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Bounded variation functions of order k on sequence spaces

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Abstract

In this paper, we generalize some results concerning bounded variation functions on sequence spaces.

Some properties of bounded variation functions on sequence spaces were investigated by Wu Congxin [1], [2], [3]. Later on, he and Zhao Linsheng in [4], [5], [6] introduced and discussed bounded variation functions of order 2 on sequence spaces. In this paper, we generalize these results to bounded variation functions of order k on sequence spaces.

Let λ be a real linear sequence space. The Köthe dual λ^* of λ is the real linear sequence space consisting of all real sequences $U = (u_1, u_2, \dots)$ satisfying $\sum\limits_{k=1}^{\infty} |u_k x_k| < \infty$ for all $X = (x_1, x_2, \dots) \in \lambda$. When $\lambda = \lambda^{**}$, we say that λ is a perfect space.

For a real function $x(t)$ defined on $[a, b]$ and $k+1$ different points $t_0, t_1, \dots, t_k \in [a, b]$, we denote

$$Q_k(x; t_0, t_1, \dots, t_k) = \sum_{i=0}^{k} \frac{x(t_i)}{\prod\limits_{\substack{j=0 \\ i \neq j}}^{k} (t_i - t_j)}$$

Definition 1 [7, p. 87]. The variation of order k of a function $x(t)$ defined on $[a, b]$ is

$$\overset{b}{\underset{a\ \ k}{V}} (x) \triangleq \sup_{\pi} \sum_{i=0}^{n-k} |Q_{k-1}(x; t_i, \dots, t_{i+k-1}) - Q_{k-1}(x; t_{i+1}, \dots, t_{i+k})| \,.$$

Where the "sup" is taken over all partitions $\pi\colon a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$. When $\overset{b}{\underset{a\ k}{V}}(x) < \infty$, we say that $x(t)$ is a bounded variation function of order k and denote by $x(t) \in V_k[a,b]$.

Lemma 1 [7, p. 88].

For any $x \in V_k[a,b]$ and k different points $a_0, a_1, \ldots, a_{k-1}$ in $[a,b]$, we have

$$|Q_{k-1}(x; t_0, 1_1, \ldots, t_{k-1})| \le |Q_{k-1}(x; a_0, \ldots, a_{k-1})| + 2 \overset{b}{\underset{a\ k}{V}}(x).$$

Lemma 2 [7, p. 79].

$$Q_{r-1}(x; t_1, t_2, \ldots, t_r) - Q_{r-1}(x; t_0, t_1, \ldots, t_{r-1}) = (t_r - t_0) Q_r(x; t_0, \ldots, t_r).$$

Lemma 3

For any $k \ge 3$, we have $(k-1)! \overset{b}{\underset{a\ k}{V}}(x) = \overset{b}{\underset{a\ 2}{V}}(x^{(k-2)})$.

Lemma 4 [11, p. 179].

Let $x\colon [a,b] \to \mathbb{R}$, $y\colon [a,b] \to \mathbb{R}$, then

$$\begin{aligned} Q_k(xy; t_0, t_1, \ldots, t_k) = {} & y(x_0) Q_k(x; t_0, t_1, \ldots, t_k) \\ & + Q_1(y; t_0, t_1) Q_{k-1}(x; t_1, \ldots, t_k) \\ & + Q_2(y; t_0, t_1, t_2) Q_{k-2}(x; t_2, t_3, \ldots, t_k) \\ & + \ldots + Q_k(y; t_0, t_1, \ldots, t_k) x(t_k). \end{aligned}$$

Lemma 5

Suppose $x\colon [a,b] \to \mathbb{R}$ and $a_0, a_1, \ldots, a_r \in [a,b] (a_i \ne a_j$ when $i \ne j)$, then

$$|Q_r(x; a_0, a_1, \ldots, a_r)| \le \frac{1}{\min\limits_{j \ne i} |a_i - a_j|^r} \sum_{i=0}^{r} |x(a_i)| \quad r = 1, 2, \ldots$$

