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Non–Standard dynamical systems.
Shadows of trajectories and abstract rivers

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Abstract

Non Standard Analysis is used to modelize some phenomenons of orbit concentration in continuous flows; the objects we introduce are studied from a topological point of view, emphasizing the notion of S-monotonicity of an orbit along its shadow, for which we give some general theorems.

0. Introduction

When looking at pictures representing trajectories of dynamical systems plotted by computers, one can often notice some phenomenons which are very evident to the eye, but rather uneasy to define precisely in mathematical terms. For example, let us have a look at Figure 1, which shows some solutions of the differential system

$$\dot{x} = 1, \ \dot{y} = 4(x^2 + y^2 - 1). \tag{1}$$

That system does not possess any critical point, nor any limit set, and in spite of appearances, the orbits induce a topologically trivial foliation of the plane. However,

Figure 1

everyone can see that all the trajectories which pass through a large portion of the plane bunch together along two very narrow zones, evoking circular arcs.

Taking for example the upper “arc”, the trajectories in its surroundings seem to gather drastically, disposing themselves very much like the tributaries of a stream flowing to the left — while the lower “arc” looks like a stream flowing to the right; we can remark that, while the solutions of (1) always move to the right, we do not need to know that at all to determine the direction in which those “rivers” seem to flow.

In Figure 2, whichs orbits of the equation $\dot{y} = -xy + y^2 + 1$, the same kind of phenomenon can be seen, but it takes place in unbounded regions of the plane (and can be related to asymptotic properties of some solutions — see the discussion at the beginning of section 2). Many other examples can be found in [1] and [2].

The attempts to find a mathematical definition of that behaviour using classical notions such as divergence, curvature etc. have failed so far; besides, in the description we made above, one cannot give a precise meaning to such expressions as “very narrow zones”, “gather drastically” etc. But in the theory of Non Standard Analysis one can give a rigorous meaning to such terms as “infinitely narrow

Figure 2

regions", "trajectories infinitely close together" and so on. One of the useful roles of N.S.A. is to provide, in precise and manageable terms, qualitative models (in the same meaning as mathematical models of a physical phenomenon) for some rather imprecise facts, especially those involving great variations of some orders of size – most often by "infinitely exagerating" some of their features.

At this stage, the relation between the phenomenon and its non-standard model is of a metamathematical nature. Yet, let us come back to our example and replace system (1) by a one-parameter family

$$\dot{x} = 1,\ \dot{y} = a(x^2 + y^2 - 1). \tag{2}$$

Taking several values of the parameter, one can see that the larger a is taken, the more accentuated the phenomenon becomes (see for example Figure 3, in which $a = 10$). It is therefore very natural to choose as non standard model a member of the family (2) with an infinitely large value of a. The properties of that non-standard differential system may be interpreted in classical terms as asymptotic properties of the family (2) when a tends to infinity (which is often related, as in the case of our example, to singular perturbations).

As far as we know, the class of asymptotic behaviours one obtains by that mean has not yet been registered by "classical" mathematicians. However we point out that, independently, the Russian mathematician S. Samborsky ([15]) wrote some papers about rivers, with different definitions but in a state of mind rather close to ours. One finds out that the presence of rivers in the non standard model can

Figure 3

be characterized by the topological behaviour, not of the trajectories themselves (in the example given, the foliation remains topologically trivial for all values of a) but of their shadows. One will see that the notion of shadow of a curve plays a central role in this paper – and we had to study it for its own sake.

In the first part of this paper, we state some general results about the relation between a (generally non-standard) arc of curve and its shadow. The second part deals with continuous dynamical systems: we give some definitions of "abstract rivers" and state some properties of those objects. Note that the results of Section 1 are not used in Section 2 before 2.5.

Preliminaries: about I.S.T.

The Non Standard background of this paper is that of the axiomatic theory I.S.T. (Internal Set Theory) settled by E. Nelson in [14].

Let we recall some of its main features: rather than adding new elements to classical structures, by ultraproducts and other model-theoretic techniques (in the style of A. Robinson), one adds a new predicate **st**() (for "standard") to the language of ZF Set Theory; its usage is ruled by 3 new axiom schemes (added to ZF Theory), namely **Idealization**, **Standardization** and **Transfert** (whence "**I.S.T.**"!).

For more details, see [11] and [14]. However we emphasize the following notions, which will be the most useful here:

— the **standardized set** of an external or internal set A is the unique standard set having exactly the same standard elements as A; its existence is asserted by the axiom of Standardization; we denote it by ${}^{st}A$.

— in a metric space E, the **halo** (elsewhere called **monad**) of a point x (resp. a subset A) of E is the (generally exernal) set of all points of E infinitely close to x (resp. A). We denote it by $\mathrm{hal}(x)$ (resp. $\mathrm{hal}(A)$).

— the **shadow** of a point x or a subset A is the standardized set of its halo. In the case of a point x it is either empty, or (if x is near standard) composed of the unique standard point infinitely close to x. That point is denoted by ${}^\circ x$, and is also called the standard part of x. The shadow of a set A is also denoted by ${}^\circ A$; if A is an internal set, then ${}^\circ A$ is a closed set.

More generally, we call pre-halo a set of the form

$$H = \left\{ \bigcap A_n / n \in \mathbb{N} \text{ and } \mathrm{st}(n) \right\},$$

and pre-galaxy a set of the form

$$G = \left\{ \bigcup A_n / n \in \mathbb{N} \text{ and } \mathrm{st}(n) \right\},$$

A_n being an internal sequence of sets. A halo is an external pre-halo, a galaxy is an external pre-galaxy. The proofs will often use the so called permanence principles, which are based on the fact that certain sets cannot be internal (such as the halo of a near standard point, or the set of all standard points of an infinite internal set), and also that a halo cannot be equal to a galaxy. From a syntactical point of view, haloes and galaxies are the simplest exernal sets, because their definition can be written using just one external quantifier ($\forall^{st}$ or $\exists^{st}$), followed by an internal formula (with only ordinary quantifiers); those external sets can be "mastered" thanks to the permanence principles, while the others (whose definition requires at least two external quantifiers of different kind, such as $\forall^{st}\exists^{st}$ or $\exists^{st}\forall^{st}$) are much harder to deal with (cf. [11, 16, 19] and part III of [17]).

1. Trace of a path along its shadow

Given an arc γ, in $\mathbb{R}^n$ for example, its shadow Γ is a standard set such that every near standard point of γ is infinitely close to some point of Γ, and even generally to an infinite number of such points. Now if $m(t)$ represents the continuous motion of a point on γ, one sometimes would like to speak about "the trace of that motion on its shadow", but this is not a well defined notion; the motion of the point induces a motion of its shadow $^\circ(m(t))$, but the function $t \to ^\circ(m(t))$ is external, and has no good properties of continuity and "intermediary values". Moreover reasoning about those external functions usually involves complicated formulas, full of external quantifiers, which forbids the use of Permanence principles; on the other hand, the standardized mapping ^{st}m has no reason to be continuous, or to remain infinitely close to m , while the shadow $^\circ m$ of m (relatively to the uniform metric) often has no reason to exist at all.

So it may be useful, but sometimes not trivial, to get an internal continuous mapping, remaining infinitely close to the given motion. We now state two theorems about the existence of such mappings, under some very simple topological hypotheses about Γ ; this is done in sections 1.1 and 1.2. 1.3 gives an elementary application, while in 1.4 and 1.5 we show how these theorems may be used to simplify some definitions, whose external complexity (from the point of view of external quantifiers) they permit to reduce, so that the only external sets involved are haloes and galaxies.

1.1 Case of a limited arc

We state a preliminary result:

Proposition 1

Let X be a connected limited subset of $\mathbb{R}^n$; if $^\circ X$ is locally connected, then it is connected and locally arcwise connected.

