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Congruences associated with inverse transversals

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Dedicated to the memory of
Monsieur le Professeur Paul Dubreil
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Abstract

An inverse transversal of a regular semigroup S is an inverse subsemigroup of S that contains a unique inverse x° of every element x of S. Here we consider the congruences on such a semigroup, considered as an algebra of type (2, 1). The structure of such semigroups being known, with 'building bricks' the inverse subsemigroup S° and the sub-bands I = $\{xx^\circ; x \in S\}, \Lambda = \{x^\circ x;\ x \in S\}$, we investigate how congruences on S are related to congruences on these building bricks.

Throughout this paper[1] we shall be concerned with a regular semigroup S with an inverse transversal. Basically, an inverse transversal is an inverse subsemigroup T of S with the property that $|T \cap V(x)| = 1$ for every $x \in S$, where $V(x)$ denotes as usual the set of inverses of x in S. Defining x° by $T \cap V(x) = \{x^\circ\}$, we can write T as $S^\circ = \{x^\circ; x \in S\}$. The structure of regular semigroups having inverse transversals has been determined by Saito [4]. Here we shall be interested in congruences on

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such semigroups. We denote by Con S the complete lattice of congruences on S. We shall say that $\vartheta \in$ Con S is a $^\circ$-*congruence* if

$$(a,b) \in \vartheta \Rightarrow (a^\circ, b^\circ) \in \vartheta\,.$$

The set of $^\circ$-congruences on S, i.e. the set of congruences on the algebra $(S, \cdot, ^\circ)$, will be denoted by $\overline{\text{Con}}\ S$. It is readily seen that $\overline{\text{Con}}\ S$ is a complete sublattice of Con S. In order to investigate $\overline{\text{Con}}\ S$ we require the following known facts.

If $E(S)$ is the set of idempotents of S then, as established by Tang [6],

$$\mathrm{I} = \{xx^\circ; x \in S\} = \{e \in E(S); e = ee^\circ\}$$

is a sub-band of S; moreover, it is left regular [i.e. $(\forall i, j \in \mathrm{I})\ iji = ij$]. Dually

$$\Lambda = \{x^\circ x; x \in S\} = \{f \in E(S); f = f^\circ f\}$$

is a sub-band of S; moreover, it is right regular [i.e. $(\forall e, f \in \Lambda)\ efe = fe$]. We have that $\mathrm{I} \cap \Lambda = E(S^\circ)$, the semilattice of idempotents of S° and an inverse transversal of both I, Λ.

Important properties of the operation $x \mapsto x^\circ$ are:

(1) $$(\forall x \in S)\ x^{\circ\circ\circ} = x^\circ\,.$$

In fact, both $x^{\circ\circ\circ}$ and x° belong to $S^\circ \cap V(x^{\circ\circ})$.

(2) *S is orthodox if and only if $(xy)^\circ = y^\circ x^\circ$ for all $x, y \in S$.*

This is established in [4].

(3) $$(\forall x, y \in S)\ (xy)^\circ = (x^\circ xy)^\circ x^\circ = y^\circ (xyy^\circ)^\circ\,.$$

This is established in [3].

(4) $$(\forall x, y \in S)\ (x^\circ y)^\circ = y^\circ x^{\circ\circ} \quad \text{and} \quad (xy^\circ)^\circ = y^{\circ\circ} x^\circ\,.$$

In view of the above result of Tang, this follows from [5, Proposition 2.2]. It may also be proved directly. For example, since I is left regular we have

$$y^\circ x^{\circ\circ} \cdot x^\circ y \cdot y^\circ x^{\circ\circ} = y^\circ yy^\circ x^{\circ\circ} x^\circ yy^\circ x^{\circ\circ} = y^\circ yy^\circ x^{\circ\circ} x^\circ x^{\circ\circ} = y^\circ x^{\circ\circ}\,,$$

and similarly $x^\circ y \cdot y^\circ x^{\circ\circ} \cdot x^\circ y = x^\circ y$. It follows that $y^\circ x^{\circ\circ} \in S^\circ \cap V(x^\circ y)$ and therefore $y^\circ x^{\circ\circ} = (x^\circ y)^\circ$.

Theorem 1

Let S be a regular semigroup with an inverse transversal S°. If $X \in \{I, S^\circ, \Lambda\}$ then Con $X = \overline{\text{Con}}\ X$.

Proof. It is well known that on an inverse semigroup every congruence ϑ is such that $(a,b) \in \vartheta$ implies $(a^{-1}, b^{-1}) \in \vartheta$. It follows immediately that Con $S^\circ = \overline{\text{Con}}\ S^\circ$.

Suppose now that $\imath \in$ Con I. If $(i,j) \in \imath$ then $(i^\circ, i^\circ j) = (i^\circ i, i^\circ j) \in \imath$ whence $(i^\circ j^\circ, i^\circ j) = (i^\circ j^\circ, i^\circ j j^\circ) \in \imath$ and therefore $(i^\circ, i^\circ j^\circ) \in \imath$. Interchanging i, j and using the fact that $i^\circ, j^\circ \in E(S^\circ)$ and therefore commute, we obtain $(i^\circ, j^\circ) \in \imath$. Hence Con I $= \overline{\text{Con}}$ I, and similarly Con $\Lambda = \overline{\text{Con}}\ \Lambda$. □

Definition. Given $\imath \in$ Con I, $\pi \in$ Con S°, $\lambda \in$ Con Λ we shall say that $(\imath, \pi, \lambda)$ is a *linked triple* if, for all $i_1, i_2 \in$ I all $x_1, x_2 \in S^\circ$, and all $l_1, l_2 \in \Lambda$,

$$(i_1, i_2) \in \imath, (l_1, l_2) \in \lambda \Rightarrow \begin{cases} \big(l_1 i_1 (l_1 i_1)^\circ,\ l_2 i_2 (l_2 i_2)^\circ\big) \in \imath & (\alpha) \\ \big((l_1 i_1)^\circ,\ (l_2 i_2)^\circ\big) \in \pi & (\beta) \\ \big((l_1 i_1)^\circ l_1 i_1,\ (l_2 i_2)^\circ l_2 i_2\big) \in \lambda & (\gamma) \end{cases}$$

$$(i_1, i_2) \in \imath, (x_1, x_2) \in \pi \Rightarrow (x_1 i_1 x_1^\circ, x_2 i_2 x_2^\circ) \in \imath \qquad (\delta)$$

$$(l_1, l_2) \in \lambda, (x_1, x_2) \in \pi \Rightarrow (x_1^\circ l_1 x_1, x_2^\circ l_2 x_2) \in \lambda \qquad (\epsilon)$$