DEFINITION 2 Let $X(t) = (x_1(t), x_2(t), \ldots)$ be an abstract function from $[a,b]$ to a sequence space λ. If for each $U = (u_1, u_2, \ldots) \in \lambda^*$, we have

$$\begin{aligned} \overset{b}{\underset{a\ k}{V}}(X, U) \triangleq \sup_{\pi} \sum_{i=0}^{n-k} \sum_{m=1}^{\infty} \big| u_m [& Q_{k-1}(x; t_i, \ldots, t_{i+k-1}) \\ & - Q_{k-1}(x; t_{i+1}, \ldots, t_{i+k})] \big| < \infty \end{aligned}$$

then $X(t)$ is called a bounded variation function of order k and denoted by $X(t) \in V_k([a,b], \lambda)$.

Theorem 1

$X(t) \in V_k([a,b],\lambda)$ *iff*

1^0 $x_m(t) \in V_k[a,b], \quad m = 1,2,\dots$ *and*

2^0 $\sum\limits_{m=1}^{\infty} \{\mathop{V}\limits_{a\ k}^{b} (x_m)\} \in \lambda^{**}$.

Proof. Necessity. 1^0 Pick $U = (\overbrace{0,0,\dots,0}^{i-1},1,0,0,\dots) \in \lambda^*$, then from

$$\sup_{\pi} \sum_{i=0}^{n-1} \sum_{m=1}^{\infty} |u_m||t_{i+k} - t_i| \; |Q_k(x_m; t_i, \dots, t_{i+k})| =$$
$$= \sup_{\pi} \sum_{i=0}^{n-k} |t_{i+k} - t_i| \; |Q_k(x_m; t_i, \dots, t_{i+k})| < \infty$$

we see $x_s \in V_k[a,b], s = 1,2,\dots$

Next we turn to 2^0. If 2^0 is not true, then there exist $U^{(0)} = (u_1^{(0)}, u_2^{(0)}, \dots) \in \lambda^*, u_m^{(0)} \neq 0,\ m = 1,2,\dots$ and $N_n \geq 1, \varepsilon_n > 0$ such that

$$\sum_{m=1}^{N_n} |u_m^{(0)}| \mathop{V}\limits_{a\ k}^{b} (x_m) = n + \varepsilon_n.$$

Since $x_m \in V_k[a,b], m = 1,2,\dots,N_n$, there exists a partition $\pi_m\colon a = t_0^{(m)} < t_1^{(m)} < \dots < t_{n_m}^{(m)} = b$ such that

$$\mathop{V}\limits_{a\ k}^{b} (x_m) \leq \sum_{i=0}^{n_m - k} |t_{i+k} - t_i| \; |Q_k(x_m; t_i^{(m)}, \dots, t_{i+k}^{(m)})| + \frac{\varepsilon_n}{2^{m+1}|u_m^{(0)}|}.$$

Let π be the partition consisting of all points $\{t_i^{(m)}\ i \leq n_m, m \leq N_n\}$: $\pi\colon a = s_0^{(N_n)} < s_1^{(N_n)} < \dots < s_{l(N_n)}^{(N_n)} = b$ then by Theorem 3 in [8], we have

$$\mathop{V}\limits_{a\ k}^{b} (x_m) \leq \sum_{i=1}^{l(N_n)-k} |s_{i+k}^{(N_n)} - s_i^{(N_n)}| \; |Q_k(x_m; s_i^{(N_n)}, \dots, s_{i+k}^{(N_n)}| + \frac{\varepsilon_n}{2^{m+1}|u_m^{(0)}|}.$$