Proof. It is known (cf. [11]) that the shadow of a connected set whose all points are near standard is a connected compact set. We conclude by the following result, due to Mazurkiewicz (cf. [13] for example): *Every metric compact set which is connected and locally connected is locally arcwise connected.* □

The main result of this section is:

Theorem 1

Let I be a compact interval in $\mathbb{R}$, γ a continuous path from I into $\mathbb{R}^n$, and Γ the shadow of $\gamma(I)^*$. If Γ is compact and locally connected, then there exists a continuous path π from I into Γ such that $(\forall t \in I)\, \pi(t) \simeq \gamma(t)$.

Moreover one can choose π so that, if $I = [a,b]$, one gets $\pi(a) = \overset{\circ}{}(\gamma(a))$ and $\pi(b) = \overset{\circ}{}(\gamma(b))$.

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* loosely speaking, we shall say that Γ is the shadow of γ.

Remarks.

- The condition "Γ compact" is equivalent to "γ limited".
- Γ is not always one-dimensional, even when γ is a homeomorphism: for example let $I = [-1, 1]$ and $\gamma(t) = (t, sin(\omega t))$, with ω infinitely large; we obtain $\Gamma = [-1, 1]^2$.
- The mapping π is generally not onto Γ.
- It is striking that the conditions of existence of π only involve the topological properties of the shadow Γ, and not those of γ itself.

Proof. Let us denote $I = [a, b]$ and $A = {}^{\circ}(\gamma(a))$, $B = {}^{\circ}(\gamma(b))$. It is sufficient to show that for any standard positive ε, there exists a continuous mapping $\pi : I \longrightarrow \Gamma$ such that $\pi(a) = A$, $\pi(b) = B$ and

$$(\forall t \in I)\, d(p(t), g(t)) < \varepsilon.$$

That statement is internal, so by permanence principle there will exist an $\varepsilon \simeq 0$ with the same properties, which is the announced result. So, for the rest of the proof let us take a fixed standard positive ε. After Proposition 1, Γ is locally arcwise connected; so to each $x \in \Gamma$ we can associate a neighbourhood $V(x)$, with diameter less than ε, such that $V(x) \cap \Gamma$ is arcwise connected, and an open neighbourhood $U(x)$ such that $\overline{U(x)} \subset V(x)$. By the axiom of Transfert, we can assume that $x \longrightarrow U(x)$ and $x \longrightarrow V(x)$ are standard mappings (so that whenever x is standard, $U(x)$ and $V(x)$ are standard sets).

Lemma 1

There exists a finite sequence (t_i) on I, $a = t_0 < t_1 < \cdots < t_{p-1} < t_p = b$, such that for all $i \in \{0, \ldots, p-1\}$, there exists some $x_i \in \Gamma$ such that $\gamma([t_i, t_{i+1}]) \subset U(x_i)$.

Remark. p is finite but not necessarily standard.

Proof of the lemma: First of all we show that the open covering $\{U(x)/x \in \Gamma\}$ of Γ is also a covering for $\gamma(I)$: for if $t \in I$, $\gamma(t)$ is near standard; let then x be its standard part; x is a standard point of Γ, so $U(x)$ is a standard open set containing x, which implies that it contains hal(x), and particularly $\gamma(t)$. So $\left\{\gamma^{-1}(U(x))/x \in \Gamma\right\}$ is an open covering of the compact set I.

After Lebesgue's Lemma there exists an $\alpha > 0$ (may be infinitesimal) such that every sub-interval of I whose length is less than α is included in one of those open sets. To prove the lemma, we just have to choose $p > (b - a)/\alpha$ and take $t_i = a + (b - a)i/p$. $\square$

Now we resume the proof of the theorem, and define the path π piecewise. Remark that for each $i = 1, \ldots, p$, ${}^{\circ}(\gamma(t_i))$ belongs to Γ and to $\overline{U(x_{i-1})} \cap \overline{U(x_i)}$, which proves that $V(x_{i-1}) \cap V(x_i) \cap \Gamma \neq \emptyset$. To each $i = 1, \ldots, p$ we can then associate (by an internal function) an element $a_i \in V(x_{i-1}) \cap V(x_i) \cap \Gamma$; we also set $a_0 = A$, $a_p = B$. In each$V(x_i) \cap \Gamma$, which is arcwise connected, there exists a path π_i from a_i to a_{i+1}; moreover for $t \in [t_i, t_{i+1}]$, $d(\gamma(t), \pi_i(t)) < \varepsilon$, as $\delta(V(x_i)) < \varepsilon$. Linking all those paths together, we obtain the desired mapping π. $\square$

EXAMPLES:

- Let $\gamma(t) = (t, f(t))$, where f is a continuous and S-continuous[*] function from $\mathbb{R}$ to $\mathbb{R}$. Then Γ is the graph of the standardized function ${}^{\circ}f$, and one can take $\pi(t) = (t, {}^{\circ}f(t))$.
- If $\gamma(t) = (t, \sin(\omega t))$ (ω an infinitely large real), then $\Gamma = [0,1] \times [-1,1]$ and one can take $\pi = \gamma$ (as, more generally, whenever $\gamma(I) \subset \Gamma$).
- We leave to the reader the construction of π when

$$\gamma(t) = (t, (1+\varepsilon)\sin(\omega t)).$$

COUNTEREXAMPLE: A case where Γ is arcwise connected, but not locally arcwise connected, and where there exists no path π with the required properties.

Let γ be the path suggested by Figure 4, where $A = (-1, -2)$, $B = (-1, 0)$, $C = (\varepsilon, 0)$ (with $\varepsilon = 1/2\pi\omega$), ω an infinitely large integer), $D = (1, \sin(1))$, $E = (1, -2)$, $F = (0, -2)$, and:

— the arc CD is the graph of $y = \sin(1/x)$ for $x \in [\varepsilon, 1]$;

— the arc BC is a part of Ox ;

— the arcs AB and DE come infinitely close to F.

The shadow of the arc BD is not arcwise connected (it is composed with $[-1,0] \times \{0\}$, the graph of $y = \sin(1/x)$ for $x \in [0,1]$, and the segment $\{0\} \times [-1,1]$) and in Γ, there is no path from hal(B) to hal(D) keeping infinitely close to $\gamma(t)$ along the arc BD.

As an exercise, we leave it to the reader to find examples of paths γ such that π exists, but Γ is not locally connected.

1.2 The non-limited case

If the arc γ is not limited, Γ needs not be compact, and not even connected: taking for example $\gamma(t) = (t, \omega(1-t^2))$ (ω infinitely large) for $t \in [-1, 1]$, the shadow Γ is composed of the half lines $\{1\} \times \mathbb{R}^+$ and $\{-1\} \times \mathbb{R}^+$, and in Γ there is no path π such that $(\forall t \in I)\, \pi(t) \simeq \gamma(t)$ (not even for the t where $\gamma(t)$ is limited). So the theorem we shall state here has a slightly weaker conclusion than the former one.

We first state a standard result:

[*] that is, $x \simeq y \Rightarrow f(x) \simeq f(y)$; a notion due to E. Nelson.

Figure 4

Proposition 2

Let K be a compact connected metric space, and a a point in K. If every point of $K \setminus \{a\}$ has a basis of connected neighbourhoods, then K is locally connected.

Proof. Obviously we just have to prove that a itself has a basis of connected neighbourhoods. Suppose not; then there exists a closed ball $X = \overline{B(a,r)}$ such that no connected neighbourhood of a is contained in X.

First we show that the ball $Y = B(a, r/2)$ intersects infinitely many components of X: were it not the case, we could denote $C_0, \dots, C_p$ those components, supposing for example $a \in C_0$; surely $p \geq 1$, for Y, being a neighbourhood of a included in X, cannot be connected; the C_i would be closed, so $d(a, C_i) > 0$ for $1 \leq i \leq p$; let α be the smallest of those distances: then C_0 would contain $B(a, \alpha)$, which would be a connected neighbourhood of a included in X.

Now, K being connected, each component of X intersects the boundary of X; so all those components intersecting Y also intersect the sphere $\Sigma = S(a, r/2)$. Let then $(y_n)_{n \geq 0}$ be a sequence of points of Σ, each one in a different component of X, and let y be a limit point for that sequence (Σ is compact).

