{p'}{1-\sigma}\big)'$. We define on $E \otimes F$ the function

$$d_{p,\sigma}(z) := \inf \left\{ \delta_{p',\sigma}\big((x_i)\big)\pi_{\bar{p}}\big((y_i)\big) \,\Big|\, z = \sum_{i=1}^{n} x_i \otimes y_i \right\}.$$

Note that $d_{p,\sigma}(z)$ can also be written as

$$\begin{aligned} d_{p,\sigma} &= \inf \left\{ \Delta_{p',\sigma}\big((x_i)\big)\pi_{\bar{p}}\big((y_i)\big) \,\Big|\, \Big| z = \sum_{i=1}^{n} x_i \otimes y_i \right\} \\ &= \inf \left\{ \Delta^t_{p',\sigma}\Big(\sum_{i=1}^{n} x_i \otimes e_i\Big)\Delta_{\bar{p}}\Big(\sum_{i=1}^{n} e_i \otimes y_i\Big) \,\Big|\, z = \sum_{i=1}^{n} x_i \otimes y_i \right\} \end{aligned}$$

where $\sum_{i=1}^{n} x_i \otimes e_i \in E \otimes \ell_{p'/(1-\sigma)}$ and $\sum_{i=1}^{n} e_i \otimes y_i \in \ell_{\bar{p}} \otimes F$.

It is easy to prove that $\alpha_{1,0,p,\sigma} = d_{p,\sigma}$. Thus, $d'_{p,\sigma}$ is the associated tensor norm to the maximal injective operator ideal $\big(\mathfrak{P}_{p',\sigma}, \pi_{p',\sigma}\big)$. We can define also $g_{p,\sigma} := d^t_{p,\sigma} = \alpha_{p,\sigma,1,0}$.

Theorem 1.11

$d_{p,\sigma} = \Delta_{\bar{p}} \otimes_{\ell_{\bar{p}}} \Delta^t_{p',\sigma}$.

Proof. We apply the calculus of traced tensor norms. Following the lines of 12.9 [1], we only need to prove that the tensor contraction

$$\hat{C}\colon \Big(E \otimes_{\Delta^t_{p',\sigma}} \ell_{p'/(1-\sigma)}\Big) \otimes_\pi \big(\ell_{\bar{p}} \otimes_{\Delta_{\bar{p}}} F\big) \longrightarrow E \otimes_{d_{p,\sigma}} F$$

is a metric surjection.

Let $S_{p'/(1-\sigma)}$ and $S_{\bar{p}}$ the subspaces of $\ell_{p'/(1-\sigma)}$ and $\ell_{\bar{p}}$ respectively of finite sequences endowed with the induced topology. Consider the map

$$D : \big(E \otimes_{\Delta^t_{p',\sigma}} S_{p'/(1-\sigma)}\big) \times \big(S_{\bar{p}} \otimes_{\Delta_{\bar{p}}} F\big) \longrightarrow E \otimes_{d_{p,\sigma}} F$$

given by $D\big(\sum_{i=1}^n x_i \otimes e_i, \sum_{i=1}^n e_i \otimes y_i\big) = \sum_{i=1}^n x_i \otimes y_i$. Let $z \in E \otimes_{d_{p,\sigma}} F$ such that $d_{p,\sigma}(z) < 1$. Then there is a representation of $z = \sum_{i=1}^n x_i \otimes y_i$ such that $\delta_{p',\sigma}\big((x_i)\big) < 1$ and $\pi_{\bar{p}}\big((y_i)\big) < 1$. Just by taking $z_0 = \sum_{i=1}^n x_i \otimes e_i \in E \otimes_{\Delta^t_{p',\sigma}} S_{p'/(1-\sigma)}$ and $z_1 = \sum_{i=1}^n e_i \otimes y_i \in S_{\bar{p}} \otimes_{\Delta_{\bar{p}}} F$ we obtain that $D\big(z_1, z_2\big) = z$ and then $D\big(\overset{\circ}{B}_{E\otimes_{\Delta_{p',\sigma}} S_{p'/(1-\sigma)}} \times \overset{\circ}{B}_{S_{\bar{p}}\otimes_{\Delta_{\bar{p}}} F} \big) = \overset{\circ}{B}_{E\otimes_{d_{p,\sigma}} F}$, since the other inclusion is obvious. This means that the tensor contraction $C : \big(E \otimes_{\Delta^t_{p',\sigma}} S_{p'/(1-\sigma)}\big) \otimes_\pi \big(S_{\bar{p}} \otimes_{\Delta_{\bar{p}}} F\big) \longrightarrow E \otimes_{d_{p,\sigma}} F$ is a metric surjection.

