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Order continuous seminorms and weak compactness in Orlicz spaces

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Abstract

Let L^φ be an Orlicz space defined by a Young function φ over a σ-finite measure space, and let φ^* denote the complementary function in the sense of Young. We give a characterization of the Mackey topology $\tau(L^*, L^{\varphi^*})$ in terms of some family of norms defined by some regular Young functions. Next, we describe order continuous (= absolutely continuous) Riesz seminorms on L^φ, and obtain a criterion for relative $\sigma(L^\varphi, L^{\varphi^*})$-compactness in L^φ. As an application we get a representation of L^φ as the union of some family of other Orlicz spaces. Finally, we apply the above results to the theory of Lebesgue spaces.

0. Introduction and preliminaries

In 1915 de la Vallée Poussin (see [12]) showed that a set Z of L^1 $\big($for a finite measure space $(\Omega, \Sigma, \mu)\big)$ has uniformly absolutely continuous L^1-norms (i.e., $\lim\limits_{\mu(E)\to 0} (\sup\limits_{x\in Z} \int\limits_E |x(t)|d\mu) = 0$) iff there exists a Young function ψ such that $\lim\limits_{u\to\infty} \psi(u)/u = \infty$ in terms of which $\sup\limits_{x\in Z} \int\limits_\Omega \psi(|x(t)|)d\mu < \infty$.

On the other hand, in view of the Dunford-Pettis criterion (on relatively compact sets in L^1)(see [3, p. 294]) the set $Z \subset L^1$ has uniformly absolutely continuous L^1-norms iff it is relatively $\sigma(L^1, L^\infty)$-compact.

Thus we have the following criterion for relative weak compactness in L^1 (for finite measures): a set Z of L^1 is relatively $\sigma(L^1, L^\infty)$-compact iff there exists a Young function ψ such that $\lim\limits_{u\to\infty} \psi(u)/u = \infty$ and $\sup\limits_{x\in Z} \int\limits_\Omega \psi(|x(t)|)d\mu < \infty$.

In 1962 T. Ando [2, Theorem 2] found similar criterion for relative $\sigma(L^\varphi, L^{\varphi^*})$-compactness in L^φ for φ being an N-function and a finite measure. This criterion was extended by the present author to the case of σ-finite measures ([15, Theorem 1.2]).

In this paper, using a different method, we extend the Ando's criterion to the case of φ belonging to a much wider class of Young functions and σ-finite measures. We can include L^φ being equal to $L^1 + L^\infty$ (so L^1 if $\mu(\Omega) < \infty$), $L^1 + L^p, L^p + L^\infty$ $(p > 1)$.

In section 1, making use of the author's results concerning the so-called modular topology $\mathcal{T}_\varphi^\wedge$ on L^φ (see [13], [14], [18], [19]), we obtain a characterization of the Mackey topology $\tau(L^\varphi, L^{\varphi^*})$ in terms of some family of norms defined by some regular Young functions, dependent on φ (see Theorem 1.5). As an application we have a description of absolutely continuous (= order continuous) Riesz seminorms on L^φ (see Corollary 1.6).

In section 2, in view of the close connection between relative $\sigma(L^\varphi, L^{\varphi^*})$-compactness in L^φ and the absolute continuity of some seminorm in L^{φ^*}, we can describe relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact sets in L^φ as norm bounded subsets of an Orlicz space L^ψ for some regular Young function ψ (see Theorem 2.4). As an application we get a representation of the Orlicz space L^φ as the union of some family of other Orlicz spaces. At last, we examine the absolute weak topology $|\sigma|(L^\varphi, L^{\varphi^*})$.

In section 3 we apply the results of sections 1 and 2 to the theory of Lebesgue spaces.

For notation and terminology concerning Riesz spaces we refer to [1], [21]. As usual, $\mathbb{N}$ stands for the set of all natural numbers.

Let (Ω, Σ, μ) be σ-finite measure space, and let L^0 denote the set of equivalence classes of all real valued measurable functions defined and a.e. finite on Ω. Then L^0 is a super Dedekind complete Riesz space under the ordering $x \leq y$ whenever $x(t) \leq y(t)$ a.e. on Ω. The Riesz F-norm

$$\|x\|_0 = \int_\Omega \frac{|x(t)|}{1 + |x(t)|} f(t) d\mu \qquad \text{for } \ x \in L^0,$$

where $f\colon \Omega \to (0, \infty)$ is measurable and $\int\limits_\Omega f(t) d\mu = 1$, determines the Lebesgue topology $\mathcal{T}_0$ on L^0, which generates the convergence in measure on subsets of Ω of finite measure. For a sequence (x_n) in L^0 we will write $x_n \to x(\mu)$ whenever $\|x_n - x\|_0 \to 0$.

For a subset A of Ω and $x \in L^0$ we will write $x_A = x \cdot \chi_A$, where χ_A stands for the characteristic function of A. We will write $E_n \searrow \emptyset$ if (E_n) is a decreasing sequence of measurable subsets of Ω such that $\mu(E_n \cap E) \to 0$ for every set $E \subset \Omega$ of finite measure.

Now we recall some notation and terminology concerning Orlicz spaces (see [6], [8], [10], [20] for more details).

By an Orlicz function we mean a function $\varphi: [0, \infty) \to [0, \infty]$ which is non-decreasing, left continuous, continuous at 0 with $\varphi(0) = 0$, not identically equal to 0.

An Orlicz function φ is called convex, whenever $\varphi(\alpha u + \beta v) \leq \alpha\varphi(u) + \beta\varphi(v)$ for $\alpha, \beta \geq 0, \alpha + \beta = 1$ and $u, v \geq 0$. A convex Orlicz function is usually called a Young function.

