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A note on absolutely p-summing operators

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Abstract

Let X, Y be Banach spaces. If X is a $\mathcal{D}_q-$space $(1 < q \leq +\infty)$ we prove that $\Pi_p^d(X, Y) \subset \Pi_p(X, Y)$ if and only if Y is isomorphic to a subspace of an L^p-space, where p is the conjugate number for q. We also prove that, if Y is a $\mathcal{D}_p-$space, $\Pi_p(X, Y) \subset \Pi_p^d(X, Y)$ if and only if X^* is isomorphic to a subspace of an L^p-space.

Throughout this note $X, Y,$ will denote Banach spaces. As usual $\Pi_p(X, Y)$ will stand for the Banach space of all absolutely $p-$summing operators T from X into Y $(1 \leq p < +\infty)$. Following [6, p. 67], $\Pi_p^d(X, Y)$ will denote the vector space of all operators $T : X \to Y$ such that $T^* \in \Pi_p(Y^*, X^*)$. If in $\Pi_p^d(X, Y)$ we consider the norm $\pi_p^d(T) = \pi_p(T^*)$, it becomes a Banach space.

We recall that a Banach space X is called a $\mathcal{D}_{q,\lambda}-$space if for each positive integer n there is a subspace X_n in X with $d(X_n, \ell_q^n) \leq \lambda$ [4]. A banach space X is called a $\mathcal{D}_q-$space if it is a $\mathcal{D}_{q,\lambda}-$space for some $\lambda \geq 1$.

In [8, Theorem 13.7] it is proved that Y is isomorphic to a subspace of an L^p-space $(1 < p < +\infty)$ if and only if $\Pi_p^d(X, Y) \subset \Pi_p(X, Y)$ for every Banach space X. We have obtained the following two results.

Theorem 1

Y is isomorphic to a subspace of an $L^p(\mu)-$space $(1 \leq p < +\infty)$ if and only if $\Pi_p^d(X, Y) \subset \Pi_p(X, Y)$ for some $\mathcal{D}_q-$space X, where q is the conjugate number for p.

Proof. $\Rightarrow$ The case $p=1$ follows easily from [7, Theorem III. 3] and the case $p \neq 0$ is just Theorem 13.7 in [8].

$\Leftarrow$ First we show that there exists a constant $c > 0$ such that

$$\pi_p(T) \leq c\, \pi_p(T^*) \quad \text{for all} \quad T \in \Pi_p^d(X, Y). \tag{1}$$

Let $M_n = \{T \in \Pi_p^d(X,Y) : \pi_p(T) \leq n\}$, for all $n \in \mathbb{N}$. Since $\Pi_p^d(X,Y) = \cup M_n$ and each set M_n is closed in $\Pi_p^d(X,Y)$, it follows that there exists $n \in \mathbb{N}$ so that $\overset{\circ}{M}_n$ is not empty. Now a standard argument yields (1).

To prove that Y is isomorphic to a subspace of an L^p-space we use the well-known characterization of J. Lindenstrauss and A. Pelczynski [3]. Let $\{y_i\}$ and $\{z_j\}$ be finite subsets of Y such that

$$\sum_i |\langle y_i, y^* \rangle|^p \leq \sum_j |\langle z_j, y^* \rangle|^p \quad \text{for all} \quad y^* \in Y^*. \tag{2}$$

We shall show that there is a constant $M > 0$ such that

$$\sum_{i=1}^n \|y_i\|^p \leq M^p \sum_{j=1}^m \|z_j\|^p .$$

Adding some zeros if necessary, we may assume that both sets have the same length, say n. By hypothesis, there is a subspace X_n in X such that $d(X_n, \ell_q^n) \leq \lambda$. Choose operators $T : X_n \to \ell_q^n$ and $S : \ell_q^n \to X_n$ such that

$$T \circ S = \mathrm{id}_{\ell_q^n} \text{ and } \|T\|\,\|S\| \leq \lambda . \tag{3}$$

If $\{e_i\}_{i=1}^n$ and $\{e_i^*\}_{i=1}^n$ are the canonical bases of ℓ_q^n and $\ell_p^n = (\ell_q^n)^*$ respectively, set $x_i = Se_i$ and $x_i^* = T^*(e_i^*)$ for $1 = 1, .., n$. There are Hahn-Banach extensions of x_i^* to all X. Let z_i^* be a such extension.Then

$$\langle x_i, z_j^* \rangle = \delta_{ij} \tag{4}$$

$$\|z_i^*\| \leq \|T\| \quad \text{and} \quad \|x_i\| \leq \|S\| \tag{5}$$

$$\begin{aligned} \sup\left\{ \Big(\sum_{i=1}^n |\langle x_i, x^* \rangle|^p \Big)^{1/p} : \|x^*\| \leq 1 \right\} &= \sup\left\{ \Big\| \sum_{i=1}^n \alpha_i x_i \Big\| : \|(\alpha_i)\|_q \leq 1 \right\} \\ &\leq \|S\| \end{aligned} \tag{6}$$