To observe that (δ) and (ϵ) are meaningful, it suffices to show that, for example, if $i \in$ I then $x^{\circ\circ} i x^\circ \in$ I for every $x \in S$. This follows from the fact that

$$\begin{aligned} x^{\circ\circ} i x^\circ (x^{\circ\circ} i x^\circ)^\circ &= x^{\circ\circ} i x^\circ x^{\circ\circ} (x^{\circ\circ} i x^\circ x^{\circ\circ})^\circ \\ &= x^{\circ\circ} i x^\circ x^{\circ\circ} (i x^\circ x^{\circ\circ})^\circ x^\circ \\ &= x^{\circ\circ} i x^\circ x^{\circ\circ} x^\circ x^{\circ\circ} i^\circ x^\circ \qquad \text{since I is orthodox} \\ &= x^{\circ\circ} i i^\circ x^\circ \\ &= x^{\circ\circ} i x^\circ . \end{aligned}$$

We shall denote by LT(S) the set of linked triples. It is clear that LT(S) is a subset of Con I $\times$ Con $S^\circ \times$ Con Λ and as such inherits the cartesian order of the latter.

Guided by property (β) above, we introduce the following notion.

Definition. We shall say that $\vartheta \in$ Con S is *braided* if, for all $i_1, i_2 \in$ I and all $l_1, l_2 \in \Lambda$,

$$(i_1, i_2) \in \vartheta|_{\text{I}}, (l_1, l_2) \in \vartheta|_\Lambda \Rightarrow \big((l_1 i_1)^\circ, (l_2 i_2)^\circ\big) \in \vartheta|_{S^\circ}.$$

We shall denote the set of braided congruences on S by BrCon S. It is readily seen that BrCon S is a complete sublattice of Con S. Clearly, we have

$$\overline{\text{Con}}\ S \subseteq \text{BrCon}\ S \subseteq \text{Con}\ S.$$

To each $(\imath, \pi, \lambda) \in$ Con I $\times$ Con $S^\circ \times$ Con Λ we associate the relation $\Psi(\imath, \pi, \lambda)$ defined on S by

$$(a,b) \in \Psi(\imath, \pi, \lambda) \iff (aa^\circ, bb^\circ) \in \imath, (a^\circ, b^\circ) \in \pi, (a^\circ a, b^\circ b) \in \lambda .$$

Theorem 2

If $(\imath, \pi, \lambda) \in$ LT(S) *then* $\Psi(\imath, \pi, \lambda) \in$ BrCon S.

Proof. Suppose that $(a,b) \in \Psi(\imath,\pi,\lambda)$. Then $(a^\circ, b^\circ) \in \pi$ and, for every $x \in S$,

$$\begin{aligned}(ax)^\circ &= x^\circ(a^\circ axx^\circ)^\circ a^\circ \\ &\overset{\pi}{\equiv} x^\circ(b^\circ bxx^\circ)^\circ b^\circ \qquad \text{by } (\beta) \\ &= (bx)^\circ .\end{aligned}$$

Similarly, $\big((xa)^\circ, (xb)^\circ\big) \in \pi$.

Now (α) gives

$$a^\circ axx^\circ(a^\circ axx^\circ)^\circ \overset{\imath}{\equiv} b^\circ bxx^\circ(b^\circ bxx^\circ)^\circ$$

whence, by (δ), we obtain

$$a^{\circ\circ}a^\circ axx^\circ(a^\circ axx^\circ)^\circ a^\circ \overset{\imath}{\equiv} b^{\circ\circ}b^\circ bxx^\circ(b^\circ bxx^\circ)^\circ b^\circ.$$

Since $(aa^\circ, bb^\circ) \in \imath$ we therefore have

$$ax(ax)^\circ = aa^\circ \cdot a^{\circ\circ}a^\circ axx^\circ(a^\circ axx^\circ)^\circ a^\circ \overset{\imath}{\equiv} bb^\circ \cdot b^{\circ\circ}b^\circ bxx^\circ(b^\circ bxx^\circ)^\circ b^\circ = bx(bx)^\circ .$$

Similarly, $\big(xa(xa)^\circ, xb(xb)^\circ\big) \in \imath$.

Using (γ) and (ϵ) we can show likewise that

$$\big((ax)^\circ ax, (bx)^\circ bx\big) \in \lambda, \quad \big((xa)^\circ xa, (xb)^\circ xb\big) \in \lambda .$$

Consequently, $\Psi(\imath,\pi,\lambda) \in \mathrm{Con}\, S$.

To prove that $\Psi(\imath,\pi,\lambda)$ is braided, suppose that $(i_1,i_2) \in \Psi(\imath,\pi,\lambda)|_{\mathrm{I}}$ and $(l_1,l_2) \in \Psi(\imath,\pi,\lambda)|_\Lambda$. Then $(i_1,i_2) \in \imath$ and $(l_1,l_2) \in \lambda$ and so, by $(\alpha),(\gamma)$ and Theorem 1, we have

$$(1) \qquad \big((l_1i_1)^{\circ\circ}(l_1i_1)^\circ, (l_2i_2)^{\circ\circ}(l_2i_2)^\circ\big) = \big([l_1i_1(l_1i_1)^\circ]^\circ, [l_2i_2(l_2i_2)^\circ]^\circ\big) \in \imath;$$

$$(2) \qquad \big((l_1i_1)^\circ(l_1i_1)^{\circ\circ}, (l_2i_2)^\circ(l_2i_2)^{\circ\circ}\big) = \big([(l_1i_1)^\circ l_1i_1]^\circ, [(l_2i_2)^\circ l_2i_2]^\circ\big) \in \lambda.$$

It follows from (1), (2), and (β) that $\big((l_1i_1)^{\circ\circ}, (l_2i_2)^{\circ\circ}\big) \in \Psi(\imath,\pi,\lambda)|_{S^\circ}$ and therefore, by Theorem 1 again, $\big((l_1i_1)^\circ, (l_2i_2)^\circ\big) \in \Psi(\imath,\pi,\lambda)|_{S^\circ}$. Hence $\Psi(\imath,\pi,\lambda)$ is braided. □

Definition. A triple $(\imath,\pi,\lambda) \in \mathrm{Con\, I} \times \mathrm{Con}\, S^\circ \times \mathrm{Con}\, \Lambda$ will be called *balanced* if

$$\imath|_{E(S^\circ)} = \pi|_{E(S^\circ)} = \lambda|_{E(S^\circ)} .$$

We shall denote the ordered set of balanced linked triples by $\mathrm{BLT}(S)$.