For a Young function φ we denote by φ^* the function complementary to φ in the sense of Young, i.e.,

$$\varphi^*(v) = \sup \left\{uv - \varphi(u): u \geq 0\right\} \quad \text{for} \quad v \geq 0.$$

For a set Ψ of Young functions we will write

$$\Psi^* = \{\psi^*: \psi \in \Psi\} .$$

We shall say that an Orlicz function ψ is *completely weaker than another* φ for all u (resp. for small u; resp. for large u), in symbols $\psi \overset{\mathrm{a}}{\triangleleft} \varphi$ (resp. $\psi \overset{\mathrm{s}}{\triangleleft} \varphi$; resp. $\psi \overset{\mathrm{l}}{\triangleleft} \varphi$), if for an arbitrary $c > 1$ there exists a constant $d > 0$ such that $\psi(cu) \leq d\varphi(u)$ for $u \geq 0$ (resp. for $0 \leq u \leq u_0$; resp. for $u \geq u_0 \geq 0$). (See [2], [20, Ch. II]).

It is seen that φ satisfies the so called Δ_2-condition for all u (resp. for small u; resp. for large u) if and only if $\varphi \overset{\mathrm{a}}{\triangleleft} \varphi$ (resp. $\varphi \overset{\mathrm{s}}{\triangleleft} \varphi$; resp. $\varphi \overset{\mathrm{l}}{\triangleleft} \varphi$).

We shall say that an Orlicz function φ *increases more rapidly* than another ψ for all u (resp. for small u; resp. for large u) in symbols $\psi \overset{\mathrm{a}}{\prec} \varphi$ (resp. $\psi \overset{\mathrm{s}}{\prec} \varphi$; resp. $\psi \overset{\mathrm{l}}{\prec} \varphi$), if for an arbitrary $c > 0$ there exists $d > 0$ such that $c\psi(u) \leq \frac{1}{d}\varphi(du)$ for all $u \geq 0$ (resp. for $0 \leq u \leq u_0$; resp. for $u \geq u_o \geq 0$).

Note that φ satisfies the so called ∇_2-condition for all u (resp. for small u; resp. for large u) if and only if $\varphi \overset{\mathrm{a}}{\prec} \varphi$ (resp. $\varphi \overset{\mathrm{s}}{\prec} \varphi$; resp. $\varphi \overset{\mathrm{l}}{\prec} \varphi$).

One can verify that for given Young functions ψ and φ the relation $\psi \overset{\mathrm{a}}{\triangleleft} \varphi$ (resp. $\psi \overset{\mathrm{s}}{\triangleleft} \varphi$; resp. $\psi \overset{\mathrm{l}}{\triangleleft} \varphi$) holds iff $\varphi^* \overset{\mathrm{a}}{\prec} \psi^*$ (resp. $\varphi^* \overset{\mathrm{s}}{\prec} \varphi^*$, resp. $\varphi^* \overset{\mathrm{l}}{\prec} \psi^*$) holds (see [2], [20, Proposition 2.2.4]).

An Orlicz function φ determines the functional $m_\varphi\colon L^0 \to [0,\infty]$ by

$$m_\varphi(x) = \int_\Omega \varphi(|x(t)|)d\mu.$$

The Orlicz space generated by φ is the ideal of L^0 defined by

$$L^\varphi = \{x \in L^0\colon m_\varphi(\lambda x) < \infty \quad \text{for some} \quad \lambda > 0\}.$$

The functional m_φ restricted to L^φ is an orthogonally additive semimodular (see [10], [11]).

L^φ can be equipped with the complete metrizable linear topology $\mathcal{T}_\varphi$ of the Riesz F-norm

$$\| x \|_\varphi = \inf\Big\{\lambda > 0\colon m_\varphi\Big(\frac{x}{\lambda}\Big) \le \lambda\Big\}.$$

Moreover, when φ is a Young function, the topology $\mathcal{T}_\varphi$ can be generated by two Riesz norms (called the Orlicz and the Luxemburg norms resp.) defined as follows:

$$\|x\|_\varphi = \sup\Big\{\int_\Omega |x(t)y(t)|d\mu\colon y \in L^{\varphi^*}, m_{\varphi^*}(y) \le 1\Big\}$$

$$|||x|||_\varphi = \inf\Big\{\lambda > 0\colon m_\varphi(\frac{x}{\lambda}) \le 1\Big\}.$$

For an Orlicz function φ let

$$E^\varphi = \{x \in L^0\colon m_\varphi(\lambda x) < \infty \quad \text{for all} \quad \lambda > 0\}$$

and

$$L^\varphi_a = \{x \in L^\varphi\colon \| x_{E_n} \|_\varphi \to 0 \quad \text{as} \quad E_n \searrow \emptyset\}.$$

It is well known that for φ taking only finite values these spaces coincide, i.e., $E^\varphi = L^\varphi_a$.

1. The Mackey topology $\tau(L^\varphi, L^{\varphi^*})$

First we recall the definition and the basic properties of the so-called modular topology on Orlicz spaces (see [13], [14]).

Let φ be an Orlicz function vanishing only at 0. For given $\varepsilon > 0$, let $U_\varphi(\varepsilon) = \{x \in L^\varphi\colon m_\varphi(x) \le \varepsilon\}$. Then the family of all sets of the form

$$\bigcup_{N=1}^{\infty} \Big(\sum_{n=1}^{N} U_\varphi(\varepsilon_n)\Big)$$

where (ε_n) is a sequence of positive numbers, forms a base of neighborhoods of 0 for a linear topology on L^φ, that will be called *the modular topology* on L^φ and will be denoted by $\mathcal{T}^\wedge_\varphi$.

The basic properties of $\mathcal{T}^\wedge_\varphi$ are included in the following theorem (see [14, Theorem 1.1], [18, Theorem 2.2], [19. Theorem 4.2]):

Theorem 1.1

Let φ be an Orlicz function vanishing only at 0. Then the following statements hold:

(i) $\mathcal{T}_\varphi^\wedge$ is the finest σ-Lebesgue topology on L^φ.

(ii) $\mathcal{T}_\varphi^\wedge \subset \mathcal{T}_\varphi$ and the equality $\mathcal{T}_\varphi^\wedge = \mathcal{T}_\varphi$ holds whenever $\varphi \in \Delta_2$.