Of course $y \neq a$; nevertheless y has no basis of connected neighbourhoods; indeed X itself (which is a neighbourhood of y) does not contain any connected neighbourhood of y, for every neighbourhood V of y contains infinitely many of the y_n, so that V intersects infinitely many components of X. $\square$

Theorem 2

Let I be a compact interval of $\mathbb{R}$, γ a path from I into $\mathbb{R}^n$, and Γ the shadow of $\gamma(I)$. If Γ is locally connected, there exists a mapping π from I into Γ such that whenever $t \in I$ and $\gamma(t)$ is near standard, π is continuous at t and $\pi(t) \simeq \gamma(t)$.

Proof. In view of Theorem 1, we may assume that γ is non-limited. We compactify the space $\mathbb{R}^n$ by a standard homeomorphism

$$G : \mathbb{R}^n \longrightarrow S^n \setminus \{\Omega\}$$

(Ω an arbitrarily chosen point of the n-dimensional sphere S^n), with $G(x) \to \Omega$ when $x \to \infty$, and reciprocally (for example G is the centered projection on S^n with center Ω). Remark that the shadow of $G(x)$ is $G({}^\circ x)$ if x is near standard in $\mathbb{R}^n$, while it is Ω if x is non-limited.

The shadow Δ of $G(\gamma(I))$ in S^n is a connected compact set (for $G(\gamma(I))$ is connected), and by the above remark $\Delta = G(\Gamma) \cup \{\Omega\}$. So every point of $\Delta \setminus \{\Omega\}$ is the image of a point of Γ by G, and as Γ is locally connected every point of $\Delta \setminus \{\Omega\}$ has a basis of connected neighbourhoods in Δ, so by Proposition 2 Δ is locally connected; now we can apply Theorem 1 to the path $G \circ \gamma$: there exists a continuous mapping $\rho : I \longrightarrow \Delta$ such that $(\forall t \in I)\, \rho(t) \simeq G \circ \gamma(t)$.

Then π may be defined as follows: we choose an arbitrary point A of Γ and set $\pi(t) = A$ if $\rho(t) = \Omega$, $\pi(t) = G^{-1} \circ \rho(t)$ otherwise.

If $\gamma(t)$ is near standard, $G \circ \gamma(t)$ is not infinitely close to Ω, so neither is $\rho(t)$: then on a neighbourhood of t we have $\pi = G^{-1} \circ \rho$, so π is continuous at t; also, as G^{-1} is a standard continuous mapping and $\rho(t) \simeq G \circ \gamma(t)$, we get $\pi(t) \simeq \gamma(t)$. $\square$

1.3 Example of application

Here and in the following subsection, we show how Theorem 1 is useful by reducing some problems to elementary properties of real-valued functions.

Proposition 3

Let $\gamma : I \longrightarrow \mathbb{R}^n$ be a path whose shadow Γ is a simple compact arc (that is, homeomorphic to an interval of $\mathbb{R}$). Then every standard subarc of Γ is the shadow of a subarc of γ.

Proof. We use the following characterization of the shadow in the case of a limited set:

Lemma 2

Let X be a limited subset of $E = \mathbb{R}^n$ and δ be the Hausdorff semi-distance on subsets of E. Then the shadow of X is the unique compact standard set Y such that $\delta(X, Y) \simeq 0$.

Proof of the lemma. We have obviously $\delta(X, {}^\circ X) \simeq 0$; on the other hand if two compact standard sets are infinitely close with respect to the Hausdorff metric, they are identical. □

Now let $\pi : I \longrightarrow \Gamma$ be as in Theorem 1, and AB a standard subarc of Γ. If we find u and v on I such that $\pi([u, v]) = AB$, we shall have $\delta(\pi([u, v]), \gamma([u, v])) \simeq 0$, so by the Lemma 2 AB is the shadow of $\gamma([u, v])$.

Let h be an homeomorphism from Γ to an interval J of $\mathbb{R}$, and let $a = h(A)$, $b = h(B)$. As $h \circ \pi$ is continuous there exist $t_1, t_2 \in I$ such that $A = \pi(t_1)$ and $B = \pi(t_2)$, and we have necessarily $h \circ \pi([t_1, t_2]) \subset [a, b]$. We want to find u and v such that one has exactly $h \circ \pi[u, v]) = [a, b]$. For that it is enough to prove the following assertion about real valued functions: *Let f be a continuous real valued function, defined on an interval I of $\mathbb{R}$; for all $a, b \in f(I)$, there exist $u, v \in I$ such that $f([u, v]) = [a, b]$.*

That is elementary proved: suppose for example $a = f(t_1)$, $b = f(t_2)$, $a < b$ and $t_1 < t_2$; just take $u = \sup\{x \in [t_1, t_2]/f(x) = a\}$, then $v = \inf\{x \in [u, t_2]/f(x) = b\}$. □

Remark. To convince oneself of the usefullness of the above theorems, one just has to try to prove that proposition without them.

1.4 S-monotonicity

Here we study the case where $M(t)$ is the motion of a point along a trajectory γ whose shadow Γ is a simple arc, so that the shadow of $M(t)$ moves always in the same direction along Γ.

Definition 1. Let $\gamma : I \longrightarrow \mathbb{R}^n$ be a path whose shadow Γ is a simple arc; γ is **S-monotonic** along Γ if and only if there exists a standard homeomorphism φ from I onto Γ with:

$$(\forall t_1, t_2 \in I)\ t_1 \leq t_2 \Rightarrow \varphi^{-1}({}^\circ\gamma(t_1)) \leq \varphi^{-1}({}^\circ\gamma(t_2)). \tag{3}$$

The following result, whose easy proof is left to the reader, is related with Proposition 3:

Proposition 4

If γ is a compact limited arc which is S-monotonic along its shadow Γ (a simple arc) and a, b are points of γ, then the shadow of the subarc $[a, b]$ of γ is the subarc $[\,{}^\circ a, {}^\circ b]$ of Γ.

Theorem I allows us to get a useful equivalent of definition 1:

Theorem 3

Let $\gamma : I \longrightarrow [a, b] \subset \mathbb{R}^n$ be a path whose shadow Γ is a simple arc. Then γ is S-monotonic along Γ if and only if there exists a one-to-one continuous mapping $\pi : I \longrightarrow \Gamma$ such that $(\forall t \in I)\, \gamma(t) \simeq \pi(t)$.

Proof. If there exists such a mapping, let us show that $t_1 \leq t_2) \Rightarrow \pi^{-1}(\,{}^\circ\gamma(t_1)) \leq \pi^{-1}(\,{}^\circ\gamma(t_2))$ — so that γ is S-monotonic along Γ.

Denoting $A = \pi(a)$, $B = \pi(b)$ the end points of Γ, π induces a continuous order relation on Γ, for which A is the first element and B the last one. Such a relation is unique, so it is standard; therefore we know that, if m and s are standard points of Γ with $\pi^{-1}(m) > \pi^{-1}(s)$, and if m' and s' are points of Γ such that $m' \simeq m$ and $s' \simeq s$, then $\pi^{-1}(m') > \pi^{-1}(s')$. Taking $m = {}^\circ\gamma(t_1)$, $s = {}^\circ\gamma(t_2)$, $m' = \pi(t_1)$ and $s' = \pi(t_2)$, we obtain the announced relation.

