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Convergence of weighted sums of random variables and uniform integrability concerning the weights

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Abstract

Let $\{a_{nk}\}, n, k \in \mathbb{N}$ be an array of real constants, and let $\{X_n\}$ be a sequence of random variables. The concept of $\{a_{nk}\}$–uniform integrability of $\{X_n\}$ is defined and two characterizations of this concept are established. Limit theorems for weighted sums $\sum_k a_{nk}(X_k - EX_k)$ are obtained, when the sequence $\{X_n\}$ is $\{a_{nk}\}$–uniformly integrable.

1. Introduction

Let $\{X_n\}$ be a sequence of random variables, i.e., a sequence of measurable functions from a probabilistic space $(\Omega, \mathcal{A}, P)$ into $\mathbb{R}$. Let $\{a_{nk}\}, k, n \in \mathbb{N}$, be an array of real numbers.

The problem of convergence in probability of the sequence of sums $S_n = \sum_k a_{nk}X_k$ has been considered by Pruitt ([12]) under the hypothesis of independence and identical distribution of the random variables and assuming that $\{a_{nk}\}$ is a Toeplitz array.

These initial conditions can be generalized in two ways: to relax the assumption of independence and replace it by pairwise independence, incorrelation, martingale assumptions, etc., or to relax the assumption of identical distribution.

In the latter case, Rohatgi ([13]), by using the double truncation method of Erdos ([7]), extends the results of Pruitt to random variables non identically distributed, but uniformly dominated by a random variable with finite moments of a certain order; this condition is implied by the uniform boundedness of moments of a suitable order (see [14]).

Wang and Bhaskara Rao ([15]) extend the Rohatgi's weak law for weighted sums to the case of random variables uniformly integrable. (It is well known ([6]) that if $\{X_n\}$ is uniformly dominated by a random variable X with $E|X| < \infty$, then $\{X_n\}$ is uniformly integrable).

Chandra ([1]) obtains several variations and extensions of the classical WLLN of Khintchine by introducing the condition of uniform integrability in the Cesàro sense, which is weaker than the uniform integrability and appears naturally related to the problem and the proof used there.

Gut ([9]) obtains a WLLN for an array $\{X_{nk}\}, 1 \le k \le k_n, n \in \mathbb{N}, k_n \to \infty$ as $n \to \infty$, of random variables such that $\{|X_{nk}|^p\}, 0 < p < 2$, is uniformly integrable in the Cesàro sense.

Analysing the above arguments, it seems natural, when we study the weak convergence of weighted sums $S_n = \sum_k a_{nk} X_k$, to require a condition which relates the random variables X_k to their respective weights a_{nk} in S_n. Thus, we introduce the condition of $\{a_{nk}\}$-uniform integrability of $\{X_n\}$, which is weaker than the uniform integrability of $\{X_n\}$ and leads to the Cesàro uniform integrability as a particular case. Both Theorem 2 and Theorem 3 provide a characterization of this concept. The function G in Theorem 3 has similar properties to function ψ in Theorem 2 of Chung ([5]), excepting the decrease of $\frac{\psi(x)}{x^2}$.

It is an open question the existence of such a function G characterizing the $\{a_{nk}\}$-uniform integrability and such that $\frac{G(t)}{t^2}$ is decreasing. In case of affirmative reply, it would be possible to obtain a SLLN for weighted sums $\sum_k a_{nk} X_k$ of pairwise independent and $\{a_{nk}\}$-uniformly integrable random variables $\{X_n\}$, similar to the SLLN of Chung ([5]) for sums $\sum_n \frac{X_n}{a_n}$ of independent random variables. Theorem 4 provides a result of convergence in L_1 and, therefore, a WLLN to this effect.

A possible field of applications of the concepts and results in this paper is the area of quality control. Lai ([10]) has used weighted sums of i.i.d. random variables to detect the change in a characteristic which assesses the quality of the output in a continuous production process. The sequence of weights in these sums $\sum_{k=1}^{n} c_{n-k} X_k$ is chosen to be $c_0 \ge c_1 \ge ... \ge c_{k-1} > 0 = c_k = c_{k+1} = ...$, that is, the sums are considered on a preassigned segment of the past.