(iii) $\mathcal{T}_\varphi^\wedge$ coincides with the Mackey topology $\tau(L^\varphi, L^{\varphi^*})$, whenever φ is a Young function.

To present the crucial for this paper characterization of the modular topology $\mathcal{T}_\varphi^\wedge$ we will distinguish some classes of Orlicz functions.

An Orlicz function φ continuous for all $u \geq 0$, taking only finite values, vanishing only at zero, and such that $\varphi(u) \to \infty$ as $u \to \infty$ is usually called a *φ-function* (see [10]). We will denote by Φ the collection of all φ-functions.

A Young function φ vanishing only at 0 and taking only finite values is called *an N-function* whenever $\lim_{u\to 0} \varphi(u)/u = 0$ and $\lim_{u\to\infty} \varphi(u)/u = \infty$ (see [6], [10]). We will denote by Φ_N the collection of all N-functions.

Let Φ_0 be the collection of all Orlicz functions φ vanishing only at 0 and such that $\varphi(u) \to \infty$ as $u \to \infty$. Let

$$\Phi_{01} = \{\varphi \in \Phi_0 \colon \varphi(u) < \infty \quad \text{for} \quad u \geq 0\}$$
$$\Phi_{02} = \{\varphi \in \Phi_0 \colon \varphi \ \text{ jumps to } \ \infty, \text{ i.e., } \varphi(u) = \infty \text{ for } u > u_0 > 0\}.$$

The following characterizations of the modular topology $\mathcal{T}_\varphi^\wedge$ will be crucial for this paper (see [13, Theorem 2.1], [14, Theorem 1.2]).

Theorem 1.2

Let $\varphi \in \Phi_{0i} (i = 1, 2)$. Then the modular topology $\mathcal{T}_\varphi^\wedge$ is generated by the family of F-norms:

$$\{\|\cdot\|_{\psi|L^\varphi} \colon \psi \in \Psi_{0i}^\varphi\}$$

where

$$\Psi_{01}^\varphi = \{\psi \in \Psi \colon \psi \overset{\mathrm{a}}{\triangleleft} \varphi\}, \quad \Psi_{02}^\varphi = \{\psi \in \Phi \colon \psi \overset{\mathrm{s}}{\triangleleft} \varphi\}.$$

Now, for φ being a Young function we are going to apply Theorem 1.2 to obtain a description of the Mackey topology $\tau(L^\varphi, L^{\varphi^*})$ in terms of some family of norms defined by some regular Young functions.

For this purpose we distinguish some classes of Young functions.

Let Φ_0^c be the collection of all Young functions φ vanishing only at 0 and such that $\lim\limits_{u\to\infty} \varphi(u)/u = \infty$.

Let

$$\begin{aligned}
\Phi_{01}^c &= \{\varphi \in \Phi_0^c : \varphi(u) < \infty \ \text{ for all } \ u \geq 0 \text{ and } \lim_{u\to 0} \tfrac{\varphi(u)}{u} = 0\},\\
\Phi_{02}^c &= \{\varphi \in \Phi_0^c : \varphi \ \text{ jumps to } \infty \ \text{ and } \lim_{u\to 0} \tfrac{\varphi(u)}{u} = 0\},\\
\Phi_{03}^c &= \{\varphi \in \Phi_0^c : \varphi(u) < \infty \ \text{ for all } \ u \geq 0 \text{ and } \lim_{u\to 0} \tfrac{\varphi(u)}{u} > 0\},\\
\Phi_{04}^c &= \{\varphi \in \Phi_0^c : \varphi \ \text{ jumps to } \infty \ \text{ and } \lim_{u\to 0} \tfrac{\varphi(u)}{u} > 0\}.
\end{aligned}$$

Then $\Phi_0^c = \bigcup\limits_{i=1}^{4} \Phi_{0i}^c$, and the sets $\Phi_{0i}^c (i = 1, 2, 3, 4)$ are pairwise disjoint. It is seen that $\Phi_{01}^c = \Phi_N$. Denote by

$$\begin{aligned}
&\Psi_{01}^{\varphi}(c) = \{\psi \in \Phi_N : \psi \overset{\mathrm{a}}{\lhd} \varphi\}, \quad \text{whenever} \quad \varphi \in \Phi_{01}^c,\\
&\Psi_{02}^{\varphi}(c) = \{\psi \in \Phi_N : \psi \overset{\mathrm{s}}{\lhd} \varphi\}, \quad \text{whenever} \quad \varphi \in \Phi_{02}^c,\\
&\Psi_{03}^{\varphi}(c) = \{\psi \in \Phi_{03} : \psi \overset{\mathrm{l}}{\lhd} \varphi\}, \quad \text{whenever} \quad \varphi \in \Phi_{03}^c,\\
&\Psi_{04}^{\varphi}(c) = \Phi_{03}^c, \quad \text{whenever} \quad \varphi \in \Phi_{04}^c.
\end{aligned}$$

The following two lemmas will be needed.

Lemma 1.3

Let $\varphi \in \Phi_{0i}^c (i = 1, 2)$ and let ψ be a φ-function such that $\psi \overset{\mathrm{a}}{\lhd} \varphi$ for $i = 1$ (resp. $\psi \overset{\mathrm{s}}{\lhd} \varphi$ for $i = 2$). Then there exists $\psi_0 \in \Psi_{0i}^{\varphi}(c)$ such that

$$\psi(u) \leq \psi_0(2u) \quad \text{for} \quad u \geq 0.$$

Proof. Take an arbitrary N-function ψ_1 such that $\psi_1 \overset{\mathrm{a}}{\lhd} \varphi$ for $i = 1$ (resp. $\psi_1 \overset{\mathrm{s}}{\lhd} \varphi$ for $i = 2$). Let us set

$$\psi_2(u) = \max\big(\psi(u), \psi_1(u)\big) \quad \text{for} \quad u \geq 0.$$

Let us put

$$p(s) = \begin{cases} 0 & \text{for} \quad s = 0,\\ \sup_{0<t\leq s} \frac{\psi_2(t)}{t} & \text{for} \quad s > 0, \end{cases}$$

and let

$$\psi_0(u) = \int_0^u p(s)ds.$$

To show that ψ_0 is an N-function we have to check that $\lim_{u\to 0} p(u) = 0$ and $\lim_{u\to\infty} p(u) = \infty$.