Hence

$$\sum_{i=0}^{l(N_n)-k} \sum_{m=1}^{\infty} |u_m^{(0)}| \, |s_{i+k}^{(N_n)} - s_i^{(N_n)}| \, |Q_k(x_m; s_i^{(N_n)}, \ldots, s_{i+k}^{(N_n)})|$$

$$\geq \sum_{i=0}^{l(N_n)-k} \sum_{m=1}^{N_n} |u_m^{(0)}| \, |s_{i+k}^{(N_n)} - s_i^{(N_n)}| \, |Q_k(x_m; s_i^{(N_n)}, \ldots, s_{i+k}^{(N_n)})|$$

$$\sum_{m=1}^{N_n} \sum_{i=0}^{l(N_n)-k} |u_m^{(0)}| \, |s_{i+k}^{(N_n)} - s_i^{(N_n)}| \, |Q_k(x_m; s_i^{(N_n)}, \ldots, s_{i+k}^{(N_n)})|$$

$$\geq \sum_{m=1}^{N_n} |u_m^{(0)}| \overset{b}{\underset{a}{V}} (x_m) - \sum_{m=1}^{N_n} \frac{\varepsilon_n}{2^{m+1}} \geq n$$

contradicting that $X(t) \in V_k([a,b], \lambda)$.

Sufficiency. Notice that

$$\sup_{\pi} \sum_{i=0}^{n-k} \sum_{m=1}^{\infty} |u_m| \, |t_{i+k} - t_i| \, |Q_k(x_m; t_i, \ldots, t_{i+k})|$$

$$\leq \sum_{m=1}^{\infty} \sup_{\pi} \sum_{i=0}^{n-k} |u_m| \, |t_{i+k} - t_i| \, |Q_k(x_m; t_i, \ldots, t_{i+k})|$$

$$= \sum_{m=1}^{\infty} |u_n| \overset{b}{\underset{a\ k}{V}} (x_m) < \infty$$

we find that $X(t)$ is bounded variation of order k. □

Theorem 2

$V_k([a,b], \lambda) \subset V_r([a,b], \lambda)$ for all $1 \leq r < k$.

Proof. It is sufficient to consider the case $r = k-1$. By Theorem 1, $X(t) = \{x_m(t)\}_{m=1}^{\infty} \in V_k([a,b], \lambda)$ implies

$$x_m(t) \in V_k[a,b] \quad \text{and} \quad \{\overset{b}{\underset{a\ k}{V}} (x_m)\} \in \lambda^{**}.$$

For k different points $a_0 < a_1 < \cdots < a_{k-1}$ in $[a,b]$, by Lemma 1, we have

$$|Q_{k-1}(x_m; t_i, \ldots, t_{i+k-1})| \leq |Q_{k-1}(x_m; a_0, \ldots, a_{k-1})| + 2 \overset{b}{\underset{a\ k}{V}} (x_m) .$$

Hence

$$\begin{aligned}
&\sup_{\pi}\sum_{i=0}^{n-k+1}\sum_{m=1}^{\infty}|u_m|\,|t_{i+k-1}-t_i|\,|Q_{k-1}(x_m;t_i,\ldots,t_{i+k-1})| \\
&\quad\le \sup_{\pi}\sum_{i=0}^{n-k+1}\sum_{m=1}^{\infty}|u_m|\,|t_{i+k-1}-t_i|\,(|Q_{k-1}(x_m;a_0,a_1,\ldots,a_{k-1})| \\
&\qquad + 2\overset{b}{\underset{a}{V}}{}_k\,(x_m)) \\
&\quad\le \sup_{\pi}\sum_{i=0}^{n-k+1}|t_{i+k-1}-t_i|\sum_{m=1}^{\infty}|u_m|\;\;(|Q_{k-1}(x_m;a_0,a_1,\ldots,a_{k-1})| \\
&\qquad + 2\overset{b}{\underset{a}{V}}{}_k\,(x_m)) \\
&\quad\le k(b-a)\Big(\frac{1}{\min\limits_{i\neq j,i,j-0,1,r,\ldots,k-1}|a_i-a_j|^{k-1}}\sum_{m=1}^{\infty}|u_m|\sum_{i=0}^{k-1}|x_m(a_i)| \\
&\qquad + 2\sum_{m=1}^{\infty}|u_m\overset{b}{\underset{a}{V}}{}_k\,(x_m)|\Big)<\infty\,.
\end{aligned}$$