Conversely, suppose γ is S-monotonic, and let φ verifying formula (3). According to Theorem 1 there exists a continuous mapping $\rho : I \longrightarrow \Gamma$ such that $(\forall t \in I)\, \gamma(t) \simeq \rho(t)$, with $\rho(a) = A$ and $\rho(b) = B$. Setting $\delta = \varphi^{-1} \circ \rho$, we can conclude by proving the following:

Let $\delta : [0, 1] \longrightarrow [0, 1]$ be a continuous mapping, such that $\delta(0) = 0$, $\delta(1) = 1$ and $t_1 < t_2 \Rightarrow (\delta(t_1) < \delta(t_2)$ or $\delta(t_1) \simeq \delta(t_2))$. Then there exists a monotonic bijective mapping $\lambda : [0, 1] \longrightarrow [0, 1]$ such that

$$(\forall t \in [0, 1])\, \lambda(t) \simeq \delta(t).$$

The proof of that assertion is easy: we define $\lambda_1(t) = \sup\{\rho(s)/s \in [0, t]\}$; λ_1 is continuous and monotonic, but may not be one-to-one; then we take an infinitesimal $\varepsilon > 0$, and set $\lambda(t) = (\lambda_1(t) + \varepsilon t)/(1 + \varepsilon)$. Finally, we just take $\pi = \varphi \circ \lambda$. $\square$

1.5 Domain of S-monotonicity

Now we show another way in which the alernate definition given by Theorem 3 is easier to handle than the first one: it allows us to reduce the syntactical complexity of the definition of certain external sets, so that we can use Permanence principles with them. To compare the "external complexity" of both definitions of S-monotonicity, let us examine how they look like in the language of I.S.T. :

Initial definition (Two external quantifiers of different kinds):
$(\exists^{st}\varphi : I \longrightarrow \Gamma$ one-to-one continuous$)(\forall s_1, s_2, t_1, t_2 \in I)$
$[(\exists^{st} a, b \in \Gamma) a \neq b$ and $\varphi(s_1) = a$ and $\varphi(s_2) = b$ and
$(\forall^{st}\varepsilon > 0)[d(\gamma(t_1), a) < \varepsilon$ and $d(\gamma(t2), b) < \varepsilon)]$ and $(t_1 \leq t_2)] \Rightarrow s_1 < s_2$.

Alternate definition (Only one external quantifier):
$(\exists\pi : I \longrightarrow \Gamma$ one-to-one continuous$)(\forall t \in I)(\forall^{st}\varepsilon > 0)\ d(\gamma(t), \pi(t)) < \varepsilon$.

If $\gamma : I \longrightarrow \mathbb{R}^n$ is a one-to-one path, and $x, y \in \gamma(I)$, we note γ_{xy} the subarc composed of the points of $\gamma(I)$ between x and y. The following lemma will be used in the proof of Theorem 8.

Lemma 3

Let $\gamma : I \longrightarrow \mathbb{R}^p$ be a one-to-one path whose shadow Γ is a simple arc; for each $x \in \gamma(I)$ we set

$$H_x = \{y \in \gamma(I)/\gamma_{xy} \text{ is S-monotonic}\}.$$

Then H_x is a pre-halo, and ${}^\circ H_x$ is a closed subarc of Γ.

Proof. Take a such that $x = \gamma(a)$ and define the following internal sequence of sets:

$$A_n = \{\gamma(b)/(\exists\pi : \mathbb{R} \longrightarrow \Gamma \text{ contin. biject.})(\forall t \in [a, b]) d(\pi(t), \gamma(t)) < 1/n\}.$$

Then in view of Theorem 3, we have $H_x = \bigcap\{A_n)/\ \mathrm{st}(n)\}$, which shows that H_x is a pre-halo.

Now we show that ${}^\circ H_x$ is a closed subarc of Γ. Setting $\gamma_x^+ = \{\gamma(t)/t \in I \text{ and } t \geq a\}$ (resp. $\gamma_x^- = \{\gamma(t)/t \in I \text{ and } t \leq a\}$) and $H_x^+ = H_x \cap \gamma_x^+$ (resp. $H_x^- = H_x \cap \gamma_x^-$) we just have to show that H_x^+ and H_x^- are closed; let us show it for H_x^+.

Γ being a simple arc, there exists on it a total ordering for which ${}^\circ H_x^+$ is a sub-interval; that interval takes one of the forms $[x, z_0)$ or $[x, z_0]$, with z_0 a standard point of Γ; we have to exclude the former possibility, and for that we show that if ${}^\circ H_x^+$ contains $[x, z_0)$, then $z_0 \in {}^\circ H_x^+$.

Setting $\Gamma_x^+ = \{y \in \gamma_x^+/\gamma_{xy} \cap \mathrm{hal}(z_0) = \emptyset\}$, we have $\Gamma_x^+ = \bigcup\{B_n/\ \mathrm{st}(n)\}$ where

$$B_n = \left\{y \in \gamma_x^+/d(z_0, \gamma_{xy}) > \frac{1}{n}\right\}.$$

It is easy to show that G_x^+ is external, so it is a galaxy; as $G_x^+ \subset H_x^+$, that inclusion is strict by Permanence principle, whence there exists $y \in H_x^+$ such that $\gamma_{xy} \cap \mathrm{hal}(z_0) = \emptyset$, which shows that $z_0 \in {}^\circ H_x^+$. □

2. Abstract rivers

Let us come back to Figure 1 (in the introduction of this paper) and try to be more accurate about the following question: why does the upper "arc of circle" look like a river running to the left? Getting a slightly nearer view of the orbit diagram, take A and B two disjoint portions of the narrow zone on which the trajectories gather together, as in Figure 5 below; the reason why we feel that A is upstream from B is that, not only the trajectories passing across A come through B but also "drains" those passing across a large zone around A. We shall offer a definition of a Non Standard relation (also called "upstream") between the points of the space to modelize that behaviour, and define rivers as standard arcs linearly ordered by that relation.

Figure 5

A usual notion in Non Standard dynamics is that of **slow-fast** vector field; when such fields are defined on the plane, the external set of the points in which the field is not of infinitely large modulus is usually contained in the halo of a so called **slow curve**; one can notice in the litterature that slow curves are most often unions of rivers. However the notion of rivers that we define is only concerned with the organization of the orbits, no matter the speed at which they are run along by the solutions (that is why we chose to place our study in the context of continuous flows).

Besides, the term of *river* is already used by M. & F. Diener, F. Blais and I.P. Van Den Berg ([4, 7, 9, 8, 10, 17, 18]), in the context of a class of standard vector fields in the plane (including the standard polynomial vector fields).

We must specify the connections between those rivers and ours — the former ones can be called **natural rivers** and ours **abstract rivers** when one wants to distinguish: the notion of *natural rivers* describes the asymptotic behaviour of some trajectories at the infinity by the fact that, under an appropriate Non Standard linear change of variables (called **macroscope**) one obtains a slow-fast vector field in which those trajectories follow the halo of a certain slow curve; those slow curves are *abstract rivers*, and the change of variables is in fact a (very rich) particular case of the method we evoked in the introduction as "adding a parameter to accentuate the phenomenon".

As the above authors point it out, note that natural rivers do not deal with the case of bounded rivers (like in the example of system we gave in the introduction) nor with spaces of higher dimensions.

To summarize, natural rivers concern Standard equations and can be given a Standard definition, in terms of polynomial growth; while abstract rivers concern Non Standard equations, their definition is essentially topological, and to give them a Standard definition, one is obliged to consider parametrized families of equations (such a Standard definition is given in the Appendix; compare its complexity with the Non Standard definition!).

In [2] we gave various examples of Non Standard changes of variables in Standard vector fields giving rise to abstract rivers (with slightly different definitions for limited shadows and rivers, but the present ones also work for those examples).

2.1 Settling the landscape

From now on, E is a standard open subset of $\mathbb{R}^n$ (n standard), and φ a continuous flow on E, that is a continuous mapping from $E \times \mathbb{R}$ into $\mathbb{R}$ verifying, for all x in E, $\varphi(x, 0) = x$ and for all t, s in $\mathbb{R}$, $\varphi(x, t+s) = \varphi(\varphi(x, t), s)$. We shall write also $\varphi_x(t)$ for $\varphi(x, t)$.

Often, the flow will be associated to a $\mathcal{C}^1$ vector field $V : E \longrightarrow \mathbb{R}^n$. Then φ will be differentiable, and we shall have

$$\frac{\partial}{\partial t}(x, t) = V(f(x, t)).$$

The flows and fields we shall consider will be internal, but most of the time non standard.

Through every $x \in E$ passes a unique trajectory $\gamma_x = \{f(x,t)/t \in \mathbb{R}\}$; three different cases may arise:

1. φ_x is constant: $\gamma_x = \{x\}$ (x is a singularity),
2. φ_x is one-to-one; in case $E = \mathbb{R}^2$, γ_x is a simple arc (homeomorphic to a line),
3. φ_x is periodic and non constant: γ_x is homeomorphic to the circle S^1.