Indeed, since $\psi \overset{\text{a}}{\lhd} \varphi$ we get that $\psi(u) \le a\varphi(u)$ for some $a > 0$ and $u \ge 0$. Hence

$$\begin{aligned} p(u) &= \sup_{0<t\le u} \frac{\psi_2(t)}{t} \le \sup_{0<t\le u} \frac{\psi(t)}{t} + \sup_{0<t\le u} \frac{\psi_1(t)}{t} \\ &\le a \sup_{0<t\le u} \frac{\varphi(t)}{t} + \sup_{0<t\le u} \frac{\psi_1(t)}{t} = a\frac{\varphi(u)}{u} + \frac{\psi_1(t)}{u}. \end{aligned}$$

Thus $\lim_{u\to\infty} p(u) = 0$, because $\lim_{u\to 0} \varphi(u)/u = 0$ and $\lim_{u\to 0} \psi_1(u)/u = 0$. Moreover, we have: $p(u) = \sup_{0<t\le u} \psi_2(t)/t \ge \sup_{0<t\le u} \psi_1(t)/t = \psi_1(u)/u$, because ψ_1 is a Young function. Hence $\lim_{u\to\infty} p(u) = \infty$, because $\lim_{u\to\infty} \psi_1(u)/u = \infty$.

Now we shall show that $\psi_0 \overset{\text{a}}{\lhd} \varphi$ if $i = 1$ (resp. $\psi \overset{\text{s}}{\lhd} \varphi$ if $i = 2$). Indeed, given $c > 0$ there exist $d > 0$ such that

$$\psi_2(u) = \psi(u) \vee \psi_1(u) \le d\varphi\Big(\frac{u}{c}\Big) \quad \text{for} \quad u \ge 0.$$

Hence

$$p(cu) = \sup_{0<t\le cu} \frac{\psi_2(t)}{t} \le \sup_{0<t\le cu} \frac{d\varphi(\frac{t}{c})}{t} = \frac{d\varphi(u)}{cu} \quad \text{for} \quad u \ge 0,$$

so

$$\psi_0(cu) \le p(cu) \cdot cu \le d\varphi(u) \quad \text{for} \quad u \ge 0.$$

Similarly we can show that $\psi_0 \overset{\text{s}}{\lhd} \varphi$ if $i = 2$.

At last we will show that $\psi(u) \le \psi_0(2u)$ for $u \ge 0$. Indeed, we have $\psi_0(2u) \ge p(u) \cdot u$ and

$$p(u) = \sup_{0<t\le u} \frac{\psi_2(t)}{t} \ge \sup_{0<t\le u} \frac{\psi(t)}{t} \ge \frac{\psi(u)}{u} \quad \text{for} \quad u \ge 0.$$

Thus

$$\psi_0(2u) \ge \psi(u) \quad \text{for} \quad u \ge 0. \ \square$$

Lemma 1.4

Let $\varphi \in \Phi_{0i}^c$ $(i = 3, 4)$ and let ψ be a φ-function such that $\psi \overset{\mathrm{a}}{\lhd} \varphi$ for $i = 3$ (resp. $\psi \overset{\mathrm{s}}{\lhd} \varphi$ for $i = 4$). Then there exists a Young function $\psi_0 \in \Psi_{0i}^{\varphi}(c)$ such that

$$\psi(u) \leq \psi_0(2u) \quad \text{for} \quad u \geq 0.$$

Proof. Take an arbitrary N-function ψ_1 such that $\psi_1(u) \leq \varphi(u)$ and $\psi_1 \overset{\mathrm{a}}{\lhd} \varphi$. Let $a > 1$ be such that $a\psi_1(1) = \varphi(1)$. Let us set

$$\psi_2(u) = \begin{cases} \max\big(\psi(u), \varphi(u)\big) & \text{for } 0 \leq u \leq 1, \\ \max\big(\psi(u), a\psi_1(u)\big) & \text{for } u \geq 1. \end{cases}$$

Let

$$p(s) = \begin{cases} 0 & \text{for } s = 0, \\ \sup_{0<t\leq s} \frac{\psi_2(t)}{t} & \text{for } s > 0, \end{cases}$$

and let

$$\psi_0(u) = \int_0^u p(s)ds \quad \text{for} \quad u \geq 0.$$

We shall show that $\psi_0 \in \Psi_{03}^c$, i.e., that $\lim\limits_{u\to 0} \psi_0(u)/u > 0$ and $\lim\limits_{u\to\infty} \psi_0(u)/u = \infty$. Indeed, for $0 \leq u \leq 1$ we have

$$p(u) = \sup_{0<t\leq u} \frac{\psi_2(t)}{t} \geq \sup_{0<t\leq u} \frac{\varphi(t)}{t} = \frac{\varphi(u)}{u},$$

so

$$\lim_{u\to 0} p(u) \geq \lim_{u\to 0} \frac{\varphi(u)}{u} > 0.$$

Since $\psi_0(u) \geq p(\frac{u}{2}) \cdot \frac{u}{2}$, we get $\lim\limits_{u\to 0} \psi_0(u)/u > 0$.

To show that $\lim\limits_{u\to\infty} \psi_0(u)/u = \infty$ it is enough to show that $\lim\limits_{u\to\infty} p(u) = \infty$. Indeed, let $u_0 > 1$ be such that $a\psi_1(u)/u \geq K = \sup\limits_{0<t\leq 1} \psi_2(t)/t$ for $u \geq u_0$. Then for $u \geq u_0$ we have:

$$\begin{aligned} p(u) = \sup_{0<t\leq u} \frac{\psi_2(t)}{t} &= \max\Big(K, \sup_{1\leq t\leq u} \frac{\psi_2(t)}{t}\Big) \geq \max\Big(K, \sup_{1\leq t\leq u} \frac{\psi_1(t)}{t}\Big) \\ &= \max\Big(K, \sup_{1\leq t\leq u} \frac{\psi_1(u)}{u}\Big) = \frac{a\psi_1(u)}{u}. \end{aligned}$$

Thus $\lim_{u\to\infty} p(u) = \infty$, because $\lim_{u\to\infty} \psi_1(u)/u = \infty$.