Theorem 3

If $(\imath,\pi,\lambda) \in \mathrm{BLT}(S)$ *then* $\Psi(\imath,\pi,\lambda) \in \overline{\mathrm{Con}}\, S$.

Proof. Given $(\imath, \pi, \lambda) \in \mathrm{BLT}(S)$ we have that $(a, b) \in \Psi(\imath, \pi, \lambda)$ implies $(a^\circ, b^\circ) \in \pi$ whence $(a^{\circ\circ}, b^{\circ\circ}) \in \pi$ and therefore $(a^{\circ\circ}a^\circ, b^{\circ\circ}b^\circ) \in \pi|_{E(S^\circ)} = \imath|_{E(S^\circ)}$ and $(a^\circ a^{\circ\circ}, b^\circ b^{\circ\circ}) \in \pi|_{E(S^\circ)} = \lambda|_{E(S^\circ)}$. Consequently we see that $(a, b) \in \Psi(\imath, \pi, \lambda)$ implies $(a^\circ, b^\circ) \in \Psi(\imath, \pi, \lambda)$, whence the result follows by Theorem 2. □

Theorem 4

The mapping $\Psi : \mathrm{BLT}(S) \to \mathrm{BrCon}\ S$ *described by* $(\imath, \pi, \lambda) \mapsto \Psi(\imath, \pi, \lambda)$ *is injective and residuated, with residual* Ψ^+ *given by* $\Psi^+(\vartheta) = (\vartheta|_{\mathrm{I}}, \vartheta|_{S^\circ}, \vartheta|_\Lambda)$.

Proof. If $\vartheta \in \mathrm{BrCon}\ S$ then taking $\imath = \vartheta|_{\mathrm{I}}, \pi = \vartheta|_{S^\circ}, \lambda = \vartheta|_\Lambda$ we see that (β), hence (α) and (γ), and $(\delta), (\epsilon)$ are satisfied. Consequently, $(\vartheta|_{\mathrm{I}}, \vartheta|_{S^\circ}, \vartheta|_\Lambda)$ is a linked triple which is clearly balanced. We can therefore define a mapping $\Phi^+ : \mathrm{BrCon}\ S \to \mathrm{BLT}(S)$ by $\Psi^+(\vartheta) = (\vartheta|_{\mathrm{I}}, \vartheta|_{S^\circ}, \vartheta|_\Lambda)$. It is clear that Ψ and Ψ^+ are isotone. Now

$$
\begin{aligned}
(a, b) \in \Psi\Psi^+(\vartheta) \Rightarrow\ & (aa^\circ, bb^\circ) \in \vartheta|_{\mathrm{I}}, (a^\circ, b^\circ) \in \vartheta|_{S^\circ}, (a^\circ a, b^\circ b) \in \vartheta|_\Lambda \\
\Rightarrow\ & a = aa^\circ \cdot a^{\circ\circ} \cdot a^\circ a \overset{\vartheta}{\equiv} bb^\circ \cdot b^{\circ\circ} \cdot b^\circ b = b
\end{aligned}
$$

so $\Psi\Psi^+(\vartheta) \subseteq \vartheta$ and therefore $\Psi\Psi^+ \leq \mathrm{id}$.

Observe next that for $i, j \in \mathrm{I}$ we have

$$(i, j) \in \Psi(\imath, \pi, \lambda) \Longleftrightarrow (i, j) \in \imath,\ (i^\circ, j^\circ) \in \pi,\ (i^\circ, j^\circ) \in \lambda\,.$$

But, by Theorem 1 and the hypothesis that $(\imath, \pi, \lambda) \in \mathrm{BLT}(S)$, we have

$$(i, j) \in \imath \Rightarrow (i^\circ, j^\circ) \in \imath|_{E(S^\circ)} = \pi|_{E(S^\circ)} = \lambda|_{E(S^\circ)}\,.$$

Hence we see that $\Psi(\imath, \pi, \lambda)|_{\mathrm{I}} = \imath$. Similarly, $\Psi(\imath, \pi, \lambda)|_\Lambda = \lambda$ and $\Psi(\imath, \pi, \lambda)|_{S^\circ} = \pi$. It follows from these observations that $\Psi^+\Psi(\imath, \pi, \lambda) = (\imath, \pi, \lambda)$ and therefore $\Psi^+\Psi = \mathrm{id}$.

Hence Ψ is injective and residuated, with residual Ψ^+. □

Corollary 1

$\mathrm{BLT}(S)$ *forms a lattice that is isomorphic to* $\overline{\mathrm{Con}}\ S = \mathrm{Im}\,\Psi$.

Proof. It follows from Theorem 3 that $\mathrm{Im}\ \Psi \subseteq \overline{\mathrm{Con}}\ S$. But for every $\vartheta \in \overline{\mathrm{Con}}\ S$ we have

$$
\begin{aligned}
(a, b) \in \vartheta \Rightarrow\ & (aa^\circ, bb^\circ) \in \vartheta|_{\mathrm{I}}, (a^\circ, b^\circ) \in \vartheta|_{S^\circ}, (a^\circ a, b^\circ b) \in \vartheta|_\Lambda \\
\Rightarrow\ & (a, b) \in \Psi\Psi^+(\vartheta)
\end{aligned}
$$

so $\vartheta \subseteq \Psi\Psi^+(\vartheta)$, whence we have equality. It follows that $\overline{\mathrm{Con}}\ S \subseteq \mathrm{Im}\ \Psi$ and therefore $\overline{\mathrm{Con}}\ S = \mathrm{Im}\ \Psi$. Now since Ψ^+ is the residual of Ψ we have $\Psi\Psi^+\Psi = \Psi$.