If $q \neq \infty$, we also have

$$\sup\left\{\Big(\sum_{i=1}^{n}|\langle z_i^*, x^{**}\rangle|^q\Big)^{1/q} : \|x^{**}\| \leq 1\right\} = \sup\left\{\Big\|\sum_{i=1}^{n}\alpha_i z_i^*\Big\| : \|(\alpha_i)\|_p \leq 1\right\}$$
$$\leq \|T\| \,. \tag{7}$$

Now we define $P : X \to Y$ by $Px = \sum_{i=1}^n \langle x, z_i^*\rangle y_i$ for all $x \in X$. Since $Px_i = y_i$ for $i = 1, .., n$, from (6) it follows that

$$\Big(\sum_{i=1}^{n}\|y_i\|^p\Big)^{1/p} \leq \pi_p(P)\,\|S\| \,. \tag{8}$$

This and (1) yields

$$\Big(\sum_{i=1}^{n}\|y_i\|^p\Big)^{1/p} \leq c\,\|S\|\,\pi_p(P^*) \,. \tag{9}$$

Finally, observe that the dual operator $P^* : Y^* \to X^*$ is defined by $P^*(y^*) = \sum_{i=1}^n \langle y_i, y^*\rangle z_i^*$ for all $y^* \in Y^*$. Thus, in case $p \neq 1$, from (2) and (7) we obtain

$$\|P^*(y^*)\| \leq \Big(\sum_{i=1}^{n}|\langle y_i, y^*\rangle|^p\Big)^{1/p} \quad \sup\left\{\Big(\sum_{i=1}^{n}|\langle z_i^*, x^{**}\rangle|^q\Big)^{1/q} : \|x^{**}\| \leq 1\right\}$$
$$\leq \|T\|\,\Big(\sum_{i=1}^{n}|\langle y_i, y^*\rangle|^p\Big)^{1/p} \leq \|T\|\,\Big(\sum_{i=1}^{n}|\langle z_i, y^*\rangle|^p\Big)^{1/p} \tag{10}$$

for all $y^* \in Y^*$. If $p = 1$, from (5) it follows that

$$\|P^*(y^*)\| \leq \big(\sup_i \|z_i^*\|\big)\sum_{i=1}^{n}|\langle y_i, y^*\rangle| \leq \|T\|\sum_{i=1}^{n}|\langle z_i, y^*\rangle| \,. \tag{10$'$}$$

In any case, by [2, p. 32] we have the following estimation for $\pi_p(P^*)$

$$\pi_p(P^*) \leq \|T\|\,\Big(\sum_{i=1}^{n}\|z_i\|^p\Big)^{1/p} .$$

Therefore

$$\Big(\sum_{i=1}^{n}\|y_i\|^p\Big)^{1/p} \leq (c\lambda)\,\Big(\sum_{i=1}^{n}\|z_i\|^p\Big)^{1/p} . \ \square$$

Theorem 2

X^* is isomorphic to a subspace of an L^p-space $(1 \leq p < +\infty)$ if and only if $\Pi_p(X,Y) \subset \Pi_p^d(X,Y)$ for some $\mathcal{D}_p-$space Y.

Proof. We only prove the "if part" of the theorem because the "only if part" can be proved using the same argument as in Theorem 1.

Let X^* be isomorphic to a subspace of an L^p-space. By Theorem 1, it follows that $\Pi_p^d(Y^*,X^*) \subset \Pi_p(Y^*,X^*)$ for every Banach space Y. Now recall that $T^{**} \in \Pi_p(X^{**},Y^{**})$ whenever $T \in \Pi_p(X,Y)$ [5]. Therefore, for every p-summing operator $T : X \to Y$ we have $T^* \in \Pi_p^d(Y^*,X^*)$ and this concludes the proof. □

Corollary

If X is a $\mathcal{D}_q-$space and Y is a $\mathcal{D}_p-$space $(1 \leq p < +\infty$, and $p^{-1} + q^{-1} = 1)$, then

$$\Pi_p(X,Y) = \Pi_p^d(X,Y)$$

if and only if X^* and Y are isomorphic to subspaces of L^p-spaces.

Remark. It is well known that if $\Pi_2(X,Y) = \mathcal{L}(X,Y)$ for all $\mathcal{L}_\infty$-space X, then Y has Orlicz's Property [3, Proposition 8.1]. Since every infinite dimensional Banach space X is a $\mathcal{D}_2$-space, by using the same argument as in Theorem 1, it can be proved that

$$\Pi_2(X,Y) = \mathcal{L}(X,Y) \ \Rightarrow Y \text{ has Orlicz's Property.}$$

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