Now, for $i = 3$ we shall show that $\psi_0 \overset{1}{\triangleleft} \varphi$. Indeed, given $c > 1$ there exists $d > 1$ such that for $u \geq 0$

$$\psi(u) \leq d\varphi\Big(\frac{u}{c}\Big) \quad \text{and} \quad a\psi_1(u) \leq d\varphi\Big(\frac{u}{c}\Big).$$

Let $u_0 > 0$ be such that $d\varphi(u_0)/u_0 \geq K = \sup_{0<t\leq 1} \psi_2(t)/t$. Then for $u \geq u_o$ we get

$$\begin{aligned} p(cu) &= \sup_{0<t\leq cu} \frac{\psi_2(t)}{t} = \max\Big(\sup_{0<t\leq 1} \frac{\psi_2(t)}{t}, \sup_{0<t\leq cu} \frac{\psi_2(t)}{t}\Big) \\ &\leq \max\Big(K, \sup_{1\leq t\leq cu} \frac{d\varphi(\frac{t}{c})}{t}\Big) = \max\Big(K, \frac{d\varphi(u)}{cu}\Big) = \frac{d\varphi(u)}{cu}. \end{aligned}$$

Thus for $u \geq u_0$

$$\psi_0(cu) \leq p(cu) \cdot cu \leq d\varphi(u)\,,$$

i.e. $\psi_0 \overset{1}{\triangleleft} \varphi$.

At last, we shall show that $\psi(u) \leq \psi_0(2u)$ for $u \geq 0$ $(i = 3, 4)$. Indeed, we have $\psi_0(2u) \geq p(u) \cdot u$ and

$$p(u) = \sup_{0<t\leq u} \frac{\psi_2(t)}{t} \geq \sup_{0<t\leq u} \frac{\psi(u)}{u} \quad \text{for} \quad u \geq 0.$$

Thus $\psi(u) \leq \psi_0(2u)$ for $u \geq 0$. □

We are now in position to present a description of the Mackey topology $\tau(L^\varphi, L^{\varphi^*})$ in terms of some family of norms defined by some regular Young functions.

Theorem 1.5

Let $\varphi \in \Phi_{0i}^c$ $(i = 1, 2, 3, 4)$. Then the Mackey topology $\tau(L^\varphi, L^{\varphi^*})$ is generated by the family of norms:

$$\{|||\cdot|||_{\psi|L^\varphi} : \psi \in \Psi_{0i}^\varphi(c)\}.$$

Proof. In view of Theorem 1.1 the equality $\mathcal{T}_\varphi^\wedge = \tau(L^\varphi, L^{\varphi^*})$ holds. Let $\varphi \in \Phi_{0i}^c$ $(i = 1,2,3,4)$. Then $\varphi \in \Phi_{01}$ for $i = 1,3$, and $\varphi \in \Phi_{02}$ for $i = 2,4$. Thus, according to Theorem 1.2 the Mackey topology $\tau(L^\varphi, L^{\varphi^*})$ is generated by the family $\{\Vert \cdot \Vert_{\psi|L^\varphi} : \psi \in \Psi_{01}^\varphi\}$ for $i = 1,3$, and by the family $\{\Vert \cdot \Vert_{\psi|L^\varphi} : \psi \in \Psi_{01}^\varphi\}$ for $i = 2,4$.

Now let $\psi \in \Psi_{01}^\varphi$ (resp. $\psi \in \Psi_{02}^\varphi$), and let $r > 0$ be given. In view of Lemma 1.3 (resp. Lemma 1.4) there exists $\psi_0 \in \Psi_{0i}^\varphi(c)$ for $i = 1,3$ (resp. $\psi_0 \in \Psi_{0i}^\varphi(c)$ for $i = 2,4$) such that $\psi(u) \le \psi_0(2u)$ for $u \ge 0$. Hence

$$\Vert x \Vert_\psi \le \Vert 2x \Vert_{\psi_0} \quad \text{for all} \quad x \in L^{\psi_0}. \tag{1}$$

Since the F-norms $\Vert \cdot \Vert_{\psi_0}$ and $|||\cdot|||_{\psi_0}$ are equivalent on L^{ψ_0}, there exists $r_1 > 0$ such that

$$B_{(\psi_0)}(r_1) \subset B_{\psi_0}(r), \tag{2}$$

where

$$B_{\psi_0}(r) = \{x \in L^{\psi_0} : \Vert x \Vert_{\psi_0} \le r\} \quad \text{and} \quad B_{(\psi_0)}(r_1) = \{x \in L^{\psi_0} : |||x|||_{\psi_0} \le r_1\}.$$

We shall show that $B_{(\psi_o)}(\frac{r_1}{2}) \cap L^\varphi \subset B_\psi(r)$. Indeed, let $x \in B_{(\psi_0)}(\frac{r_1}{2}) \cap L^\varphi$. Then $|||2x|||_{\psi_0} \le r_1$; hence by (2), $\Vert 2x \Vert_{\psi_0} \le r$. Next, by (1) we get that $\Vert x \Vert_\psi \le r$.

Thus we proved that the topology τ_φ^* generated by the family of norms $\{|||\cdot|||_\psi : \psi \in \Psi_{0i}^\varphi(c)\}$ is finer than $\tau(L^\varphi, L^{\varphi^*})$.

On the other hand, since for $\psi \in \Psi_{0i}^\varphi(c)$ the F-norms $\Vert \cdot \Vert_\psi$ and $|||\cdot|||_\psi$ are equivalent on L^φ, we get that $\tau(L^\varphi, L^{\varphi^*})$ is finer than τ_φ^*. Thus the proof is completed. □

As an application of Theorem 1.5 we obtain a characterization of absolutely continuous (order continuous) seminorms on L^φ (see [2, Theorem 3]).