Since $\varepsilon \leq \Delta_{p,\sigma} \leq \Delta_p$, then $E \otimes_{\Delta^t_{p',\sigma}} S_{p'/(1-\sigma)}$ and $S_{\bar{p}} \otimes_{\Delta_{\bar{p}}} F$ are dense on $E \mathbin{\hat{\otimes}}_{\Delta^t_{p',\sigma}} \ell_{p'/(1-\sigma)}$ and $\ell_{\bar{p}} \mathbin{\hat{\otimes}}_{\Delta_{\bar{p}}} F$ respectively, and so

$$\hat{C}\colon \Big(E \otimes_{\Delta^t_{p',\sigma}} \ell_{p'/(1-\sigma)}\Big) \otimes_\pi \big(\ell_{\bar{p}} \otimes_{\Delta_{\bar{p}}} F\big) \longrightarrow E \otimes_{d_{p,\sigma}} F$$

is a metric surjection, (see [1] 7.4 and 12.9). This means that $\Delta_{\bar{p}} \otimes_{\ell_{\bar{p}}} \Delta^t_{p',\sigma} = d_{p,\sigma}$ and concludes the proof. □

Corollary 1.12

$$g_{p,\sigma} = \Delta_{p',\sigma} \otimes_{\ell_{p'/(1-\sigma)}} \Delta^t_{\bar{p}}.$$

Corollary 1.13

$$\alpha'_{p,\sigma,p,\nu} = g'_{q,\nu} \otimes d'_{p,\sigma}.$$

Proof. Theorem 0.3 gives $\mathfrak{D}_{q',\nu,p',\sigma} = \mathfrak{P}^{\text{dual}}_{p',\sigma} \circ \mathfrak{P}_{q',\nu}$. Since $\mathfrak{D}_{q',\nu,p',\sigma}$ is maximal normed, 29.8 [1] implies that $\alpha'_{p,\sigma,q,\nu}$ is the associated tensor norm to $\mathfrak{P}^{\text{dual}}_{p',\sigma} \otimes \mathfrak{P}_{q',\nu}$ and then the result holds. □

The next corollary is an application of the main theorem of traced tensor norms in order to obtain a characterization of the adjoint ideal $\mathfrak{P}^*_{p,\sigma,q,\nu}$. Note that the characterization of $\mathfrak{P}^*_{p,\sigma}$ can be obtained as a particular case of the following:

Corollary 1.14

Let $1 \leq p, q \leq \infty$ *band* $0 \leq \sigma, \nu < 1$. *Then* $\mathfrak{D}^*_{p,\sigma,q,\nu} = \big(\mathfrak{P}^{\mathrm{dual}}_{q,\nu} \otimes \mathfrak{P}_{p,\sigma}\big)^*$. *Moreover, if* $E, F \in \mathrm{BAN}$, *the following assertions are equivalent.*
i) $T \in \mathfrak{D}^*_{p,\sigma,q,\nu}(E,F)$.
ii) For every $G \in \mathrm{BAN}$ *(or only for the Johnson space* C_2*) the map*

$$id_G \otimes T : G \otimes_{g'_{p',\sigma}} E \longrightarrow G \otimes_{\overleftarrow{g}_{q',\nu}} F$$

is continuous and $D^*_{p,\sigma,q,\nu}(T) = \|id_{C_2} \otimes T\|$.