It follows that $X(t)\in V_{k-1}([a,b],\lambda)$. $\square$

Corollary 1

If $\{\overset{b}{\underset{a}{V}}{}_k\,(x_m)\}\in\lambda^{**}$, *then* $\{\overset{b}{\underset{a}{V}}{}_r\,(x_m)\}\in\lambda^{**}$ *for all* $1\le r<k$.

Theorem 3

Suppose $k\ge 3$, *then* $X(t)\in V_k([a,b],\lambda)$ *iff* $X'(t)\equiv\{x'_m(t)\}\in V_{k-1}([a,b],\lambda)$.

Proof. Necessity. By Theorem 1, $x'_m(t)\in V_{k-1}[a,b]$. Moreover, by Lemma 3, when $k=3$, from $\{\overset{b}{\underset{a}{V}}{}_3\,(x_m)\}\in\lambda^{**}$, we have $\{\overset{b}{\underset{a}{V}}{}_2\,(x'_m)\}\in\lambda^{**}$, and when $k>3$, we have

$$(k-2)|\overset{b}{\underset{a}{V}}{}_{k-1}\,(x'_m)=\overset{b}{\underset{a}{V}}{}_2\,(x_m^{(k-2)}).$$

Hence, $\{\overset{b}{\underset{a}{V}}{}_{k-1}\,(x'_m)\}\in\lambda^{**}$ and the conclusion follows from Theorem 1.

Sufficiency. By Theorem 1 in [9], we have $x_m(t) \in V_k[a,b]$. Observing that $k = 3$ implies $2!\overset{b}{\underset{a}{V}}{}_3(x_m) = \overset{b}{\underset{a}{V}}{}_2(x'_m)$ and $k > 3$ implies

$$(k-2)!\overset{b}{\underset{a}{V}}{}_{k-1}(x'_m) = \overset{b}{\underset{a}{V}}{}_2(x_m^{(k-2)}) = (k-1)!\overset{b}{\underset{a}{V}}{}_k(x_m)$$

we find that $\{\overset{b}{\underset{a}{V}}{}_k(x_m)\} \in \lambda^*(k \geq 3)$ and that Theorem 1 implies $X(t) \in V_k[a,b],\lambda)$. □

Theorem 1 and Theorem 3 imply

Corollary 2

Let $k \geq 3$ then the following are equivalent

1^0 $X(t) \in V_k([a,b],\lambda)$;

2^0 $\forall 2 \leq r < k,\ X^{(k-r)}(t) - X^{(k-r)}(a) \in V_r([a,b],\lambda)$;

3^0 $\exists 2 \leq r < k$, *such that* $X^{(k-r)}(t) - X^{(k-r)}(a) \in V_r([a,b],\lambda)$;

4^0 $\forall 2 \leq r < k$, *we have*

(i) $x_m^{(k-r)}(t) \in V_r[a,b], m = 1,2,\ldots$

(ii) $\{\overset{b}{\underset{a}{V}}{}_r(x_m^{(k-r)})\} \in \lambda^{**}$

5^0 $\exists 2 \leq r < k$ *such that*

(i) $x_m^{(k-r)}(t) \in V_r[a,b], m = 1,2,\ldots$

(ii) $\{\overset{b}{\underset{a}{V}}{}_r(x_m^{(k-r)}\} \in \lambda^{**}$.

Theorem 4

Assume that λ is a perfect space, then $X(t) \in V_k([a,b],\lambda)$ iff there exist convex functions $X^{(i)}(t) \in V_k([a,b],\lambda)$ $(i = 1,2)$ of order k such that

$$X(t) = X^{(1)}(t) - X^{(2)}(t) \quad (t \in [a,b])$$

($X(t)$ is called a convex function of order k, if for each natural number $m, x_m(t)$ is a usual convex function of order k and $x(t)$ is called a usual convex function of order k, if for any partition $\pi\colon a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$, we have $Q_k(x;t_0,t_1,\ldots,t_k) \geq 0$).