If, in our paper, we consider that each random variable X_n is a statistic computed from the nth sample, the sums $S_n = \sum_k a_{nk} X_k$ could be used in detecting the variation in the quality of the output, summing over the entire past, since the sequence of weights does not have to be eventually zero. The condition of $\{a_{nk}\}$-uniform integrability relates the statistic X_k to their respective weight in S_n, assigning weights on the remote past which are different from the weights on the immediate past.

In this framework, the above open question leads to the possibility of using the characterization of $\{a_{nk}\}$-uniform integrability in Theorem 3 in order to obtain rules of detection of changes in the quality of the output similar to those based in the first passage times.

2. Uniform integrability concerning an array

DEFINITION. Let $\{a_{nk}\}, k, n \in \mathbb{N}$, be an array of real constants satisfying

$$\sum_k |a_{nk}| \leq C \quad \text{for every} \;\; n \in \mathbb{N}\,,$$

where C is some positive constant. A sequence $\{X_n\}$ of integrable random variables is said to be $\{a_{nk}\}$-uniformly integrable (or uniformly integrable concerning the array $\{a_{nk}\}$) if

$$\lim_{a\to\infty} \quad \sup_n \left(\sum_k |a_{nk}| \int_{[|X_k|>a]} |X_k|\, dP \right) = 0\,.$$

Remark. In the particular case of the array

$$a_{nk} = \begin{cases} \dfrac{1}{n} & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}$$

the condition of $\{a_{nk}\}$-uniform integrability is the uniform integrability in the Cesàro sense of Chandra ([1]).

The following assertion is easy to check:

Theorem 1

Let $\{X_n\}$ be a sequence of uniformly integrable random variables. Then, $\{X_n\}$ is $\{a_{nk}\}$-uniformly integrable for all arrays $\{a_{nk}\}$ such that $\sum_k |a_{nk}| \leq C$ for every $n \in \mathbb{N}$, where C is some positive constant.

The following example, due to B.V. Rao (in [1]), shows that the above condition is weaker than the uniform integrability: there exist non uniformly integrable sequences of random variables which are $\{a_{nk}\}$-uniformly integrable for some array $\{a_{nk}\}$.

EXAMPLE 1: Let $X_n = \pm 1$ with probability $\frac{1}{2}$, if n is not a perfect cube, and $X_n = \pm n^{\frac{1}{3}}$ with probability $\frac{1}{2}$, if n is a perfect cube.

The sequence $\{X_n\}$ is not uniformly integrable, since $\sup_n E|X_n| = \infty$.

However, $\{X_n\}$ is uniformly integrable relative to the array

$$a_{nk} = \begin{cases} \dfrac{1}{n} & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

In fact, if $a \geq 1$, we have:

$$\sum_k |a_{nk}| \int_{[|X_k|>a]} |X_k|\, dP \leq \sum_k |a_{nk}| \int_{[|X_k|>1]} |X_k|\, dP$$
$$\leq \frac{1}{n} \sum_{\substack{k=j^3 \\ k\leq n}} k^{\frac{1}{3}} \leq \frac{\left(n^{\frac{1}{3}}+1\right) n^{\frac{1}{3}}}{2n} \to 0 \quad \text{when } n \to \infty\,.$$

A sequence $\{X_n\}$ of integrable random variables is uniformly integrable if, and only if, $\{E|X_n|\}$ is uniformly bounded and the maps $A \to E\left(|X_n| I_A\right)$ are uniformly continuous ([6]). In the same sense, we obtain the following characterization of the $\{a_{nk}\}$-uniform integrability:

Theorem 2

A sequence $\{X_n\}$ of random variables is $\{a_{nk}\}$-uniformly integrable if, and only if:

a) $\sup_n \left(\sum_k |a_{nk}| E|X_k|\right) = M < \infty$

b) for each $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $\{A_k\}$ is a sequence of events satisfying

$$\sup_n \left(\sum_k |a_{nk}| P(A_k)\right) < \delta$$

then

$$\sup_n \left(\sum_k |a_{nk}| \int_{A_k} |X_k|\, dP\right) < \varepsilon\,.$$

Proof. Let $\{X_n\}$ be a sequence of $\{a_{nk}\}$-uniformly integrable random variables. Given $\varepsilon > 0$, there exists $a > 0$ such that

$$\sup_n \left(\sum_k |a_{nk}| \int_{[|X_k|>a]} |X_k| \, dP \right) < \frac{\varepsilon}{2} \, .$$