Proof. The isometry $\mathfrak{D}_{p,\sigma,q,\nu} = \mathfrak{P}^{\mathrm{dual}}_{q,\nu} \circ \mathfrak{P}_{p,\sigma}$ and the fact that $\mathfrak{P}_{p,\sigma}$ is injective imply the first assertion just by applying 29.8 [1]. The equivalence between i) and ii) holds by a direct application of 29.4 [1]. □

2. The Chevet-Persson-Saphar inequalities for the interpolated norm $\Delta_{p,\sigma}$.

DEFINITION 2.1. Let $E, F \in \mathrm{BAN}$, $1 \leq p \leq \infty$ and $0 \leq \sigma < 1$. We define on $E \otimes F$ the function

$$b_{p,\sigma}(z) := \inf \left\{ w_{\left(p/(1-\sigma)\right)'}\big((x_i)\big)\delta_{p,\sigma}\big((y_i)\big) \;\Big|\; z = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

It is easy to proof that $b_{p,\sigma}$ defines a tensor norm of the class $\alpha_{p,\sigma,q,\nu}$.

Proposition 2.2

$b_{p,\sigma} \leq \Delta_{p,\sigma} \leq b'_{p',\sigma}$ on $L_{p/(1-\sigma)}(\mu) \otimes E$.

Proof. The second inequality follows by duality just by applying the Chevet-Persson-Saphar inequalities (see e.g. [1]). Note that $b_{p,\sigma} \leq d_{p/(1-\sigma)}$. Thus

$$b_{p,\sigma} \leq d_{p/(1-\sigma)} \leq \Delta_{p/(1-\sigma)} \leq d'_{\left(p/(1-\sigma)\right)'} \leq b'_{p',\sigma}.$$

To prove the other inequality, first we claim that if $z = \sum_{i=1}^n f_i \otimes x_i \in L_{p/(1-\sigma)}(\mu) \otimes E$ and $(z_k)_{k=1}^\infty$ is a sequence of E-valued step functions which converges to z for $\Delta_{p/(1-\sigma)}$, then $\forall\, \varepsilon > 0$ there exists a k_0 such that $\forall\, k \geq k_0$

$$\Bigg| \sup_{x' \in B_{E'}} \left(\int_\Omega \left(\big|\langle z_k, x'\rangle\big|^{1-\sigma} \|z_k\|^\sigma \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p} - \sup_{x' \in B_{E'}} \left(\int_\Omega \left(\big|\langle z, x'\rangle\big|^{1-\sigma} \|z\|^\sigma \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p} \Bigg| < \varepsilon. \quad (1)$$

Indeed, for every $x' \in B_{E'}$ and every $k \in \mathbb{N}$, the following inequalities hold.

$$\left| \left(\int_\Omega \left(\left| \langle z_k, x' \rangle \right|^{1-\sigma} \|z_k\|^\sigma \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p} - \left(\int_\Omega \left(\left| \langle z, x' \rangle \right|^{1-\sigma} \|z\|^\sigma \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p} \right|$$

$$\le \left(\int_\Omega \left(\left| \langle z_k, x' \rangle \right|^{1-\sigma} \|z_k\|^\sigma - \left| \langle z, x' \rangle \right|^{1-\sigma} \|z\|^\sigma \right| \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p}$$

$$= \left(\int_\Omega \left(\Big| |\langle z_k, x' \rangle|^{1-\sigma} \|z_k\|^\sigma - |\langle z_k, x' \rangle|^{1-\sigma} \|z\|^\sigma + |\langle z_k, x' \rangle|^{1-\sigma} \|z\|^\sigma \right.\right.$$

$$\left.\left. -|\langle z, x' \rangle|^{1-\sigma} \|z\|^\sigma \Big| \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p}$$

$$\le \left(\int_\Omega \left(|\langle z_k, x' \rangle|^{1-\sigma} \left(\big| \|z_k\|^\sigma - \|z\|^\sigma \big| \right) \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p}$$

$$+ \left(\int_\Omega \left(\Big| |\langle z_k, x' \rangle|^{1-\sigma} - |\langle z, x' \rangle|^{1-\sigma} \Big| \|z\|^\sigma \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p}$$

$$\le \left(\int_\Omega \left(|\langle z_k, x' \rangle|^{1-\sigma} \|z_k - z\|^\sigma \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p}$$

$$+ \left(\int_\Omega \left(\|z_k - z\|^{1-\sigma} \|z\|^\sigma \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p}. \tag{2}$$