Corollary 1.6

Let $\varphi \in \Phi_{0i}^c$ $(i = 1,2,3,4)$. Then for a Riesz seminorm p on L^φ the following statements are equivalent:

(i) *p is order continuous (i.e., $p(x_n) \downarrow 0$ whenever $x_n \downarrow 0$ in L^φ).*

(ii) *p is absolutely continuous (i.e. $p(x_{E_n}) \to 0$ whenever $E_n \searrow \emptyset$ and $x \in L^\varphi$).*

(iii) *There exists $\psi \in \Psi_{0i}^\varphi(c)$ and a number $a > 0$ such that*

$$p(x) \le a|||x|||_\psi \quad \text{for} \quad x \in L^\varphi.$$

Proof. (i)$\Leftrightarrow$(ii) See [9, Theorem 2.1].

(i)$\Rightarrow$(iii) Let $\varphi \in \Phi_{0i}^c$ $(i = 1, 2, 3, 4)$. Since $\tau(L^\varphi, L^{\varphi^*})$ is the finest σ-Lebesgue topology on L^φ (see Theorem 1.1) in view of Theorem 1.5 and [5, Ch. 4, § 18, (4)] there exist $\psi_1, \dots, \psi_n \in \Psi_{0i}^\varphi(c)$ and a number $a > 0$ such that

$$p(x) \le a \max\{|||x|||_{\psi_1}, \dots, |||x|||_{\psi_n}\} \quad \text{for all} \quad x \in L^\varphi.$$

Let us put

$$\psi(u) = \max\big(\psi_1(u), \dots, \psi_n(u)\big) \quad \text{for} \quad u \ge 0.$$

Then $\psi \in \Psi_{0i}^\varphi(c)$ and $|||x|||_{\psi_i} \le |||x|||_\psi$ for $x \in L^\varphi$, so $p(x) \le a|||x|||_\psi$ for $x \in L^\varphi$.

(iii)$\Rightarrow$(i) Since $\tau(L^\varphi, L^{\varphi^*})$ is a σ-Lebesgue topology, by Theorem 1.5, for each $\psi \in \Psi_{0i}^\varphi(c)$ the norm $|||\cdot|||_\psi$ is order continuous on L^φ; so p is also order continuous on L^φ. □

2. Weak compactness in Orlicz spaces

Throughout this section we assume that (Ω, Σ, μ) is a σ-finite measure space.

For any Young function φ the following criterion for relative $\sigma(L^\varphi, L^{\varphi^*})$-compactness is well known (see [11, §28], [8, Ch. I, §3, Theorem 5], [20, Corollary 4.5.2]):

Theorem 2.1

Let φ be a Young function. For a subset Z of L^φ the following statements are equivalent:

(i) Z is relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact.

(ii) Z is $\sigma(L^\varphi, L^{\varphi^*})$-bounded and for each $y \in L^{\varphi^*}$

$$\lim_{n\to\infty} \sup_{x \in Z} \int_{E_n} |x(t)y(t)| d\mu = 0 \quad \text{whenever} \quad E_n \searrow \emptyset.$$

The next theorem presents conditions for relative $\sigma(L^\varphi, L^{\varphi^*})$-compact embeddings of Orlicz spaces. This theorem was proved in a different way in [20, Theorem 5.3.3] for φ being an N-function.

Theorem 2.2

Let φ and ψ be Young functions.

1^0. If $\varphi \overset{\mathrm{a}}{\prec} \psi$ (resp. $\varphi \overset{\mathrm{l}}{\prec} \psi$ if $\mu(\Omega) < \infty$, resp. $\varphi \overset{\mathrm{s}}{\prec} \psi$ if μ is the counting measure on $\mathbb{N}$), then the embedding

$$i\colon L^{\psi} \hookrightarrow L^{\varphi}$$

is relatively $\sigma(L^{\varphi}, L^{\varphi^*})$-compact (i.e., every norm bounded subset of L^{φ} is relatively $\sigma(L^{\varphi}, L^{\varphi^*})$-compact).

2^0. Let $L^{\psi} \subset L^{\varphi}$ with $\lim_{u\to\infty} \psi(u)/u = \infty$, and let the measure space (Ω, Σ, μ) be infinite and atomless (resp. finite and atomless; resp. $\Omega = \mathbb{N}$ with μ being the counting measure). If the embedding

$$i\colon L^{\psi} \hookrightarrow L^{\varphi}$$

is relatively $\sigma(L^{\varphi}, L^{\varphi^*})$-compact, then $\varphi \overset{\mathrm{a}}{\prec} \psi$ (resp. $\varphi \overset{\mathrm{l}}{\prec} \psi$; resp. $\varphi \overset{\mathrm{s}}{\prec} \psi$).

Proof. 1^0. We have $L^{\psi} \subset L^{\varphi}$ and the Young function ψ^* is finite valued because $\lim_{u\to\infty} \psi(u)/u = \infty$. Let the set $Z \subset L^{\psi}$ be norm bounded, i.e., $\sup\{|||x|||_{\psi}\colon x \in Z\} < \infty$. For $y \in L^{\varphi^*}$ let us put

$$p_Z(y) = \sup\Big\{ \int_{\Omega} |x(t)y(t)|d\mu\colon x \in Z \Big\}.$$

In view of Theorem 2.1 we have to show that the seminorm p_Z is absolutely continuous on L^{φ^*}, i.e., $p_Z(y_{E_n}) \to 0$, as $E_n \searrow \emptyset$ for $y \in L^{\varphi^*}$. Indeed, let $y \in L^{\varphi^*}$ and $E_n \searrow \emptyset$. Since $\varphi \overset{\mathrm{a}}{\prec} \psi$ (resp. $\varphi \overset{\mathrm{l}}{\prec} \psi$; resp. $\varphi \overset{\mathrm{s}}{\prec} \psi$) we get that $\psi^* \overset{\mathrm{a}}{\triangleleft} \varphi^*$ (resp. $\psi^* \overset{\mathrm{l}}{\triangleleft} \varphi^*$; resp. $\psi^* \overset{\mathrm{s}}{\triangleleft} \varphi^*$). Hence $L^{\varphi^*} \subset E^{\psi^*} = L_a^{\psi^*}$ (see [20, Theorem 5.3.1]).