Then

$$E|X_k| = \int_{[|X_k| \leq a]} |X_k| \, dP + \int_{[|X_k|>a]} |X_k| \, dP \leq a + \int_{[|X_k|>a]} |X_k| \, dP$$

which implies that

$$\begin{aligned} \sum_k |a_{nk}| E|X_k| &\leq a \sum_k |a_{nk}| + \sum_k |a_{nk}| \int_{[|X_k>a]} |X_k| \, dP \\ &\leq aC + \frac{\varepsilon}{2} < \infty \quad \text{for every } n \in \mathbb{N}. \end{aligned}$$

Thus a) holds. In order to prove b), let $\{A_n\}$ be a sequence of events such that $\sum_k |a_{nk}| P(A_k) < \frac{\varepsilon}{2a} = \delta$. Then:

$$\begin{aligned} &\sum_k |a_{nk}| \int_{A_k} |X_k| \, dP \\ &= \sum_k |a_{nk}| \left(\int_{A_k \cap [|X_k| \leq a]} |X_k| \, dP + \int_{A_k \cap [|X_k|>a]} |X_k| \, dP \right) \\ &\leq a \sum_k |a_{nk}| P(A_k) + \sum_k |a_{nk}| \int_{[|X_k|>a]} |X_k| \, dP \\ &< a \frac{\varepsilon}{2a} + \frac{\varepsilon}{2} = \varepsilon \, . \end{aligned}$$

Conversely, for each $a > 0$:

$$\sum_k |a_{nk}| P \left[|X_k| > a \right] \leq \frac{1}{a} \sum_k |a_{nk}| E|X_k| \leq \frac{M}{a}$$

for every $n \in \mathbb{N}$.

Given $\varepsilon > 0$, we have, for each $a \geq a_0 = \frac{2M}{\delta}$:

$$\sum_k |a_{nk}| P \left[|X_k| > a \right] \leq \sum_k |a_{nk}| P \left[|X_k| > a_0 \right] \leq \frac{M}{a_0} = \frac{\delta}{2}.$$

Therefore, the events $A_k = [|X_k| > a]\,, k \in \mathbb{N}$, verify condition b) for each $a \geq a_0$.

Thus:

$$\sum_k |a_{nk}| \int_{[|X_k|>a]} |X_k|\, dP < \varepsilon$$

for each $a \geq a_0$ and for every $n \in \mathbb{N}$, i.e., the sequence $\{X_n\}$is $\{a_{nk}\}$-uniformly integrable. □

The following lemma provides an interesting consequence of Theorem 2:

Lemma 1

If $\{X_n\}$ is a sequence of $\{a_{nk}\}$-uniformly integrable random variables, then $S_n = \sum_k a_{nk}\,(X_k - EX_k)$ is uniformly integrable.

Proof. a) For every $n \in \mathbb{N}$:

$$\begin{aligned} E|S_n| &\leq \sum_k |a_{nk}| E|X_k - EX_k| \\ &\leq 2 \sum_k |a_{nk}| E|X_k| \leq 2M < \infty \end{aligned}$$

according to a) of Theorem 2.

b) According to b) of Theorem 2, given $\varepsilon > 0$, there exists $\delta^* > 0$ such that if $\sup_n \sum_k |a_{nk}| P(A_k) < \delta^*$, then:

$$\sup_n \left(\sum_k |a_{nk}| \int_{A_k} |X_k|\, dP \right) < \frac{\varepsilon}{2}\,.$$

Let $0 < \delta < \min\{\frac{\delta^*}{C}, \frac{\varepsilon}{2M}\}$, and let A be an event with $P(A) < \delta$. Then:

$$\begin{aligned} \int_A |S_n|\, dP &\leq \sum_k |a_{nk}| \int_A |X_k - EX_k|\, dP \\ &\leq \sum_k |a_{nk}| \int_A |X_k|\, dP + P(A) \sum_k |a_{nk}| E|X_k| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2M} M = \varepsilon \quad \text{for every } \ n \in \mathbb{N} \end{aligned}$$

as $\sum_k |a_{nk}| P(A) < \delta C < \delta^*$.

Therefore, the sequence $\{S_n\}$ is uniformly integrable. □

The next theorem is an analytic characterization of the $\{a_{nk}\}$-uniform integrability analogous to the classical characterization of the uniform integrability due to La Vallée-Poussin. This result is used in practice with the function $G(t) = t^p, p > 1$. The proof extends the classical proof in [11] in the same sense of [3] for the characterization of the Cesàro uniform integrability.