Now, using Hölder's inequality with indexes $1/\sigma$ and $1/(1-\sigma)$ we have

$$(2) \le \left(\int_\Omega \|z_k\|^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)^2/p} \left(\int_\Omega \|z_k - z\|^{p/(1-\sigma)} d\mu \right)^{\sigma(1-\sigma)/p}$$

$$+ \left(\int_\Omega \|z\|^{p/(1-\sigma)} d\mu \right)^{\sigma(1-\sigma)/p} \left(\int_\Omega \|z_k - z\|^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)^2/p}.$$

This inequalities mean that the expression

$$\sup_{x' \in B_{E'}} \Big| \left(\int_\Omega \left(|\langle z_k, x' \rangle|^{1-\sigma} \|z_k\|^\sigma \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p}$$

$$- \left(\int_\Omega \left(|\langle z, x' \rangle|^{1-\sigma} \|z\|^\sigma \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p} \Big|$$

converges to 0 if $z_k \longrightarrow z$ on $\Delta_{p/(1-\sigma)}$. Since

$$\sup_{x' \in B_{E'}} \left(\int_\Omega \left(|\langle z_k, x' \rangle|^{1-\sigma} \|z_k\|^\sigma \right)^{p/(1-\sigma)} d\mu \right)^{(1-\sigma)/p} < \infty \quad \text{for each} \quad k, \quad \text{and}$$

$$\sup_{x'\in B_{E'}} \left(\int_\Omega \left(|\langle z, x'\rangle|^{1-\sigma}\|z\|^\sigma\right)^{p/(1-\sigma)} d\mu\right)^{(1-\sigma)/p} < \infty \text{, the inequality (1) holds.}$$

Now, let $z \in L_{p/(1-\sigma)}(\mu) \otimes E$ and $\varepsilon > 0$. Then we can find a representation of z as $z = \sum_{j=1}^n z_j$ satisfying

$$\sum_{j=1}^n \sup_{x'\in B_{E'}} \left(\int_\Omega \left(|\langle z_j, x'\rangle|^{1-\sigma}\|z_j\|^\sigma\right)^{p/(1-\sigma)} d\mu\right)^{(1-\sigma)/p} \leq \Delta_{p,\sigma}(z) + \varepsilon.$$

Since the set of step functions is dense on $L_{p/(1-\sigma)}(\mu) \overset{\wedge}{\otimes}_{\Delta_{p/(1-\sigma)}} E$, for every $1 \leq j \leq n$ there exists a step function s_j such that

$$\sup_{x'\in B_{E'}} \left(\int_\Omega \left(|\langle z_j - s_j, x'\rangle|^{1-\sigma}\|z_j - s_j\|^\sigma\right)^{p/(1-\sigma)} d\mu\right)^{(1-\sigma)/p} \leq \Delta_{p/1-\sigma}(z_j - s_j) \leq \frac{\varepsilon}{2^j}.$$

Thus, if $s = \sum_{j=1}^n s_j$, then $b_{p,\sigma}(z - s) \leq d_{p/(1-\sigma)}(z - s) \leq \Delta_{p/(1-\sigma)}(z - s) \leq \varepsilon$.

Note that we can take the step functions s_j satisfying also the conditions of the previous claim for $\frac{\varepsilon}{2^j}$. On the other hand, for every $s_j = \sum_{m=1}^{l_j} \chi_{A_{m,j}} \otimes x_{m,j}$ – where $\left\{A_{m,j} \,\middle|\, 1 \leq m \leq l_j\right\}$ is a class of pairwise disjoint sets – we can consider the representation $s_j = \sum_{m=1}^{l_j} \chi_{A_{m,j}} \mu(A_{m,j})^{(\sigma-1)/p} \otimes x_{m,j}\mu(A_{m,j})^{(1-\sigma)/p}$.