(cf for example [3] Theorems 1.9, 1.13 and 2.14).

2.1.1 Arc γ_{xy}

Suppose φ_x is non constant, and let $y \in \gamma_x$, $y \neq x$. In case 2 (φ_x one-to-one), we denote by γ_{xy} the unique subarc of γ_x with ends x and y. In case 3 (φ_x periodic), there are exactly two such subarcs, that we shall denote γ^1_{xy} and γ^2_{xy}.

If φ_x is constant, of course we cannot have $y \neq x$, but we shall take $\gamma_x = \{x\}$.

2.1.2 Limited shadow

Let $\mathcal{G}$ be the principal galaxy of E (that is the external set of all near standard points of E). If $x \in \mathcal{G}$ the following external set will be called the limited shadow of x:

$$LS(x) = \{ {}^\circ x\} \text{ if } \varphi_x \text{ is constant;}$$

$$LS(x) = \bigcup \{ {}^\circ\gamma_{xy}/y \in \gamma_x \text{ and } \gamma_x y \subset \mathcal{G}\} \text{ if } \varphi_x \text{ is one-to-one;}$$

$$LS(x) = \bigcup \{ {}^\circ\gamma^i_{xy}/y \in \gamma_x \text{ and } \gamma^i_{xy} \subset \mathcal{G},\ i = 1,2\} \text{ if } \varphi_x \text{ is periodic.}$$

For completeness we shall take $LS(x) = \emptyset$ when x is not in $\mathcal{G}$. In other words, ${}^\circ y \in LS(x)$ means that ${}^\circ x$ and ${}^\circ y$ are in the same connected component of ${}^\circ\gamma_x$, and the solution is not "lost from sight" between x and y.

EXAMPLES: Figures 6 and 7 suggest various situations (the region below the dotted line represents the principal galaxy).

Remarks.

- $LS(x)$ is always contained in $\mathcal{G}$.
- ${}^\circ x \in LS(x)$ but generally $x \notin LS(x)$.
- If $y \in LS(x)$, ${}^\circ y \in LS(x)$ but generally $LS(y) \neq LS(x)$.
- If $y \in \gamma_x$ and $g_{xy} \subset \mathcal{G}$, ${}^\circ y \in LS(x)$ and $LS(y) = LS(x)$.

2.1.3 Draining

Let $X \subset \mathcal{G}$ and $y \in \mathcal{G}$. We shall say that y **drains** X iff $(\forall x \in X) y \in LS(x)$.

${}^{\circ}y \in LS(x)$

Figure 6

${}^{\circ}y \notin LS(x)$

Figure 7

Remarks.

- Every standard point drains its own halo.
- If φ is standard and x_0 is a standard isolated sink or source, then it drains a standard open set (the set of all points whose orbit has x_0 as α- or ω-limit point).
- If φ is standard, $y \in \gamma_x$ and $\gamma_{xy} \subset \mathcal{G}$, then ${}^{\circ}y$ drains $\mathrm{hal}(x)$.
- If y drains X and z drains the halo of y, then z drains X.

In the case of a $\mathcal{C}^1$ standard flow, it is easy to show that only points infinitely close to a singularity may drain an open non-void standard set. This is not the case for non standard flows, in which we shall be mostly interested in the following.

2.1.4 Upstream, downstream

We denote by $\mathcal{A}$ the standard relation* which, for standard x and y, satisfies:

$$x \, \mathcal{A} \, y \Longleftrightarrow y \text{ drains a standard neighbourhood of } x.$$

In such a case we shall say that x is **upstream** from y, or that y is **downstream** from x.

We shall note $\mathcal{A}_=$ for "$\mathcal{A}$ or $=$ ".

Being a standard relation, $\mathcal{A}$ is uniquely determined by its values on pairs of standard points; we point out that it is defined by standardization of an external

* $\mathcal{A}$ from the french word "Amont", i.e. upstream.

relation, so that *a priori* its precise meaning is known only for standard points; we may have to work it out for non standard ones.

$\mathcal{A}$ is a transitive relation: it is sufficient to check it for standard pairs of points, and that results from the third remark in 2.1.3. However in most cases it is neither reflexive, nor symmetric, nor anti-symmetric, except when restricted to special subsets, as in the following.

2.1.5 Rivers

A **river** (or **abstract river**) F is a standard arc in E, which possesses a continuous parametrization $\psi : I \longrightarrow F$ (I an interval of $\mathbb{R}$), such that

$$(\forall t_1, t_2 \in I)\ t_1 < t_2 \Longleftrightarrow y(t_1)\,\mathcal{A}\,y(t_2).$$

We shall say that such a parametrization goes **down** the river; it is necessarily one-to-one, and by Transfert principle we can always choose it standard. If $x, y \in F$ and $x\,\mathcal{A}\,y$ we shall say that F **flows from** x **to** y.

A **maximal river** is a river which is not contained in any longer one. Of course any river is a part of a maximal one.

The definition implies that the restriction of $\mathcal{A}$ to F is a strict total ordering; that last property is not sufficient for what we want to describe, for that ordering must also coincide with a natural (continuous) one on the arc F; in [2] we exhibited an example where that was not the case.

EXAMPLES:

1. Let $E = \mathbb{R}^2$, φ the flow associated with $dy/dx = -y/\varepsilon$ ($\varepsilon \simeq 0$, positive). The axis Ox is a river; each point $(x, 0)$ drains any open set of the form $]a, b[\times]c, d[$, $a < b \ll x$*. So $x_1 < x_2 \Rightarrow (x_1, 0)\,\mathcal{A}\,(x_2, 0)$, and the river flows to the right.
2. A well known case: Liouville equation $dy/dx = 1/\varepsilon(y^2 - x)$. On Figure 8, we have $a\,\mathcal{A}\,b\,\mathcal{A}\,s$, $b\,\mathcal{A}\,c$, $d\,\mathcal{A}\,e$.
   We find two rivers:
   $F_1 = \{(t^2, t)/t \geq 0\}$, which flows to the left;
   $F_2 = \{(t^2, t)/t < 0\}$, which flows to the right.

Remarks.

- Points such as c are downstream from every point of $F_1 \setminus \mathrm{hal}(s)$, but do not belong to any river.
- Note that the opposite field ($dx/dt = -1, dy/dt = t - y^2$) not only has the same rivers, but those rivers flow in the same direction (F_1 to the left, F_2 to the right),

---

* We note $a \gg b$ (resp $a \ll b$) for $a > b$ and not $a \simeq b$ (resp. $a < b$ and not $a \simeq b$).

Figure 8

as the direction of flowing depends only on the arrangement of trajectories as set of points, not on the way they are described by the flow.

3. **Confluence**: it may happen that two distinct maximal rivers have a common part; in all the explicit cases we know, that common part is a downstream section of both rivers — and in the case of monotonic rivers (a notion to be defined in section 2.5) Theorem 8 easily implies that it always holds. Figure 9 represents such a situation, for the equation (!) :

$$\frac{dy}{dx} = -\left[\pi + \arctan(\omega(y^2 - x^2)) + \arctan(\omega x)\right] . \arctan(\omega y) \ (\omega \simeq +\infty).$$

2.1.6 Invariance

We indicate under what sort of transformations the above notions are invariant.

Definition 2. A mapping $H : E \longrightarrow F$ is an **S-homeomorphism** iff:

- H is bijective;
- $(\forall x, y \text{ limited})\, x \simeq y \Leftrightarrow H(x) \simeq H(y)$;
- $(\forall x \in E)\, x \text{ limited } \Leftrightarrow H(x) \text{ limited}$.

Typically, a standard homeomorphism is an S-homeomorphism. A notion invariant under every S-homeomorphism is an **S-topologic** notion. We show that the notions we introduced so far are S-topologic.

We recall that if φ is a flow on E, and H a bijection from E onto E, one denotes by $y = H_*\varphi$ the flow defined by $\psi(x,t) = H \circ \varphi(H^{-1}(x), t)$.