Theorem 3

Let $\{a_{nk}\}, k, n \in \mathbb{N}$, be an array of real constants satisfying $\sum_k |a_{nk}| \leq C$ for every $n \in \mathbb{N}$, where C is some positive constant. A sequence of integrable random variables $\{X_n\}$ is $\{a_{nk}\}$-uniformly integrable if, and only if, there exists a measurable function $G : (0, \infty) \longrightarrow (0, \infty)$, $G(0) = 0$, such that $\frac{G(t)}{t} \to \infty$ when $t \to \infty$ and $\sup_n \sum_k |a_{nk}| EG(|X_k|) < \infty$. Moreover, G can be selected convex and such that $\frac{G(t)}{t}$ is increasing.

Proof. a) Assume the existence of such a function G. Let M defined as $M = \sup_n \sum_k |a_{nk}| EG(|X_k|)$. Given $\varepsilon > 0$, there exists $a > 0$ such that $\frac{G(t)}{t} \geq \frac{M+1}{\varepsilon}$ for $t > a$.

Then:

$$\sum_k |a_{nk}| \int_{[|X_k|>a]} |X_k| \, dP \leq \frac{\varepsilon}{M+1} \sum_k |a_{nk}| \int_{[|X_k|>a]} G(|X_k|) \, dP \leq \frac{M\varepsilon}{M+1} < \varepsilon$$

for every $n \in \mathbb{N}$.

Note that the convexity of G and the increase of $\frac{G(t)}{t}$ are not necessary for this implication.

b) Suppose $\{X_n\}$ is $\{a_{nk}\}$-uniformly integrable.

Let $\{g_n\}, n \in \mathbb{N}$ a sequence of nonnegative constants with $g_n \uparrow \infty$. Define $g : (0, \infty) \longrightarrow (0, \infty)$ as $g(x) = g_n$ if $x \in [n-1, n), n \in \mathbb{N}$, and let

$$G(t) = \int_0^t g(x) \, dx, \quad t > 0 \, ; \quad G(0) = 0 \, .$$

G is convex and $\frac{G(t)}{t} \uparrow \infty$ (see [3]).

Next, we prove the existence of a sequence $\{g_n\}$ satisfying the above conditions and such that $\sup_n \sum_k |a_{nk}| EG(|X_k|) < \infty$.

We can choose a sequence of positive integers $\{n_j\}, j \in \mathbb{N}$, such that

$$\sup_n \sum_k |a_{nk}| \int_{[X_k|\geq n_j]} |X_k| \, dP \leq 2^{-j} \quad \text{for each } j \in \mathbb{N}$$

and $n_j \to \infty$ when $j \to \infty$.

Let $g_n = \text{card}\,\{j \in \mathbb{N} : n_j < n\}$ for every $n \in \mathbb{N}$.

Evidently, $g_n < \infty$ and $g_n \uparrow \infty$.

We can write $G(t) = \sum_{i=0}^{n-1} g_i + (t-n+1) g_n \quad (g_0 = 0)$ for every $t \in (n-1, n]$. This implies $G(t) \leq \sum_{i=1}^{n} g_i$ for every $t \in (n-1, n], n \in \mathbb{N}$.

Then, for each $k \in \mathbb{N}$:

$$\begin{aligned} EG(|X_k|) &\leq E\left[\sum_{n=1}^{\infty} I(n-1 < |X_k| \leq n) \sum_{i=1}^{n} g_i\right] \\ &= E\left[\sum_{i=1}^{\infty} g_i \left(\sum_{n=i}^{\infty} I(n-1 < |X_k| \leq n)\right)\right] = \sum_{i=1}^{\infty} g_i P\left[|X_k| > i-1\right] \end{aligned}$$

where I(A) denotes the indicator of A.