By applying Hölder's inequality (see [20, Ch. III, §3]) we get

$$\begin{aligned} p_Z(y_{E_n}) &= \sup\Big\{ \int_{\Omega} |x(t)y_{E_n}(t)|d\mu\colon x \in Z \Big\} \\ &\le \|y_{E_n}\|_{\psi^*} \cdot \sup\Big\{|||x|||_{\psi}\colon x \in Z\Big\}. \end{aligned}$$

Thus $p_Z(y_{E_n}) \to 0$, because $y \in L_a^{\psi^*}$.

2^0. Since $L^{\psi} \subset L^{\varphi}$ we have $L^{\varphi^*} \subset L^{\psi^*}$, and ψ^* is finite valued. To prove that $\varphi \overset{\mathrm{a}}{\prec} \psi$ (resp. $\varphi \overset{\mathrm{s}}{\prec} \psi$, resp. $\varphi \overset{\mathrm{l}}{\prec} \psi$) it is enough to show that $L^{\varphi^*} \subset E^{\psi^*}$, because this inclusion implies that $\psi^* \overset{\mathrm{a}}{\triangleleft} \varphi^*$ (resp. $\psi^* \overset{\mathrm{s}}{\triangleleft} \varphi^*$; resp. $\psi^* \overset{\mathrm{l}}{\triangleleft} \varphi^*$) (see [20, Theorem 5.3.1]).

Indeed, let $y \in L^{\varphi^*}$. Since the unit ball $B_\psi(1) = \{x \in L^\psi : |||x|||_\psi \leq 1\} \subset L^\varphi$ is $\sigma(L^\varphi, L^{\varphi^*})$-compact, in view of Theorem 2.1 we get that

$$\|y_{E_n}\|_{\psi^*} = \sup\left\{ \int_\Omega |x(t)y_{E_n}(t)|d\mu : x \in L^\psi, x \in B_\psi(1)\right\} \to 0 \quad \text{as} \quad E_n \searrow \emptyset.$$

This means that $y \in L_a^{\psi^*} = E^{\psi^*}$. □

Corollary 2.3

Let φ be a Young function, and let the measure space (Ω, Σ, μ) be infinite and atomless (resp. finite and atomless; resp. $\Omega = \mathbb{N}$ with μ being the counting measure). Then the following statements are equivalent:

(i) *φ satisfies the ∇_2-condition for all u (resp. for large u; resp. for small u).*

(ii) *Every norm bounded subset of L^φ is relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact.*

The main aim of this section is to show that a relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact subset of L^φ (for φ being a finite valued Young function) is norm bounded in L^ψ for some regular Young function ψ dependent on φ.

This result extends the well-known Ando's criterion for relative weak compactness in L^φ obtained for φ being an N-function and finite measures (see [2, Theorem 2]). For this purpose we distinguish some classes of Young functions.

Let Φ_1^c be the collection of Young functions taking only finite values and such that $\lim_{u\to 0} \varphi(u)/u = 0$.

Let

$$\Phi_{11}^c = \{\varphi \in \Phi_1^c : \varphi(u) > 0 \quad \text{for } u > 0, \text{ and } \lim_{u\to\infty} \tfrac{\varphi(u)}{u} = \infty\}\ ,$$

$$\Phi_{12}^c = \{\varphi \in \Phi_1^c : \varphi(u) > 0 \quad \text{for } u > 0, \text{ and } \lim_{u\to\infty} \tfrac{\varphi(u)}{u} < \infty\}\ ,$$

$$\Phi_{13}^c = \{\varphi \in \Phi_1^c : \varphi(u) = 0 \quad \text{near } u > 0, \text{ and } \lim_{u\to\infty} \tfrac{\varphi(u)}{u} = \infty\}\ ,$$

$$\Phi_{14}^c = \{\varphi \in \Phi_1^c : \varphi(u) = 0 \quad \text{near } u > 0, \text{ and } \lim_{u\to\infty} \tfrac{\varphi(u)}{u} < \infty\}\ .$$

Then $\Phi_1^c = \bigcup_{i=1}^\infty \Phi_{1i}^c$, and the sets Φ_{1i}^c are pairwise disjoint. It is seen that $\Phi_{11}^c = \Phi_N$.

Denote by

$$\Psi_{11}^\varphi(c) = \{\psi \in \Phi_N : \varphi \overset{\mathrm{a}}{\prec} \psi\}, \quad \text{whenever} \quad \varphi \in \Phi_{11}^c,$$

$$\Psi_{12}^\varphi(c) = \{\psi \in \Phi_N : \varphi \overset{\mathrm{s}}{\prec} \psi\}, \quad \text{whenever} \quad \varphi \in \Phi_{12}^c,$$

$$\Psi_{13}^\varphi(c) = \{\psi \in \Phi_{13}^c : \varphi \overset{\mathrm{l}}{\prec} \psi\}, \quad \text{whenever} \quad \varphi \in \Phi_{13}^c,$$

$$\Psi_{14}^\varphi(c) = \Phi_{13}^c, \quad \text{whenever} \quad \varphi \in \Phi_{14}^c.$$

The next important lemma shows the relation between the sets Φ_{0i}^c and Φ_{1i}^c, and the sets $\Psi_{0i}^\varphi(c)$ and $\Psi_{1i}^\varphi(c)$ $(i = 1, 2, 3, 4)$.