Figure 9

Proposition 5

If $H : E \longrightarrow E$ is an S-homeomorphism and φ a flow on E then:

- *if x, y are standard points of E, such that $x\,\mathcal{A}\,y$ for φ, then for $H_*\varphi$ one has $^\circ H(x)\,\mathcal{A}\,^\circ H(y)$;*
- *if F is a river for φ, then $^\circ H(F)$ is a river for $H_*\varphi$.*

The proof is straightforward.

2.2 Conditions of existence

2.2.1 An existence theorem

Often, at the place where they come infinitely close to a river, the trajectories undergo a sudden change of direction ; the following theorem states condition for the existence of such rivers in the case of a differential equation in the real plane.

Theorem 4

Consider the differential equation $y' = f(x, y)$, where f is a $\mathcal{C}^1$ mapping from $\mathbb{R}^2$ into $\mathbb{R}$. Suppose there exists a differentiable standard function $g : I \longrightarrow \mathbb{R}$ (I an open interval of $\mathbb{R}$), U a standard open set containing the graph of g, a and b real valued functions on I such that for all x in I:

1. *$a(x) \ll g'(x) \ll b(x)$;*
2. *if (x, y) U, $y \gg g(x) \Rightarrow f(x, y) < a(x)$ and $y \ll g(x) \Rightarrow f(x, y) > b(x)$.*

Then the graph of g is a river flowing to the right—i.e

$$x_1 < x_2 \Rightarrow (x_1, g(x_1))\,\mathcal{A}\,(x_2, g(x_2)).$$

Remark. If we change condition 2) in an obvious way we obtain a river flowing to the left.

Proof. After the change of variable $Y = y - g(x)$ we are reduced to the case $g = 0$. The conditions become, for all $x \in I$:

$$a(x) \ll 0 \ll b(x)$$

and, if $(x, y) \in U$, $y \gg 0 \Rightarrow f(x, y) < a(x)$ and $y \ll 0 \Rightarrow f(x, y) > b(x)$.

Figure 10

Let $\alpha = \sup\{a(x)/x \in I\}$ and $\beta = \inf\{b(x)/x \in I\}$; we have $\alpha \ll 0 \ll \beta$ (even if a and b are not continuous ! This is merely an application of a Permanence principle).

We show that $I \times 0$ is a river flowing to the right: it is enough to prove that if x_1 and x_2 are distinct standard points of I, with $x_1 < x_2$, then $(x_2, 0)$ drains a standard neighbourhood of $(x_1, 0)$: by compacity of $[x_1, x_2]$, U contains an open standard rectangle $R = (c, d) \times (-h, h)$ with $h > 0$ and $[x_1, x_2] \subset (c, d) \subset I$. Let W be the open set bounded by the lines:

$$y = \pm h,\ x = c,\ y = a(x - x_2),\ y = b(x - x_2)$$

(cf. Figure 10).

W contains a standard neighbourhood of $(x_1, 0)$, and on its boundary, except in $\mathrm{hal}(x_2, 0)$, the field points towards the interior of W. So, a trajectory passing through a point $M \in W$ is obliged to get out through $\mathrm{hal}(x_2, 0)$ — which shows that $(x_2, 0) \in LS(M)$. So finally $(x_2, 0)$ drains W. $\square$

2.2.2 Example of application: a bounded river

Consider the equation:

$$\frac{dy}{dx} = \arctan\left(\frac{1}{\varepsilon}(x^2 - y)\right).$$

If $y \gg x^2$, $dy/dx \simeq -\pi/2$; if $y \ll x^2$, $dy/dx \simeq \pi/2$. We set $g(x) = x^2$, and, choosing a positive standard number s, we set $a(x) = -\pi/2 + s$, $b(x) = \pi/2 - s$. Condition 2) of the Theorem 4 holds for all x; on the other hand, condition 1) holds only when $2x \in] - \pi/2 + s, \pi/2) - s[$.

As s may be chosen arbitrarily small, the theorem states the existence of a river along the open parabolic arc $AB = \{(x, x^2)/x \in] - \pi/4, \pi/4[\}$.

Furthermore in the present case, one can easily show that the river may be extended to the closed arc AB, but not beyond it: here we get a compact maximal river. There are essentially four different types of trajectories, as represented on Figure 11. Similarly, Figure 12 shows orbits of the equation $\dot{y} = \arctan(20(x^4 - y))$.

Figure 11 *Figure 12*

2.2.3 Tangentially approached rivers

As we pointed out, the rivers whose existence is stated by Theorem 4 have a particular aspect; other cases may arise, for example rivers upon which the shadows of trajectories arrive smoothly, being tangent to the river, and without sudden change of slope.

For example we can obtain such a river by regular infinitesimal perturbation of an equation which does not possess the unicity property:

Let g be the inverse function of $f(x) = x(x^2+\varepsilon)$; the equation (4) $dy/dx = g(y)$ is obtained by a perturbation of (5) $dy/dx = y^{1/3}$; g is $1/\varepsilon$ -lipschitzian, so (4) has the unicity property.

Using the Short Shadow Lemma ([11]) outside the halo of Ox (where (4) and (5) are locally lipschitzian), and the fact that $g(y)$ has the sign of y, one easily shows that every trajectory of (4) passing through a limited point has a standard trajectory of (5) as its shadow. Some of them remain in the halo of Ox ; for the others, there exists a standard real number x_0 such that $x \geq x_0 \Rightarrow \simeq \pm 2/3(x - x_0)^{3/2}$, and $x \leq x_0 \Rightarrow \simeq 0$. So, Ox is a river with tangential approach. Figure 13 shows the aspect of the orbits.

Figure 13

More accurately, one can show that if $x \ll x_0$, $y = \pm(1 + ø)\varepsilon \exp((x - x_0)/\varepsilon)$ ($ø$ stands for any infinitesimal number); so that, going to the left inside the halo of the river, the trajectories get closer to each other with exponential speed.

2.3 In midstream

We give here some precisions about the meaning of the relation $\mathcal{A}$ for standard and non standard points of a river.

Proposition 6

Let F be a river and x, y standard points of F; then $x\,\mathcal{A}_=\,y$ if and only if y drains hal(x).

Proof. If y drains a standard neighbourhood of x, it drains its halo. Reciprocally, suppose $x \neq y$; if $x\,\mathcal{A}\,y$ is false, then $y\,\mathcal{A}\,x$, so x drains a standard neighbourhood U of y; if y would drain hal(x), y would also drain U (cf. 2.1.3) so we should have $y\,\mathcal{A}\,y$, which is impossible. □

Remark. If the points x, y are not both standard, it may happen that $x\,\mathcal{A}\,y$ but y does not drain hal(x). For example, in the field defined by $dx/dt = x$, $dy/dt = y/\varepsilon$ ($\varepsilon \simeq 0$), which possesses a saddle point at $(0,0)$, the half line $\{(x,0)/x > 0\}$ is a river flowing to the right, but the point $(1,0)$ does not drain the halo of $(\varepsilon, 0)$. Besides, even when y is near standard, it may happen that the river cannot be extended down to ${}^\circ y$; take for example a river F flowing to the left, on the graph of $y = \sin(1/x)\,(x > 0)$: a point $(\varepsilon, sin(1/\varepsilon))$ is in F, but its shadow is not. Such a point is downstream from any standard point of F.

The following proposition shows that those complications cannot happen in midstream of a river.

Proposition 7

Let F be a river, x and y points of F, such that there exist two standard points x_1 and y_1 on F with $x_1\,\mathcal{A}\,x\,\mathcal{A}\,y\,\mathcal{A}\,y_1$. Then:

- ${}^\circ x \in F$, ${}^\circ y \in F$ *and* ${}^\circ x\,\mathcal{A}_=\,{}^\circ y$;
- *if* ${}^\circ x \neq {}^\circ y$, *then* ${}^\circ y$ *drains a standard neighbourhood of* x.