Therefore, for every $n \in \mathbb{N}$:

$$\begin{aligned} \sum_{k=1}^{\infty} |a_{nk}| EG(|X_k|) &\leq \sum_{k=1}^{\infty} |a_{nk}| \left(\sum_{j=1}^{\infty} \sum_{i=n_j}^{\infty} P\left[|X_k| > i\right]\right) \\ &\leq \sum_{k=1}^{\infty} |a_{nk}| \left(\sum_{j=1}^{\infty} E|X_k| I(|X_k| \geq n_j)\right) \\ &= \sum_{j=1}^{\infty} \left(\sum_{k=1}^{\infty} |a_{nk}| \int_{[|X_k| \geq n_j]} |X_k| \, dP\right) \leq \sum_{j=1}^{\infty} 2^{-j} < \infty\,. \end{aligned}$$

To obtain the last bound, we have used the following well known inequality:

Let X be an integrable random variable; then:

$$\sum_{k=n}^{\infty} P\left[|X| \geq k\right] \leq E|X| I(|X| \geq n) \quad \text{for every } \ n \in \mathbb{N}. \ \square$$

3. Convergence of weighted sums of random variables

The following lemma will be used later:

Lemma 2

Let $\{X_n\}$ be a sequence of uniformly bounded pairwise independent random variables.

Let $\{a_{nk}\}, k, n \in \mathbb{N}$, be an array of real constant satisfying:

a) $\sum_k |a_{nk}| \leq C$ *for every $n \in \mathbb{N}$, where C is some positive constant.*

b) $\lim_n \sup_k |a_{nk}| = 0$.

Then:

$$\sum_k a_{nk} \left(X_k - EX_k\right) \to 0 \quad \text{in} \quad L_2\,.$$

Proof. Let $H > 0$ an uniform bound of $\{X_n\}$.

Then:

$$E\left(\sum_k a_{nk}\left(X_k - EX_k\right)\right)^2$$
$$= \sum_k a_{nk}^2 E\left(X_k - EX_k\right)^2 \leq 4H^2 \left(\sup_k |a_{nk}|\right) \sum_k |a_{nk}|$$
$$\leq 4CH^2 \sup_k |a_{nk}| \to 0 \quad \text{when } \ n \to \infty\,. \ \Box$$

The following theorem provides a result of convergence in L_1 of a weighted sum of random variables when these variables are uniformly integrable relative to the array of weights.

Theorem 4

Let $\{a_{nk}\}, k, n \in \mathbb{N}$, *be an array of real constants satisfying:*
a) $\sum_k |a_{nk}| \leq C$ *for every* $n \in \mathbb{N}$, *where* C *is some positive constant.*
b) $\lim_n \sup_k |a_{nk}| = 0$.

Let $\{X_n\}$ *be a sequence of pairwise independent and* $\{a_{nk}\}$*-uniformly integrable random variables.*

Then:

$$S_n = \sum_k a_{nk}\left(X_k - EX_k\right) \to 0 \quad \text{in} \quad L_1.$$

Proof. Given $\varepsilon > 0$, there exists $a > 0$ such that:

$$\sup_n \left(\sum_k |a_{nk}| \int_{[|X_k|>a]} |X_k|\, dP\right) < \frac{\varepsilon}{4}\,.$$

We define, for each $k \in \mathbb{N}$:

$$W_k = X_k I(|X_k| \leq a)$$
$$Y_k = X_k - W_k.$$

We write:

$$S_n = \sum_k a_{nk}\big((W_k - EW_k) + (Y_k - EY_k)\big)\,.$$

$\{W_n - EW_n\}$ is a sequence of pairwise independent random variables such that:

$$|W_n - EW_n| \leq 2a \quad \text{for every } \ n \in \mathbb{N}.$$

Lemma 2 assures the convergence of $\sum_k a_{nk}(W_k - EW_k)$ to zero in L_1, when $n \to \infty$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$E\Big|\sum_k a_{nk}(W_k - EW_k)\Big| < \frac{\varepsilon}{2} \quad \text{for every } n \geq n_0.$$

On the other hand:

$$\begin{aligned}\sum_k |a_{nk}|E|Y_k - EY_k| &\leq 2\sum_k |a_{nk}|E|Y_k| \\ &= 2\sum_k |a_{nk}| \int_{[|X_k|>a]} |X_k|\, dP < \frac{\varepsilon}{2} \quad \text{for every } n \in \mathbb{N}.\end{aligned}$$

Then, $E|S_n| < \varepsilon$ for every $n \geq n_0$, i.e., $S_n \to 0$ in L_1, and the theorem is proved. □

We can drop the assumption on pairwise independence at the price of slightly strengthening the other assumptions:

Theorem 5

Let $\{a_{nk}\}, k, n \in \mathbb{N}$, *be an array of real constants satisfying:*
a) $\sum_k |a_{nk}|^r \leq C$ *for some* $r \in (0,1)$ *and for every* $n \in \mathbb{N}$, *where* C *is some positive constant*
b) $\lim_n \sup_k |a_{nk}| = 0$.