Thus $\Psi\Psi^+$ acts as the identity on Im Ψ. More precisely, if Ψ_*^+ is the restriction of Ψ^+ to Im Ψ and if $\Psi_* : \mathrm{BLT}(S) \to \mathrm{Im}\,\Psi$ is the mapping induced by Ψ [i.e. $\Psi_*(\imath, \pi, \lambda) = \Psi(\imath, \pi, \lambda)$] then Ψ_*^+ and Ψ_* are mutually inverse isomorphisms. Consequently we have the order isomorphism Im $\Psi \simeq \mathrm{BLT}(S)$. □

Corollary 2

The relation $\sim$ *defined on* BrCon S *by*

$$\vartheta \sim \varphi \iff \vartheta|_{\mathrm{I}} = \varphi|_{\mathrm{I}}\,,\ \vartheta|_{S^\circ} = \varphi_{S^\circ},\ \vartheta|_\Lambda = \varphi|_\Lambda$$

is a dual closure equivalence. The smallest element in the $\sim$ *-class of* ϑ *is* $\Psi\Psi^+(\vartheta)$.

Proof. Since Ψ is residuated, $\Psi\Psi^+$ is a dual closure on BrCon S and the equality $\Psi^+ = \Psi^+\Psi\Psi^+$ gives

$$\vartheta \sim \varphi \iff \Psi^+(\vartheta) = \Psi^+(\varphi) \iff \Psi\Psi^+(\vartheta) = \Psi\Psi^+(\varphi)\,.$$

Also, the equality $\Psi\Psi^+\Psi = \Psi$ gives Im Ψ = Im $\Psi\Psi^+$. It follows by Corollary 1 that the fixed points of the dual closure $\Psi\Psi^+$ are precisely the elements of $\overline{\mathrm{Con}}\ S$. If $\vartheta \in$ BrCon S then the smallest element in the $\sim$-class of ϑ relative to this dual closure is clearly $\Psi\Psi^+(\vartheta)$. □

Corollary 3

There is a lattice isomorphism $\overline{\mathrm{Con}}\ S \simeq (\mathrm{BrCon}\ S)/\sim$. □

As Corollary 1 above shows, every $\vartheta \in$ Con S determines uniquely, and is uniquely determined by, a balanced linked triple. Moreover, given $\pi \in$ Con S°, there is a balanced linked triple whose middle component is π if and only if π can be extended to a $^\circ$-congruence on S; and a similar statement holds for a given $\imath \in$ Con I or $\lambda \in$ Con Λ.

It is instructive at this juncture to give an example of a congruence on S° that does not extend to a $^\circ$-congruence on S.

EXAMPLE 1: Let $\mathrm{Sing}_{2\times2}\mathbb{R}$ be the semigroup of singular real 2×2 matrices and let $\mathrm{Sing}^*_{2\times2}\mathbb{R}$ be the subsemigroup of those matrices whose leading element (i.e. that in the (1,1)-position) is non-zero. Observe that $\mathrm{Sing}^*_{2\times2}\mathbb{R}$ consists of matrices of the form

$$\begin{bmatrix} a & b \\ c & a^{-1}bc \end{bmatrix}$$

where $a, b, c, \in \mathbb{R}$ with $a \neq 0$. Let M be the set $\mathrm{Sing}^*_{2\times 2}\mathbb{R}$ with the 2×2 zero matrix adjoined. Then, as is shown in [1, Example 9], M is a regular semigroup and if we define

$$\begin{bmatrix} a & b \\ c & a^{-1}bc \end{bmatrix}^\circ = \begin{bmatrix} a^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^\circ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then the subset

$$M^\circ = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}; x \neq 0 \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

is a inverse transversal of M. If M^1 denotes M with the 2×2 identity matrix adjoined then an inverse transversal of M^1 is $(M^1)^\circ = (M^\circ)^1$. Consider the partition of $(M^1)^\circ$ with classes

$$\left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}; x \neq 0 \right\} \cup \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

Clearly, this defines a congruence $\equiv$ on $(M^1)^\circ$. However, $\equiv$ has no extension that is a congruence on M^1, hence no extension that is a $^\circ$-congruence on M^1. To see this, suppose that there is such an extension which we denote also by $\equiv$. Observe that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gives, on multiplication on the right by $\begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix}$, the equivalence $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix}$. Thus in particular, for every x, $\begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ x-1 & 0 \end{bmatrix}$. Multiplying on the left by $\begin{bmatrix} x & -1 \\ 0 & 0 \end{bmatrix}$, we obtain the contradiction $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Definition. We shall say that $\pi \in \mathrm{Con}\, S^\circ$ is *special* if it has an extension in $\overline{\mathrm{Con}}\, S$; equivalently, if it is the middle component of some balanced linked triple.

The set of special congruences on S° will be denoted by $\mathrm{SpCon}\, S^\circ$

Theorem 5

$\pi \in \mathrm{Con}\, S^\circ$ is special if and only if

$$(x_1, x_2) \in \pi \Rightarrow (\forall i \in \mathrm{I})(\forall l \in \Lambda) \quad \big((lx_1i)^\circ, (lx_2i)^\circ\big) \in \pi\,.$$

Proof. If π is special let $\imath \in \mathrm{Con}\, \mathrm{I}$ and $\lambda \in \mathrm{Con}\, \Lambda$ be a such that $(\imath, \pi, \lambda) \in \mathrm{BLT}(S)$. For all $i \in \mathrm{I}$, $x = x^{\circ\circ} \in S^\circ$, $l \in \Lambda$ we have, since I is left regular and Λ is right regular,

$$lxi = lxx^\circ xi = xx^\circ lxx^\circ xix^\circ x = x \cdot x^\circ lx \cdot x^\circ \cdot xix^\circ \cdot x\,.$$

If now $(x_1, x_2) \in \pi$ then it follows, using properties $(\delta), (\epsilon)$ and the fact that the restrictions of $\Psi(\imath, \pi, \lambda)$ to I, S°, Λ are $\imath, \pi, \lambda$ respectively, that

$$(lx_1 i, lx_2 i) \in \Psi(\imath, \pi, \lambda)\,,$$

and consequently $\big((lx_1 i)^\circ, (lx_2 i)^\circ\big) \in \pi\,.$

Conversely, suppose that π satisfies the above condition and consider the relation $\hat{\pi}$ defined on S by

$$(a, b) \in \hat{\pi} \iff (\forall i \in \mathrm{I})(\forall l \in \Lambda) \quad \big((lai)^\circ, (lbi)^\circ\big) \in \pi\,.$$