Proof. Let $\psi : I \longrightarrow F$ be a standard continuous parametrization going down F; there exist s_1, s, t_1, t such that $s_1 < s < t < t_1$ and $x_1 = \psi(s_1)$, $x = \psi(s)$, etc. Necessarily, s_1 and t_1 are standard, so ${}^\circ s$ and ${}^\circ t$ are in I, and as ψ is standard continuous, $\psi({}^\circ s) = {}^\circ x$ and $\psi({}^\circ t) = {}^\circ y$. So ${}^\circ x$ and ${}^\circ y$ are in F, and while ${}^\circ s \leq {}^\circ t$, we have ${}^\circ x\,\mathcal{A}_=\,{}^\circ y$.

Furthermore, if ${}^\circ x \neq {}^\circ y$ then ${}^\circ x\,\mathcal{A}\,{}^\circ y$, so ${}^\circ y$ drains a standard neighbourhood of ${}^\circ x$ — which is also a neighbourhood of x. □

2.4 Source and mouth

Now we study the upstream and downstream ends of a river. Let F be a river and $x \in F$. We note $F^x = \{y \in F / x\,\mathcal{A}_=\,y\}$ the part of F downstream from x.

If x and y are on F, with $x\,\mathcal{A}_=\,y$, we note $F^x_y = \{z \in F / x\,\mathcal{A}_=\,z\,\mathcal{A}_=\,y\}$ the part of F between x and y.

Proposition 8

Let F be a river, x_0 a standard point of F and $x \simeq x_0$. Then $F^{x_0} \subset {}^{st}LS(x) \subset {}^\circ\gamma_x$.

Proof. F^{x_0} being standard, it is sufficient for the first inclusion to check that if y_0 is a standard point of F^{x_0}, then $y_0 \in LS(x)$; but that is obvious, for y_0 drains the halo of x_0; the second inclusion is evident. Note that sometimes those inclusions may be strict ones. □

Remark. If x and y are distinct points of F, then F^x_y is homeomorphic to a compact interval of $\mathbb{R}$; that is due to the fact that any parametrization going down F is an homeomorphism, when restricted to F^x_y (by Proposition 7 when x and y are standard, then by transfer in the other cases).

The following result shows that such a parametrization remains an homeomorphism on the whole of any part of F downstream a given point (compare with Theorem 1.25 of [3]) .

Theorem 5 (going downstream)

On a river, the part downstream from any given point is homeomorphic, either to a compact interval or to a closed half line.

Proof. Let x_0 be a point of F. By Transfer Principle, we can assume that x_0 is standard. If there exists a point z_0 which is most downstream on F, then $F^{x_0} = F^{x_0}_{z_0}$, so by the above remark F^{x_0} is homeomorphic to a compact interval.

Otherwise, we can always assume that $F^{x_0} = \psi([a, +\infty))$ (ψ a parametrization going down F). Let us show that ψ is bicontinuous on that interval: it is so on any interval $[a, b]$, $b > a$; if it was not so on $[a, +\infty)$, there would exist $x_1 \in F^{x_0}$, limit point of $\psi(t)$ when $t \to +\infty$; in the present case there exists y strictly downstream from x_1; we can choose x_1 and y standard, so that y drains a standard open neighbourhood V of x_1.

But by the choice of x_1 there exists a point $z \in V$ strictly downstream from y; then V would be a standard neighbourhood of z, drained by y, and one would have $y\,\mathcal{A}\,z\,\mathcal{A}\,y$, which is impossible on a river (see Figure 14). □

On the other hand, a whole river is not always homeomorphic to an interval of $\mathbb{R}$; for example consider the flow associated to the system defined (in polar coordinates) by:

$$\frac{d\rho}{dt} = \frac{1}{\varepsilon}(1-\rho), \quad \frac{d\theta}{dt} = (1-\rho\cos\theta)^2.$$

It possesses a river on the unit circle, flowing counterclockwise from $(0,1)$ (excluded) down to $(0,1)$ (included).

Besides, the downstream part of a river may not be closed, as in the second example of following Proposition 6.

Figure 14

2.4.1 Definitions

A river F is said to be **closed downstream** (resp. **compact downstream**) if for a point x of F, F^x is closed (resp. compact).

As F_y^x is compact for all $x, y \in F$, those properties do not depend on the choice of x in F; particularly it may be chosen standard. F is compact downstream if and only if it has a most downstream point; by transfer such a point is necessarily standard.

One says that F **tends to infinity downstream** if for all compact K, $(\exists x \in F)\, F^x \cap K = \emptyset$.

The following proposition is a consequence of Theorem 5.

Proposition 9

A river tends to infinity downstream if and only if it is closed downstream but not compact downstream.

Proof. In view of Theorem 5, it is enough to show that in a metric space E, any mapping $\psi : \mathbb{R}^+ \longrightarrow E$ which is bicontinuous and tends to infinity has a closed image, and that any subset homeomorphic to a half line that is not closed cannot be contained within a compact set — which are easy standard results. □

Theorem 6 (going upstream)

Let F be a river. There exists a point x_0 of F such that:

- *x_0 is upstream from any standard point of F;*
- *$F \subset {}^\circ\gamma_{x_0}$.*

Remark. This result shows that a river may always be described as a part of the shadow of a single trajectory, issued from a point of the river.

Proof. We use a Permanence principle. Setting $\Gamma = \bigcup \{F^x / st(x)\}$, Γ is a pre-galaxy. If F has a point x_0 most upstream, it is standard and $F = F^{x_0}$; in that case $F \subset {}^\circ\gamma_{x_0}$ by Proposition II.4.1.

Otherwise, $(\forall^{st} x \in F)(\exists^{st} y \in F)\, y \,\mathcal{A}\, x$; in that case Γ cannot be an internal set, so it is a galaxy. Then let us set

$$H = \{x \in F / (\forall^{st} z \in F)\, x \,\mathcal{A}\, z \Rightarrow z \in {}^\circ\gamma_x\}.$$

We can also define H as the set of all x such that

$$(\forall^{st} z \in F)(\forall^{st} \varepsilon > 0)\, x \,\mathcal{A}\, z \Rightarrow d(z, \gamma_x) < \varepsilon,$$

which shows that H is a pre-halo.

Moreover $\Gamma \subset H$, for if $x \in \Gamma$ there exists a standard point $y \in F$ upstream from x ; then for any standard z downstream from x on F, we have $x \in F_z^y$, and by Proposition II.4.1 this implies ${}^\circ x \in F_z^y$ and z drains $\mathrm{hal}(x)$, whence $x \in H$.

Now by Permanence Principle Γ cannot be equal to H, so there exists $x \in H \setminus \Gamma$; it is easy to check that such a point has the announced properties. □

2.5 Stability and monotonicity

The property of being a river is not always kept by restriction of the domain E: in the example presented in Figure 15, the axis Ox is a river flowing to the right, but if we restrict the domain to $E' = \mathbb{R}^+ \times \mathbb{R}$, then $Ox \cap E' = \mathbb{R}^+ \times 0$ is not a river in E'. The points M_1, M_2, M_3 represented on the figure are of the form $(x_1, 1/\omega))$, $(x_2, 2/\omega)$, $(x_3, 0)$, with $x_1 \ll 0 \ll x_2 \ll x_3$. Remark that in E' the relation $(x_2, 0)\,\mathcal{A}\,(x_3, 0)$ is false, while it is true in E.

One can see here that the lack of "stability" of the notion of river is due to the fact that the trajectories change of direction within the halo of the river; a related fact is that, for example, no arc of trajectory issued from M_2 has for shadow the part of the river between $({}^\circ x_2, 0)$ and $({}^\circ x_3, 0)$.

$x'=\arctan(\omega y-1),\, y'=\omega y|y|.$

Figure 15

The purpose of this chapter is to study the special properties of the rivers whose restrictions to "reasonable" domains remain rivers, in relation to the other properties evoked in the above remarks.