Proof. The sufficiency is obvious. Now we prove the necessity. The necessity is already known for $k = 1$ and $k = 2$. Suppose that the condition is necessary for $k = m - 1$, we investigate the case $k = m$. Since by Theorem 3, $X'(t) \in V_{m-1}([a, b], \lambda)$ by the assumption, there exist convex functions $Y^{(i)}(t) \in V_{m-1}([a, b], \lambda)$ of order $m - 1$ $(i = 1, 2)$ such that $X'(t) = Y^{(1)}(t) - Y^{(2)}(t)$. For any $c \in (a, b)$, we have

$$X(t) = \int_0^t X'(s)ds - X(c) = \int_0^t Y^{(1)}(s)ds - \int_0^t Y^{(2)}(s)ds - X(c).$$

Set

$$X^{(1)}(t) = \int_0^t Y^{(1)}(s)ds, \quad X^{(2)}(t) = \int_0^t Y^{(2)}(s)ds + X(c)$$

then by Theorem 13 in [8], $X^{(i)}(t)$ is convex of order $m, i = 1, 2$. But $(X^{(i)}(t))' \in V_{m-1}([a, b], \lambda)$, by Theorem 3, $X^{(i)}(t) \in V_m([a, b], \lambda)$ $i = 1, 2$. Clearly, $X(t) = X^{(1)}(t) - X^{(2)}(t)$, and $X^{(1)}(t) \in \lambda$ (and thus $X^{(2)}(t) \in \lambda$) can be deduced as follows

$$\begin{aligned} |x_m^{(1)}(t)| \le \int_a^b |y_m^{(1)}(s)|ds \le \int_a^b |y_m^{(1)}(a)| + \overset{b}{\underset{a}{V}} (y_m^{(1)})ds \\ + (b - a)\big(|y_m^{(1)}(a)| + \overset{b}{\underset{a}{V}} (y_m^{(1)})\big) \end{aligned}$$

therefore, $\{X_m^{(1)}(t)\} \in \lambda^{**} = \lambda$. $\square$

Theorem 5

Let λ be perfect, then $X(t), Y(t) \in V_k([a, b], \lambda)$ implies $X(t)Y(t) \in V_k([a, b], \lambda)$ iff for any $Z(t) \in V_k([a, b]), \lambda), U \in \lambda^*$ and $c \in [a, b]$, we have

$$\Big\{|u_m|\big(|Z_m(c)| + \overset{b}{\underset{a}{V}} (Z_m) + \overset{b}{\underset{a\ 2}{V}} (Z_m) + \cdots + \overset{b}{\underset{a\ k}{V}} (Z_m)\big)\Big\} \in \lambda^*\,.$$

Proof. Sufficiency. Let $X(t) \in V_k([a, b], \lambda), Y(t) \in V_k([a, b], \lambda)$, by Lemma 4,