Let $\{X_n\}$ *be a sequence of random variables such that* $\{|X_n|^r\}$ *is* $\{|a_{nk}|^r\}$*-uniformly integrable.*

Then:

$$S_n = \sum_k a_{nk} X_k \to 0 \quad \text{in} \quad L_r\,.$$

Proof. Given $\varepsilon > 0$, there exists $a > 0$ such that:

$$\sup_n \left(\sum_k |a_{nk}|^r \int_{[|X_k|>a]} |X_k|^r\, dP\right) < \frac{\varepsilon}{2}\,.$$

We define W_k and Y_k as in Theorem 4; then:

$$\begin{aligned}E\Big|\sum_k a_{nk} W_k\Big| &\leq \sum_k |a_{nk}|E|W_k| \\ &\leq a\left(\sup_k |a_{nk}|^{1-r}\right)\sum_k |a_{nk}|^r \\ &\leq aC\left(\sup_k |a_{nk}|^{1-r}\right) \to 0\,.\end{aligned}$$

Therefore $E\left|\sum_k a_{nk}W_k\right|^r < \frac{\varepsilon}{2}$ for n sufficiently large.
Moreover:

$$E\left|\sum_k a_{nk}Y_k\right|^r \leq \sum_k |a_{nk}|^r E|Y_k|^r$$
$$= \sum_k |a_{nk}|^r \int_{[|X_k|>a]} |X_k|^r \, dP < \frac{\varepsilon}{2} \quad \text{for every } n \in \mathbb{N}.$$

Then $S_n \to 0$ in L_r. □

4. Some complements

The following example shows that Theorem 4 does not hold, as stated, for random elements in separable Banach spaces:

Example 2: Let $l^1 = \left\{x \in \mathbb{R}^\infty : \|x\| = \sum_n |x_n| < \infty\right\}$.
$(l^1, \|.\|)$ is a real separable Banach space.

Let $\{e_n\}$ be the standard basis of l^1. Let $\{X_n\}$ be the sequence of independent random elements in l^1 defined by:

$$X_n = \begin{cases} e_n & \text{with probability } \frac{1}{2} \\ -e_n & \text{with probability } \frac{1}{2} \end{cases}$$

Then $\|X_n\| = 1$ with probability one and $EX_n = 0$, for every $n \in \mathbb{N}$.
Therefore, for every $a \geq 1$ and for every $n \in \mathbb{N}$:

$$\int_{[\|X_n\|>a]} \|X_n\| \, dP = 0\,.$$

Thus, $\{X_n\}$ is uniformly integrable, and so $\{a_{nk}\}$-uniformly integrable for any array $\{a_{nk}\}$ such that $\sum_k |a_{nk}| \leq C$ for every $n \in \mathbb{N}$, where C is a positive constant.

But $E\|S_n\| = E\|\sum_k a_{nk}X_k\| = \sum_k |a_{nk}|$, and the sequence $\{E\|S_n\|\}$ does not necessarily tend to zero. For instance, if

$$a_{nk} = \begin{cases} \dfrac{1}{n} & \text{if } 1 \leq k \leq n \\ 0 & \text{if } k > n \end{cases}$$

then $E\|\sum_k a_{nk}X_k\| = 1$ for every $n \in \mathbb{N}$.

However, the proof of Theorem 5 can be extended to separable Banach spaces, and so the following result is valid:

Theorem 6

Let $\{a_{nk}\}, k, n \in \mathbb{N}$ *be an array of real constants satisfying:*
a) $\sum_k |a_{nk}|^r \leq C$ *for some* $r \in (0,1)$ *and for every* $n \in \mathbb{N}$*, where* C *is some positive constant*
b) $\lim_n \sup_k |a_{nk}| = 0$.

Let $\{X_n\}$ *be a sequence of random elements in a real separable Banach space* $(B, \|.\|)$*, such that* $\{\|X_n\|^r\}$ *is* $\{|a_{nk}|^r\}$*-uniformly integrable.*

Then:

$$E\Big\| \sum_k a_{nk} X_k \Big\|^r \to 0 \quad \text{when} \;\; n \to \infty\,.$$

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