Clearly, we have $\pi \subseteq \hat{\pi}|_{S^\circ}$. Given $(a, b) \in \hat{\pi}$ and $x \in S$ we have, for all $i \in \mathrm{I}$ and $l \in \Lambda$,

$$(laxi)^\circ = (xi)^\circ \big(laxi(xi)^\circ\big)^\circ \stackrel{\pi}{\equiv} (xi)^\circ \big(lbxi(xi)^\circ\big)^\circ = (lbxi)^\circ;$$

$$(lxai)^\circ = \big((lx)^\circ lxai\big)^\circ (lx)^\circ \stackrel{\pi}{\equiv} \big((lx)^\circ lxbi\big)^\circ (lx)^\circ = (lxbi)^\circ\,.$$

Consequently $(ax, bx) \in \hat{\pi}$ and $(xa, xb) \in \hat{\pi}$, so we have that $\hat{\pi} \in \mathrm{Con}\, S$.

Observe now that

$$(a, b) \in \hat{\pi} \Rightarrow (a^\circ, b^\circ) \in \pi\,.$$

In fact, if $(a, b) \in \hat{\pi}$ then taking $i = e \in E(S^\circ)$ and $l = f \in E(S^\circ)$ in the definition of $\hat{\pi}$ we obtain $(e^\circ a^\circ f^\circ, e^\circ b^\circ f^\circ) \in \pi$. Choosing in particular $e^\circ = a^\circ a^{\circ\circ}$ and $f^\circ = a^{\circ\circ} a^\circ$, we have $(a^\circ, a^\circ a^{\circ\circ} b^\circ a^{\circ\circ} a^\circ) \in \pi$ whence $(a^{\circ\circ}, a^{\circ\circ} b^\circ a^{\circ\circ}) \in \pi$. Interchanging a and b we have likewise $(b^{\circ\circ}, b^{\circ\circ} a^\circ b^{\circ\circ}) \in \pi$ whence $(b^\circ, b^\circ a^{\circ\circ} b^\circ) \in \pi$. Since S°/π is an inverse semigroup we see, on passing to quotients, that $[b^\circ] = [a^{\circ\circ}]^{-1} = [a^\circ]$ and hence that $(a^\circ, b^\circ) \in \pi$.

It follows from this implication that $\hat{\pi}|_{S^\circ} \subseteq \pi$, whence $\hat{\pi}|_{S^\circ} = \Pi$, and that $\hat{\pi} \in \overline{\mathrm{Con}}\, S$. Hence π is special. □

Corollary 1

Given $\pi \in \mathrm{SpCon}\, S^\circ$ there is a biggest $\vartheta \in \overline{\mathrm{Con}}\, S$ that corresponds to a balanced linked triple of the form $(-, \pi, -)$, namely the relation $\hat{\pi}$ defined on S by

$$(a, b) \in \hat{\pi} \iff (\forall i \in \mathrm{I})(\forall l \in \Lambda) \quad \big((lai)^\circ, (lbi)^\circ\big) \in \pi\,.$$

Proof. If π is special let $\imath \in \mathrm{Con}\, \mathrm{I}$ and $\lambda \in \mathrm{Con}\, \Lambda$ be such that $(\imath, \pi, \lambda) \in \mathrm{BLT}(S)$. For all $i_1, i_2 \in \mathrm{I}$ we have, by (β),

$$(i_1, i_2) \in \imath \Rightarrow (\forall i \in \mathrm{I})\ (i_1 i, i_2 i) \in \imath \Rightarrow (\forall i \in \mathrm{I})(\forall l \in \Lambda)\ \big((li_1 i)^\circ), (li_2 i)^\circ\big) \in \pi$$

which shows that $\imath \subseteq \hat{\pi}|_{\mathrm{I}}$. Similarly we have $\lambda \subseteq \hat{\pi}|_\Lambda$. It follows that

$$(\imath, \pi, \lambda) \leq \big(\hat{\pi}|_{\mathrm{I}}, \hat{\pi}|_{S^\circ}, \hat{\pi}|_\Lambda\big) = \Psi^+(\hat{\pi})$$

and therefore $\Psi(\imath, \pi, \lambda) \subseteq \Psi\Psi^+(\hat{\pi}) = \hat{\pi}$, whence the result follows. □

Corollary 2

If $\pi \in \mathrm{SpCon}\, S^\circ$ then the biggest balanced linked triple with middle component π is $(\hat{\pi}|_{\mathrm{I}}, \pi, \hat{\pi}|_\Lambda)$.

Theorem 6

The mapping $\Phi_{S^\circ} : \overline{\text{Con}}\ S \to \text{SpCon}\ S^\circ$ *given by* $\Phi_{S^\circ}(\vartheta) = \vartheta|_{S^\circ}$ *is surjective and residuated, with residual* $\Phi^+_{S^\circ}$ *given by* $\Phi^+_{S^\circ}(\pi) = \hat{\pi}$.

Proof. Clearly, both Φ_{S° and $\Phi^+_{S^\circ}$ are isotone. For every $\pi \in$SpCon S° we have

$$\Phi_{S^\circ}\Phi^+_{S^\circ}(\pi) = \hat{\pi}|_{S^\circ} = \pi$$

so that $\Phi_{S^\circ}\Phi^+_{S^\circ} = \text{id}$; and for every $\vartheta \in \overline{\text{Con}}\ S$ we have, by Corollary 1 of Theorem 5,

$$\Phi^+_{S^\circ}\Phi_{S^\circ}(\vartheta) = \widehat{\vartheta|_{S^\circ}} \geq \vartheta$$

so that $\Phi^+_{S^\circ}\Phi_{S^\circ} \geq \text{id}$. Hence Φ_{S° is surjective and residuated with residual $\Phi^+_{S^\circ}$. □

Corollary

The relation $\equiv_{S^\circ}$ *defined on* $\overline{\text{Con}}\ S$ *by*

$$\vartheta \equiv_{S^\circ} \varphi \iff \vartheta|_{S^\circ} = \varphi|_{S^\circ}$$

is a closure equivalence. The biggest element in the $\equiv_{S^\circ}$ *-class of* ϑ *is* $\widehat{\vartheta|_{S^\circ}}$. *Moreover, there is a lattice isomorphism* SpCon $S^\circ \simeq (\overline{\text{Con}}\ S)/\equiv_{S^\circ}$.