$$
\begin{aligned}
&\sup_{\pi}\sum_{i=1}^{n-k}\sum_{m=1}^{\infty}|u_m|\ |t_{i+k}-t_i|\ |Q_k(x_my_m;t_i,t_{i+1},\dots,t_{i+k})| \\
&=\sup_{\pi}\sum_{i=1}^{n-k}\sum_{m=1}^{\infty}|u_m|\ |t_{i+k}-t_i|\ |y_m(t_i)Q_k(x_m;t_i,\dots,t_{i+k}) \\
&\quad+Q_1(y_m;t_i,t_{i+1})Q_{i-1}(x_m;t_{i+1},\dots,t_{i+k}) \\
&\quad+\cdots+Q_k(y_m;t_i,\dots,t_{i+k})x_m(t_{i+k})| \\
&\le\sup_{\pi}\sum_{i=1}^{n-k}\sum_{m=1}^{\infty}|u_m|\ |t_{i+k}-t_i|\big[\big|y_m(t_i)Q_k(x_m;t_i,\dots,t_{i+k}) \\
&\quad+Q_1(y_m;t_i,t_{i+1})Q_{i-1}(x_m;t_{i+1},\dots,t_{i+k}) \\
&\quad+\cdots+Q_k(y_m;t_i,\dots,t_{i+k})x_m(t_{i+k})\big|\big] \\
&\le\sup_{\pi}\sum_{i=1}^{n-k}\sum_{m=1}^{\infty}|u_m|\ |t_{i+k}-t_i|\big(|y_m(a_0)| \\
&\quad+2\overset{b}{\underset{a}{V}}(y_n)\big)|Q_k(x_m;t_i,\dots,t_{i+k})| \\
&\quad+\sup_{\pi}\sum_{i=1}^{n-k}\sum_{m=1}^{\infty}|u_m|\ |t_{i+k}-t_i|(|Q_1(y_m;a_0,a_1)| \\
&\quad+\overset{b}{\underset{a}{V}}{}_2\,(y_m)(|Q_{k-1}(x_m;a_0,\dots,a_{t-1})|+\overset{b}{\underset{a}{V}}{}_k\,(x_m) \\
&\quad+\cdots+\sup_{\pi}\sum_{i=1}^{n-k}\sum_{m=1}^{\infty}|u_m|\ |t_{i+k}-t_i|\ |Q_k(y_m;t_i,\dots,t_{i+k})|(|x_m(a_0)|+2\overset{b}{\underset{a}{V}}(x_m)),
\end{aligned}
$$

where $\{a_i\}_{i=0}^{k-1}$ are different points in (a,b). For the first term, we have

$$
\begin{aligned}
&\sup_{\pi}\sum_{i=1}^{n-k}\sum_{m=1}^{\infty}|u_m|\ |t_{i+k}-t_i|\big(|y_m(a_0)|+2\overset{b}{\underset{a}{V}}(y_m)\big)|Q_k(x_m;t_i,\dots,t_{i+k})| \\
&\qquad\le\sum_{m=1}^{\infty}|u_m|c|y_m(a_0)|+2\overset{b}{\underset{a}{V}}(y_m))\sup_{\pi}\sum_{i=0}^{n-k}|t_{i+k}-t_i|\ |Q_k(x_m;t_1,\dots,t_{i+k})| \\
&\qquad=\sum_{m=1}^{\infty}|u_m|\big(|y_m(a_0)|+2\overset{b}{\underset{a}{V}}(y_m)\big)\overset{b}{\underset{a}{V}}{}_k\,(x_m)<\infty
\end{aligned}
$$

(note that $\{|u_m|(|y_m(a_0)|+2\,V_a^b\,(y_m))\}\in\lambda^*,\{V_{a\,k}^b\,(x_m)\}\in\lambda^{**}$) .

Similarly, for the last term, we have

$$\sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k} - t_i|\ |Q_k(y_m; t_i, \ldots, t_{i+k})| \big(|x_m(a_0)| + 2 \overset{b}{\underset{a}{V}} (x_m)\big) < \infty.$$

Now, we show that the other terms are also bounded. Without loss of generality, we only consider the term

$$\begin{aligned}\sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k} - t_i| (|Q_1((y_m; a_0, a_1)| \\ + 2 \overset{b}{\underset{a\ 2}{V}} (y_m))(|Q_{k-1}(x_m; a_0, \ldots, a_{i-1})| + 2 \overset{b}{\underset{a\ k}{V}} (x_m)) \,.\end{aligned}$$