2.5.1 Definitions

The restriction of a flow to a subset E' of E may not be a flow, which raises a problem : what is the meaning of the assertion "*$E' \cap F$ is (or is not) a river, relatively to E'*"? Rather than dealing with delicate problems or reparametrization, we think the simplest way is to define notions such as "relative limited shadow", and so on, as follows:

We denote by $\mathcal{G}_{E'} = \{x \in E'/x \text{ is near standard and } {}^\circ x \in E'\}$ the principal galaxy of E' ; then if E' is a part of E and $x \in E'$, the **limited shadow of** x

relatively to E' is

$$LS_{E'}(x) = \bigcup \{ {}^{\circ}\gamma_{xy} / y \in \gamma_x \cap E' \text{ and } \gamma_{xy} \subset \mathcal{G}_{E'} \}$$

in the case where φ_x is one-to-one, and similarly in the other cases (cf. 2.1.2).

In an obvious way, that relative notion induces a relation of *draining relatively to* E' (that we can denote $\mathcal{A}_{E'}$), and the notion of *being a river relatively to* E', just replacing LS by $LS_{E'}$ and $\mathcal{A}$ by $\mathcal{A}_{E'}$ in the definitions of 2.1.

If F is a river, we call **interval** of F a subset F' such that whenever x and z belong to F', we have $(y \in F \text{ and } x \,\mathcal{A}\, y \,\mathcal{A}\, z) \Rightarrow y \in F'$.

We shall say that a river F is **stable** if, for any standard open set $U \subset E$, and any interval F' of F included in U, F' is a river relatively to U.

The examples given in the preceding sections are all stable rivers; so are those obtained by application of Theorem 4. The river in Figure 16 is stable, while those in Figures 15 and 17 are not.

Stable river

$x'=\varepsilon y,\, y'=\varepsilon^2-y^2.$

Figure 16

Also, we shall say that a river F is **monotonic** if it has the following property: if x_0 and y_0 are standard points of F, with $x_0 \,\mathcal{A}\, y_0$, then

$$(\forall x \simeq x_0)(\exists y \simeq y_0) y \in \gamma_x \text{ and } \overset{\circ}{(\gamma_{xy})} = F^{x_0}_{y_0}. \tag{6}$$

(that terminology may not be evident for the moment, but it will be made clear by Theorem 8).

Unstable river

$x'=4\arctan[y/\varepsilon(y^2/\varepsilon-1)],\ y'=9\varepsilon-y^4/\varepsilon.$

Figure 17

Remark. By Proposition 6, for any river, (6) implies $x_0\,\mathcal{A}\,y_0$.

Theorem 7

A stable river is monotonic.

Proof. Let x_0 and y_0 be standard points of a stable river F, such that $x_0\,\mathcal{A}\,y_0$, and suppose (6) does not hold. Then there exists $x \simeq x_0$ such that

$$(\forall y \in \gamma_x \cap \mathrm{hal}(y_0))\ \overset{\circ}{\vphantom{x}}(\gamma_{xy}) \neq F^{x_0}_{y_0}$$

(that is, no arc of trajectory issued from x has $F^{x_0}_{y_0}$ as its shadow).

$F^{x_0}_{y_0}$ is homeomorphic to a real interval, and $\overset{\circ}{\vphantom{x}}(\gamma_{xy})$ is connected (for it is the shadow of a connected set all points of which are near standard — see the proof of proposition 1) and it contains the points x_0 and y_0, that is the end points of $F^{x_0}_{y_0}$; so $\overset{\circ}{\vphantom{x}}(\gamma_{xy})$ cannot be strictly included in $F^{x_0}_{y_0}$. The only possibility that they can be different is that there exists a point $t \in \gamma_{xy}$ such that ${}^{\circ}t \notin F^{x_0}_{y_0}$ — that is $(\exists^{st}\varepsilon > 0)d(t, F^{x_0}_{y_0}) > \varepsilon$.

But that is true for every $y \in \gamma_{xy} \cap \mathrm{hal}(y_0)$, so by Permanence there exists a standard positive ε_0 such that

$$(\forall y \in \gamma_{xy} \cap \mathrm{hal}(y_0))(\exists t \in \gamma_{xy})\, d(t, F^{x_0}_{y_0}) > \varepsilon_0.$$

Then the set $U = \left\{t \in E/d(t, F^{x_0}_{y_0}) < \varepsilon_0/2\right\}$ is an open standard set containing $F^{x_0}_{y_0}$, but y does not drain $\mathrm{hal}(x_0)$ relatively to U — which contradicts the stability of F. $\Box$

2.5.2 S-monotonicity of trajectories along a river

Definition. Let I be an internal or external interval in $\mathbb{R}$, let $x \in E$ and F be a river; an arc of trajectory $\gamma = \varphi_x(I)$ is **S-monotonic** along F iff:

- every point of γ is in the halo of a standard point of F;
- one of the following properties holds:
  a) $(\forall u, s \in I)\left[u \leq s \Rightarrow \overset{\circ}{}(\varphi_x(u))\,\mathcal{A}_=\, {}^\circ(\varphi_x(s)\right]$ (**S-descending** arc);
  b) $(\forall u, s \in I)\left[u \leq s \Rightarrow \overset{\circ}{}(\varphi_x(u))\,\mathcal{A}_=\, {}^\circ(\varphi_x(s))\right]$ (**S-ascending** arc).

(this is a particular case of the notion studied in 1.4).

For any $x \in E$ we shall note $\gamma^+_x = \varphi_x(\mathbb{R}^+)$ and $\gamma^-_x = \varphi_x(\mathbb{R}^-)$ the half trajectories issued from x.

The following theorem relates that notion to that of monotonic river.

Theorem 8 (about monotonicity)

A river F is monotonic if and only if it has the following property:

for all standard $x_0 \in F$, and all $x \simeq x_0$, there exists an arc of trajectory γ issued from x, S-descending along F and such that

$$ {}^\circ\gamma = F^{x_0}. \tag{7}$$

Furthermore if F is compact downstream, γ can be chosen internal and compact; otherwise γ is necessarily external.

Remarks.

- In some cases there are two such arc issued from the same point x, one contained in the half trajectory γ^+_x, the other in γ^-_x, as in Figure 16.
- When F is compact downstream, there exists on F a standard point a_0 which is most downstream on the river. Then the above theorem asserts the existence of arcs of trajectories, issued from every point in the halo of a standard point of F, "S-descending" down to the mouth of F, that is which are S-monotonic along F down to the halo of a_0.

Proof. We shall only develop the proof that a monotonic river satisfies (7), the converse being straightforward by Proposition 6.

So let F be a monotonic river. If x_0 is the most downstream point of F, then $F^{x_0} = \{x_0\}$, so the conclusion is trivial; from now on we suppose it is not the case, and we choose a point $x \simeq x_0$.

Then we can take y_0 and z_0 two different points on F such that $x_0 \,\mathcal{A}\, y_0 \,\mathcal{A}\, z_0$; by the definition of a monotonic river there exists an $y \simeq y_0$ such that $y \in \gamma_x$ and $\overset{\circ}{(\gamma_{xy})} = F^{x_0}_{y_0}$.

Let us now state a first lemma:

Lemma 4

If y_0 is a standard point of F downstream from x_0, but not most downstream on F, take $y \in \gamma_x$ such that $\overset{\circ}{(\gamma_{xy})} = F^{x_0}_{y_0}$; then γ_{xy} is S-monotonic along F.

Proof. Suppose for example $y \in \gamma_x^+$. Let z_0 be a standard point of F, strictly downstream from y_0; let also t be a positive real number such that $y = \varphi_x(t)$ and s_1, s_2 such that $0 < s_1 < s_2 < t$; we set $m_1 = \varphi_x(s_1)$ and $m_2 = \varphi_x(s_2)$ and show that ${}^\circ m_1 \,\mathcal{A}_=\, {}^\circ m_2$ (cf. Figure 18): otherwise we should have ${}^\circ m_2 \,\mathcal{A}\, {}^\circ m_1$; but in that case, by the choice of y, the shadow of $\gamma_{m_1}^-$ could not go strictly downstream y_0 before passing by x_0; on the other hand for the same reason, the shadow of $\gamma_{m_1}^+$ would go at least up to ${}^\circ m_2$, before eventually passing by z_0; but that contradicts property (6) for the pair of points $({}^\circ m_1, z_0)$. $\square$

Figure 18