Proof. Since Φ_{S° is residuated, $\Phi^+_{S^\circ}\Phi_{S^\circ}$ is a closure with associated equivalence $\equiv_{S^\circ}$. Moreover, since Φ_{S° is surjective we have that $\Phi^+_{S^\circ}$ is injective, and therefore SpCon $S^\circ \simeq \text{Im}\ \Phi^+_{S^\circ} = \text{Im}\ \Phi^+_{S^\circ}\Phi_{S^\circ}$, the set of closed elements. □

Given now $\imath \in$ Con I, consider the relation $\hat{\imath}$ defined on S by

$$(a,b) \in \hat{\imath} \iff (\forall x \in S)\ \big(ax(ax)^\circ, bx(bx)^\circ\big) \in \imath\,.$$

We observe in passing that in this definition the range of the quantifier can be reduced to I. In fact, if $\big(ai(ai)^\circ, bi(bi)^\circ\big) \in \imath$ for all $i \in$ I then for every $x \in S$ we have

$$\big(ax(ax)^\circ, bx(bx)^\circ\big) = \big(axx^\circ(axx^\circ)^\circ, bxx^\circ(bxx^\circ)^\circ\big) \in \imath\,.$$

Theorem 7

If $\imath \in$ Con I *then* $\hat{\imath}|_{\text{I}} = \imath$.

Proof. For all $i, j \in$ I we have $ji = ji(ji)^\circ$ and so, on the one hand, $(i_1, i_2) \in \imath$ implies $(i_1, i_2) \in \hat{\imath}$, whence $\imath \subseteq \hat{\imath}|_{\mathrm{I}}$. On the other hand, if $(i_1, i_2) \in \hat{\imath}|_{\mathrm{I}}$ then for all $i \in \mathrm{I}$ we have $(i_1 i, i_2 i) \in \imath$. Taking $i = i_1$ we obtain $(i_1, i_2 i_1) \in \imath$ which, on left multiplication by i_1, gives $(i_1, i_1 i_2) \in \imath$; and taking $i = i_2$ we obtain $(i_1 i_2, i_2) \in \imath$. Hence $(i_1, i_2) \in \imath$, and so we have the reverse inclusion $\hat{\imath}|_{\mathrm{I}} \subseteq \imath$. □

As the following example shows, not every congruence on I extends to a congruence on S.

Example 2: Relative to the inverse transversal M° of example 1 we have

$$\mathrm{I} = \left\{ \begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix} ; x \in \mathbb{R} \right\} \cup \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

Consider the partition of I whose classes are:

$$\left\{ \begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix} ; x \in \mathbb{R} \right\} \text{ and } \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$

Clearly, this defines a congruence $\equiv$ on I. However, $\equiv$ has no extension that is a congruence on M, hence no extension that is a $^\circ$-congruence on M. To see this, suppose that there is such an extension which we denote also by $\equiv$. Observe that $\begin{bmatrix} 1 & 0 \\ x & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ x-1 & 0 \end{bmatrix}$ gives, on multiplication on the left by $\begin{bmatrix} x & -1 \\ 0 & 0 \end{bmatrix}$, the contradiction $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Definition. We shall say that $\imath \in \mathrm{Con}\,\mathrm{I}$ is *special* if it an extension in $\overline{\mathrm{Con}}\ S$; equivalently, if it is the first component of some balanced linked triple.

The set of special congruences on I will be denoted by SpCon I.

Theorem 8

If $\imath \in \mathrm{Con}\,\mathrm{I}$ *the following statements are equivalent:*

(1) $\imath \in \mathrm{SpCon}\,\mathrm{I}$*;*

(2) $(i, j) \in \imath \Rightarrow (\forall x \in S)\big(xi(xi)^\circ, xj(xj)^\circ\big) \in \imath$*;*

(3) $\hat{\imath} \in \overline{\mathrm{Con}}\ S$*.*

Proof. (1) ⇒ (2): If $\imath$ is special then it is the first component of some balanced linked triple and so, for every $x \in S$, we have

$$\begin{aligned} i \stackrel{\imath}{\equiv} j &\Rightarrow\ x^\circ xi(x^\circ xi)^\circ \stackrel{\imath}{\equiv} x^\circ xj(x^\circ xj)^\circ \quad \text{by } (\alpha) \\ &\Rightarrow\ x^{\circ\circ}x^\circ xi(x^\circ xi)^\circ x^\circ \stackrel{\imath}{\equiv} x^{\circ\circ}x^\circ xj(x^\circ xj)^\circ x^\circ \quad \text{by } (\delta) \\ &\Rightarrow\ xi(xi)^\circ = xx^\circ x^{\circ\circ}x^\circ xi(x^\circ xi)^\circ x^\circ \stackrel{\imath}{\equiv} xx^\circ x^{\circ\circ}x^\circ xj(x^\circ xj)^\circ x^\circ = xj(xj)^\circ\,. \end{aligned}$$

(2) $\Rightarrow$ (3): It is clear that $\hat\imath$ is a right congruence. If $(a,b) \in \hat\imath$ then $\big(ax(ax)^\circ, bx(bx)^\circ\big) \in \imath$ whence (2) gives, for every $y \in S$,

$$yax(yax)^\circ = yax(ax)^\circ\big(yax(ax)^\circ\big)^\circ \stackrel{\imath}{\equiv} ybx(bx)^\circ\big(ybx(bx)^\circ\big)^\circ = ybx(ybx)^\circ$$

and so $\hat\imath$ is also a left congruence.