It can be easily proved that $w_{\left(p/(1-\sigma)\right)'}\left((\chi_{A_{m,j}}\mu(A_{m,j})^{(\sigma-1)/p})_{m=1}^{l_j}\right) \leq 1$ and

$$\delta_{p,\sigma}\left((x_{m,j}\mu(A_{m,j})^{(1-\sigma)/p})\right) = \sup_{x'\in B_{E'}} \left(\int_\Omega \left(|\langle s_j, x'\rangle|^{1-\sigma}\|s_j\|\right)^{p/(1-\sigma)} d\mu\right)^{(1-\sigma)/p}.$$

Finally, as

$$\begin{aligned}
b_{p,\sigma}(z) &\leq \sum_{j=1}^n b_{p,\sigma}(s_j) + b_{p,\sigma}(s - z) \\
&\leq \sum_{j=1}^n w_{(p/(1-\sigma))'}\left(\left(\chi_{A_{m,j}}\mu(A_{m,j})^{(\sigma-1)/p}\right)_{m=1}^{l_j}\right)\delta_{p,\sigma}\left(\left(x_{m,j}\mu(A_{m,j})^{(1-\sigma)/p}\right)_{m=1}^{l_j}\right) \\
&\quad + \varepsilon \\
&\leq \sum_{j=1}^n \sup_{x'\in B_{E'}} \left(\int_\Omega \left(|\langle s_j, x'\rangle|^{1-\sigma}\|s_j\|^\sigma\right)^{p/(1-\sigma)} d\mu\right)^{(1-\sigma)/p} + \varepsilon \\
&\leq \sum_{j=1}^n \sup_{x'\in B_{E'}} \left(\int_\Omega \left(|\langle z_j, x'\rangle|^{1-\sigma}\|z_j\|^\sigma\right)^{p/(1-\sigma)} d\mu\right)^{(1-\sigma)/p} + 2\varepsilon \leq \Delta_{p,\sigma}(z) + 3\varepsilon
\end{aligned}$$

the result holds. $\square$

Corollary 2.3

$b_{p,\sigma} = \Delta_{p,\sigma} \otimes_{\ell_{p/(1-\sigma)}} \varepsilon$.

Proof. The inequality $\Delta_{p,\sigma} \otimes_{\ell_{p/(1-\sigma)}} \varepsilon \leq b_{p,\sigma}$ is obvious by the definition of $b_{p,\sigma}$. The converse follows from 2.2.: $b_{p,\sigma} \leq b_{p,\sigma} \otimes_{\ell_{p/(1-\sigma)}} \varepsilon \leq \Delta_{p,\sigma} \otimes_{\ell_{p/(1-\sigma)}} \varepsilon$. $\square$

The following proposition is a generalization of the result of Gordon and Saphar for the interpolated case [2]. The proof is exactly the same that the one that holds for proposition 15.11 [1].

Proposition 2.4

Let α be a tensor norm and $C \geq 1$. If $\alpha \leq C\Delta_{p,\sigma}$ on $L_{p/(1-\sigma)}(\mu) \otimes E$ for all normed spaces E, then $\alpha \leq C\, b_{p,\sigma}$.

Proof. We only need to observe that the following diagram commutes for the sequence space $\ell_{(p/(1-\sigma))'}$ and for every $E, F \in$ BAN, where C_1 and C_2 are the respective tensor contractions.

$$\begin{array}{ccc}
\left(F \otimes_\varepsilon \ell_{(p/(1-\sigma))'}\right) \otimes_\pi \left(\ell_{p/(1-\sigma)} \otimes_{\Delta_{p,\sigma}} E\right) & \xrightarrow{C_1} & F \otimes_{b_{p,\sigma}} E \\
\downarrow id & & \downarrow id \\
\left(F \otimes_\varepsilon \ell_{(p/(1-\sigma))'}\right) \otimes_\pi \left(\ell_{p/(1-\sigma)} \otimes_\alpha E\right) & \xrightarrow[C_2]{} & F \otimes_\alpha E
\end{array}$$

Note that C_1 is a metric surjection – this is corollary 2.3 – and that $\|C_2\| \leq 1$. Since the left identity has norm $\leq C$, the same holds for the right identity. Then $\alpha \leq C\, b_{p,\sigma}$ on $F \otimes E$. $\square$

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