By Lemma 5,

$$\begin{aligned}&\sup_{\pi} \sum_{i=1}^{n-k} \sum_{m=1}^{\infty} |u_m|\ |t_{i+k} - t_i| (|Q_1((y_m; a_0, a_1)| \\ &\quad + 2 \overset{b}{\underset{a\ 2}{V}} (y_m))(|Q_{k-1}(x_m; a_0, \ldots, a_{i-1})| + 2 \overset{b}{\underset{a\ k}{V}} (x_m)) \\ &\leq k(b-a) \sum_{m=1}^{\infty} |u_n| (|Q_1(y_m; a_0, a_1)| \\ &\quad + 2 \overset{b}{\underset{a\ 2}{V}} (y_m))(|Q_{k-1}(x_m; a_0, \ldots, a_{k-1})| + 2 \overset{b}{\underset{a\ k}{V}} (x_m)) < \infty.\end{aligned}$$

Thus, $X(t)Y(t) \in V_k([a, b], \lambda)$.

Necessity. By Theorem 2.6 in [7], the condition is necessary for $k = 1$. Now, suppose $k \geq 2$. Define

$$\begin{aligned}x_m(t) &= \big(|Z_m(c)| + \overset{b}{\underset{a}{V}} (Z_m) + \ldots + \overset{b}{\underset{a\ k}{V}} (Z_m)\big) t^{k-1} \\ y_m(t) &= |x_m^{(0)}| t\end{aligned} \qquad (a \leq t \leq b)$$

then from

$$x_m^{(k-1)}(t) = (k-1)! \big(|Z_m(c)| + \overset{b}{\underset{a}{V}} (Z_m) + \cdots + \overset{b}{\underset{a\ k}{V}} (Z_m)\big)$$

$$y_m^{(k-1)}(t) = \begin{cases} |x_m^{(0)}|, & k = 2 \\ 0, & k > 2 \end{cases}$$

we have

$$x_m^{(k-2)} \in V_2[a,b], \quad y_m^{(k-2)} \in V_2[a,b].$$

Hence, Theorem 1 in [9] implies $x_m, y_m \in V_k[a,b]$, and Proposition 3.4 in [7] claims $\overset{b}{\underset{a}{V}}{}_k (x_m) = \overset{b}{\underset{a}{V}}{}_k (y_m) = 0$. Thus, $\{\overset{b}{\underset{a}{V}}{}_k (x_m)\} \in \lambda^{**}$ and $\{\overset{b}{\underset{a}{V}}{}_k (y_m)\} \in \lambda^{**}$, and so, by Theorem 1,

$$X(t) = \{x_m(t)\} \in V_k([a,b],\lambda)$$

$$y(t) = \{y_m(t)\} \in V_k([a,b],\lambda).$$

Therefore $X(t)y(t) \in V_k([a,b]),\lambda)$.

For any $t_0 \neq t_1 \neq \cdots \neq t_k$ in (a,b), by Proposition 3.5 in [7] p. 82,

$$Q_k(x_m y_m; t_0, \dots, t_k) = |x_m^{(0)}|\big(|Z_m(c)| + \overset{b}{\underset{a}{V}} (Z_m) + \cdots + \overset{b}{\underset{a}{V}}{}_k (Z_m)|\big)$$

and from

$$\begin{aligned} \infty > \overset{b}{\underset{a}{V}}{}_k (XY;U) &\geq \sum_{m=1}^{\infty} |u_m|\ |t_k - t_0|\ |Q_k(x_m y_m; t_0, \dots, t_k)| \\ &= |t_k - t_0| \sum_{m=1}^{\infty} |u_m|\big(|Z_m(c) + \overset{b}{\underset{a}{V}} (Z_m) + \cdots + \overset{b}{\underset{a}{V}}{}_k (Z_k)|\big)\ |x_m^{(0)}| \end{aligned}$$

we find

$$\big\{|u_m|\big(|Z_m(c)| + \overset{b}{\underset{a}{V}} (Z_m) + \cdots + \overset{b}{\underset{a}{V}}{}_k (Z_m)\big)\big\} \in \lambda^*. \ \square$$

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