To see that $\hat\imath \in \overline{\mathrm{Con}}\, S$, let $(a,b) \in \hat\imath$. Then

$$\big(aa^\circ, ba^\circ a^{\circ\circ}b^\circ\big) = \big(aa^\circ(aa^\circ)^\circ, ba^\circ(ba^\circ)^\circ\big) \in \imath\,,$$

from which it follows on the one hand by Theorem 1 that

$$(1') \qquad \big(a^{\circ\circ}a^\circ, b^{\circ\circ}a^\circ a^{\circ\circ}b^\circ\big) \in \imath\,,$$

and on the other hand, taking $x = b^{\circ\circ}b^\circ$ in (2) using the fact that I is left regular and Λ is right regular, that $(b^{\circ\circ}b^\circ aa^\circ, b^{\circ\circ}a^\circ a^{\circ\circ}b^\circ) \in \imath$. It follows by Theorem 1 that

$$(2') \qquad \big(a^{\circ\circ}a^\circ b^{\circ\circ}b^\circ, b^{\circ\circ}a^\circ a^{\circ\circ}b^\circ\big) \in \imath\,.$$

We deduce from $(1')$ and $(2')$ that $(a^{\circ\circ}a^\circ, a^{\circ\circ}a^\circ b^{\circ\circ}b^\circ) \in \imath$. In a similar way we can show that $(b^{\circ\circ}b^\circ, b^{\circ\circ}b^\circ a^{\circ\circ}a^\circ) \in \imath$. Since $E(S^\circ)$ is a semilattice, it follows that

$$(3') \qquad (a^{\circ\circ}a^\circ, b^{\circ\circ}b^\circ) \in \imath.$$

We now have, using $(3')$ and (δ),

$$(4') \qquad (a^\circ a^{\circ\circ}, a^\circ b^{\circ\circ}b^\circ a^{\circ\circ}) = (a^\circ a^{\circ\circ}a^\circ a^{\circ\circ}, a^\circ b^{\circ\circ}b^\circ a^{\circ\circ}) \in \imath\,.$$

Since $\hat\imath$ is a congruence we have that $(a^\circ a, a^\circ b) \in \hat\imath$, whence

$$\big(a^\circ ab^\circ b^{\circ\circ}(a^\circ ab^\circ b^{\circ\circ})^\circ, a^\circ bb^\circ b^{\circ\circ}(a^\circ bb^\circ b^{\circ\circ})^\circ\big) \in \imath$$

from which, using Theorem 1 again, we obtain

$$(5') \qquad \big(a^\circ a^{\circ\circ}b^\circ b^{\circ\circ}, a^\circ b^{\circ\circ}b^\circ a^{\circ\circ}\big) \in \imath.$$

It follows from (4′) and (4′) that $(a^\circ a^{\circ\circ}, a^\circ a^{\circ\circ} b^\circ b^{\circ\circ}) \in \imath$. Similarly, we have $(b^\circ b^{\circ\circ}, b^\circ b^{\circ\circ} a^\circ a^{\circ\circ}) \in \imath$. Since $E(S^\circ)$ is a semilattice it follows that

$$(a^\circ a^{\circ\circ}, b^\circ b^{\circ\circ}) \in \imath\,. \tag{6'}$$

Combining (3′), (6′) and the hypothesis that $(a,b) \in \hat{\imath}$ we obtain

$$(a^{\circ\circ}, b^{\circ\circ}) = \big(a^{\circ\circ}a^\circ \cdot a \cdot a^\circ a^{\circ\circ}, b^{\circ\circ}b^\circ \cdot b \cdot b^\circ b^{\circ\circ}\big) \in \hat{\imath}|_{S^\circ}\,.$$

It now follows by Theorem 1 that $(a^\circ, b^\circ) \in \hat{\imath}$ and hence that $\hat{\imath} \in \overline{\text{Con}}\ S$.

(3) ⇒ (1) : This is immediate from Theorem 7. □

Corollary

Given $\imath \in \text{SpCon I}$ *there is a biggest* $\vartheta \in \overline{\text{Con}}\ S$ *that corresponds to a balanced linked triple of the form* $(\imath, -, -)$, *namely* $\hat{\imath}$.

Proof. Suppose that $\vartheta \in \overline{\text{Con}}\ S$ is such that $\vartheta|_{\text{I}} = \imath$. If $(a,b) \in \vartheta$ then for every $x \in S$ we have $(ax, bx) \in \vartheta$ whence $\big((ax)^\circ, (bx)^\circ\big) \in \vartheta$ and therefore

$$\big(ax(ax)^\circ, bx(bx)^\circ\big) \in \vartheta|_{\text{I}} = \imath$$

which gives $(a,b) \in \hat{\imath}$. Hence $\vartheta \subseteq \hat{\imath}$. □

Theorem 9

The mapping $\Phi_{\text{I}} : \overline{\text{Con}}\ S \to \text{SpCon I}$ *given by* $\Phi_{\text{I}}(\vartheta) = \vartheta|_{\text{I}}$ *is surjective and residuated with residual* Φ_{I}^+ *given by* $\Phi_{\text{I}}^+(\imath) = \hat{\imath}$.

Proof. Given $\imath \in \text{SpCon I}$ we have, by Theorem 8, $\hat{\imath} \in \overline{\text{Con}}\ S$. Also, by Theorem 7, $\hat{\imath}|_{\text{I}} = \imath$. It is clear that both Φ_{I} and Φ_{I}^+ are isotone. Now since, for every $\imath \in \text{SpCon I}$

$$\Phi_{\text{I}}\Phi_{\text{I}}^+(\imath) = \Phi_{\text{I}}(\hat{\imath}) = \hat{\imath}|_{\text{I}} = \imath$$

we have $\Phi_{\text{I}}\Phi_{\text{I}}^+ = \text{id}$. Also, for every $\vartheta \in \overline{\text{Con}}\ S$, it follows by Theorem 8 that

$$\Phi_{\text{I}}^+\Phi_{\text{I}}(\vartheta) = \Phi_{\text{I}}^+(\vartheta|_{\text{I}}) = \widehat{\vartheta|_{\text{I}}} \supseteq \vartheta\,,$$

so $\Phi_{\text{I}}^+\Phi_{\text{I}} \geq \text{id}$. Hence Φ_{I} is surjective and residuated with residual Φ_{I}^+. □

Corollary

The relation $\equiv_{\text{I}}$ *defined on* $\overline{\text{Con}}\ S$ *by*

$$\vartheta \equiv_{\text{I}} \varphi \iff \vartheta|_{\text{I}} = \varphi|_{\text{I}}$$

is a closure equivalence. The biggest element in the $\equiv_{\text{I}}$*-class of* ϑ *is* $\widehat{\vartheta|_{\text{I}}}$. *Moreover, there is a lattice isomorphism* $\text{SpCon I} \simeq (\overline{\text{Con}}\ S)/\equiv_{\text{I}}$.

We can of course consider likewise special congruences on Λ. In so doing we obtain dual results to Theorem 7, 8, 9.

We recall now that an inverse transversal S° is said to be *multiplicative* [2] if $\Lambda \mathrm{I} = E(S^\circ)$. When S° is multiplicative, certain simplifications arise. For example, in this case we have $li = (li)^\circ$ for all $l \in \Lambda,\ i \in \mathrm{I}$ whence it follows immediately that every $\vartheta \in \mathrm{Con}\ S$ braided, so that $\mathrm{BrCon}\ S = \mathrm{Con}\ S$. Combining this observation with Corollaries 2, 3 of Theorem 4, we obtain:

Theorem 10

Let S be a regular semigroup with an inverse transversal S°. If S° is multiplicative then $\overline{\mathrm{Con}}\ S \simeq (\mathrm{Con}\ S)/\sim$ where $\sim$ is the dual closure equivalence given by

$$\vartheta \sim \varphi \Longleftrightarrow \vartheta|_\mathrm{I} = \varphi|_\mathrm{I},\ \vartheta|_{S^\circ} = \varphi|_{S^\circ},\ \vartheta|_\Lambda = \varphi|_\Lambda\,.$$

Definition. The elements of $\mathrm{SpCon\ I} \times \mathrm{SpCon}\ S^\circ \times \mathrm{SpCon}\ \Lambda$ will be called *special triples*.

We shall denote the set of balanced special triples by $\mathrm{BSpT}(S)$. Clearly, we have the inclusion $\mathrm{BLT}(S) \subseteq \mathrm{BSpT}(S)$. As the following result shows, when S° is multiplicative the reverse inclusion holds.

Theorem 11

Let S be regular semigroup with an inverse transversal S°. If S° is multiplicative then every balanced special triple is a balanced linked triple.

Proof. Suppose that $(\imath, \pi, \lambda) \in \mathrm{BSpT}(S)$ and consider the balanced linked triples that correspond to $\hat\imath, \hat\pi, \hat\lambda \in \overline{\mathrm{Con}}\ S$, namely

$$(\imath, \hat\imath|_{S^\circ}, \hat\imath|_\Lambda),\ (\hat\pi|_\mathrm{I}, \pi, \hat\pi|_\Lambda),\ (\hat\lambda|_\mathrm{I}, \hat\lambda|_{S^\circ}, \lambda)\,.$$

Observe that

(1) $\pi \subseteq \hat\imath|_{S^\circ}$ and $\pi \subseteq \hat\lambda|_{S^\circ}$

In fact, if $(a,b) \in \pi|_{S^\circ} = \pi$ then by (δ), for every $i \in \mathrm{I}$ we have $(aia^\circ, bib^\circ) \in \hat\pi|_\mathrm{I}$. Since S° is in particular a quasi-ideal, i.e. $S^\circ S S^\circ \subseteq S^\circ$ [3], we have $aia^\circ \in E(S^\circ)$ and therefore $(aia^\circ, bib^\circ) \in \hat\pi|_{E(S^\circ)} = \pi|_{E(S^\circ)} = \imath|_{E(S^\circ)}$ whence it follows that $(a,b) \in \hat\imath|_{S^\circ}$. Thus we see that $\pi \subseteq \hat\imath|_{S^\circ}$. Similarly, using (ε), we have $\pi \subseteq \hat\lambda|_{S^\circ}$.

(2) $\imath \subseteq \hat\pi|_\mathrm{I}$ and $\lambda \subseteq \hat\pi|_\Lambda$.

In fact, if $(i_1, i_2) \in \imath$ then, for every $i \in \mathrm{I}$, we have $(i_1 i, i_2 i) \in \imath$ and so, by (β), for every $l \in \Lambda$ we have $\big((li_1 i)^\circ, (li_2 i)^\circ\big) \in \pi \subseteq \hat\imath|_{S^\circ}$ by (1). Since S° is multiplicative this gives $((li_1i)^\circ,(li_2i)^\circ) \in \hat{\imath}|_{E(S^\circ)} = \imath|_{E(S^\circ)} = \pi|_{E(S^\circ)}$ whence $(i_1,i_2) \in \hat{\pi}|_{\mathrm{I}}$. Thus we see that $\imath \subseteq \hat{\pi}|_{\mathrm{I}}$; and similarly $\lambda \subseteq \hat{\pi}|_\Lambda$.

(3) $\imath \subseteq \hat{\lambda}|_{\mathrm{I}}$ and $\lambda \subseteq \hat{\imath}|_\Lambda$.

In fact, if $(i_1,i_2) \in \imath$ then, by (γ), for every $l \in \Lambda$ we have $((li_1)^\circ li_1,(li_2)^\circ li_2) \in \hat{\imath}|_\Lambda$. Since S° is multiplicative this gives $((li_1)^\circ li_1,(li_2)^\circ li_2) \in \hat{\imath}|_{E(S^\circ)} = \imath|_{E(S^\circ)} = \lambda|_{E(S^\circ)}$ whence $(i_1,i_2) \in \hat{\lambda}|_{\mathrm{I}}$. Thus we see that $\imath \subseteq \hat{\lambda}|_{\mathrm{I}}$. Similarly, using (α), we have $\lambda \subseteq \hat{\imath}|_\Lambda$.

It now follows from (1), (2), (3) that

$$(\imath,\pi,\lambda) = (\imath,\hat{\imath}|_{S^\circ},\hat{\imath}|_\Lambda) \wedge (\hat{\pi}|_{\mathrm{I}},\pi,\hat{\pi}|_\Lambda) \wedge (\hat{\lambda}|_{\mathrm{I}},\hat{\lambda}|_{S^\circ},\lambda) \in \mathrm{BLT}(S)$$

as required. □

Corollary 1

If S° is multiplicative then $\overline{\mathrm{Con}}\ S \simeq BSpT(S)$

Proof. This follows from Corollary 1 of Theorem 4. □

Corollary 2

If S° is multiplicative and $(\imath,\pi,\lambda) \in BLT(S)$ then $\Psi(\imath,\pi,\lambda) = \hat{\imath} \cap \hat{\pi} \cap \hat{\lambda}$.

Corollary 3

If S° is multiplicative and $\vartheta \in \overline{\mathrm{Con}}\ S$ then $\vartheta = \widehat{\vartheta|_{\mathrm{I}}} \cap \widehat{\vartheta|_{S^\circ}} \cap \widehat{\vartheta|_\Lambda}$.

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