Lemma 2.4

1^0. Let $\varphi \in \Phi^c_{0i}$ $(i = 1,2,3,4)$. Then $\varphi^* \in \Phi^c_{1i}$ and

$$\left(\Psi^{\varphi}_{0i}(c)\right)^* = \Psi^{\varphi^*}_{1i}(c).$$

2^0. Let $\varphi \in \Phi^c_{1i}$ $(i = 1,2,3,4)$. Then $\varphi^* \in \Phi^c_{0i}$ and

$$\left(\Psi^{\varphi}_{1i}(c)\right)^* = \Psi^{\varphi^*}_{0i}(c).$$

Proof. In view of [17, Lemma 3.1] $\varphi^* \in \Phi^c_{1i}$ whenever $\varphi \in \Phi^c_{0i}$, and $\varphi^* \in \Phi^c_{0i}$ whenever $\varphi \in \Phi^c_{1i}$ $(i = 1,2,3,4)$.

But it is known that for Young functions ψ and φ the relation $\psi \overset{\mathrm{a}}{\lhd} \varphi$ (resp. $\psi \overset{\mathrm{s}}{\lhd} \varphi$; resp. $\psi \overset{\mathrm{l}}{\lhd} \varphi$) holds if and only if the relation $\varphi^* \overset{\mathrm{a}}{\prec} \psi^*$ (resp. $\varphi^* \overset{\mathrm{s}}{\prec} \psi^*$; resp. $\varphi^* \overset{\mathrm{l}}{\prec} \psi^*$) holds (see [20, proposition 2.2.4]). $\square$

Now we are ready to obtain our desired description of relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact sets in L^φ.

Theorem 2.5

Let $\varphi \in \Phi^c_{1i}$ $(i = 1,2,3,4)$. For a subset Z of L^φ the following statements are equivalent:

(i) Z is relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact.

(ii) There exists $\psi \in \Psi^{\varphi}_{1i}(c)$ such that $Z \subset L^\psi$ and

$$\sup\left\{\|x\|_\psi : x \in Z\right\} < \infty\,.$$

Proof. (i)$\Rightarrow$(ii) Since the set $Z \subset L^\varphi$ is relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact, in view of Theorem 2.1 the seminorm $p_Z(y) = \sup\{\int_\Omega |x(t)y(t)|d\mu : x \in Z\}$ is absolutely continuous on L^{φ^*}. Hence by Corollary 1.6 there exist $\psi_0 \in \Psi^{\varphi^*}_{0i}(c)$ (so $L^{\varphi^*} \subset E^{\psi_0}$) and a number $a > 0$ such that

$$p_Z(y) \le a|||y|||_{\psi_0} \quad \text{for} \quad y \in L^{\varphi^*}. \tag{1}$$

We shall show that $Z \subset L^{\psi_0^*}$ and $\sup\{\|x\|_{\psi_0^*} : x \in Z\} \le a$.

Indeed, let $x \in Z$. Then by (1), for $y \in L^{\varphi^*}, |||y|||_{\psi_0} \leq 1$ we get that

$$\int_\Omega |x(t)y(t)|d\mu \leq a. \tag{2}$$

Since the measure space (Ω, Σ, μ) is σ-finite, there exists a sequence (Ω_n) of measurable subsets of Ω such that $\Omega_n \uparrow, \Omega = \bigcup_{n=1}^{\infty} \Omega_n, \mu(\Omega_n) < \infty$. Let $z \in L^{\psi_0}$ and $z \neq 0$. For $n = 1, 2, \ldots$ denote by

$$z^{(n)}(t) = \begin{cases} z(t) & \text{if } |z(t)| \leq n \text{ and } t \in \Omega_n, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $|x(t)z^{(n)}(t)| \uparrow_n |x(t)z(t)|$ on Ω, so by Fatou's lemma and (2) we obtain

$$\begin{aligned} \frac{1}{|||z|||_{\psi_0}} \int_\Omega |x(t)z(t)|d\mu &\leq \frac{1}{|||z|||_{\psi_0}} \sup_n \int_\Omega |x(t)z^{(n)}(t)|d\mu \\ &\leq \sup\Big\{ \int_\Omega |x(t)y(t)|d\mu \colon y \in L^{\varphi^*}, |||y|||_{\psi_0} \leq 1 \Big\} \leq a. \end{aligned}$$

Hence $x \in (L^{\psi_0})^\times = L^{\psi_0^*}$, where $(L^{\psi_0})^\times$ denotes the Köthe dual of L^{ψ_0}. Moreover, since

$$\|x\|_{\psi_0^*} = \sup\Big\{ \Big| \int_\Omega x(t)z(t)d\mu \Big| \colon z \in L^{\psi_0}, |||z|||_{\psi_0} \leq 1 \Big\}$$

we get that $\|x\|_{\psi_0^*} \leq a$. Putting $\psi = \psi_0^*$ and using Lemma 2.4 we get that $\psi \in \Psi_{1i}^{\varphi}(c)$ and $Z \subset L^\psi$ with $\sup\{\|x\|_\psi \colon x \in Z\} \leq a$.

(ii)$\Rightarrow$(i) It follows from Theorem 2.2. $\square$

As an application of Theorem 2.5 we obtain a representation of L^φ as the union of some family of other Orlicz spaces.

Corollary 2.6

Let $\varphi \in \Phi_{1i}^c$ $(i = 1, 2, 3, 4)$. Then the following equality holds:

$$L^\varphi = \bigcup \{L^\psi \colon \psi \in \Psi_{1i}^{\varphi}(c)\}.$$

Proof. From Theorem 2.5 we obtain that $L^\varphi \subset \bigcup\{L^\psi \colon \psi \in \Psi_{1i}^{\varphi}(c)\}$. On the other hand, $L^\psi \subset L^\varphi$ for each $\psi \in \Psi_{1i}^{\varphi}(c)$. $\square$

Remark. The equality from Corollary 2.6 for $i = 1, 2$ was obtained in a different way in [14, Theorem 2.6].

At last, we apply Theorem 2.5 to examination of the absolute weak topology $|\sigma|(L^\varphi, L^{\varphi^*})$ (see [8, Definition 2, p. 27]).

Theorem 2.7

Let $\varphi \in \Phi_{1i}^c$ $(i = 1, 2, 3, 4)$. For a sequence (x_n) in L^φ the following statements are equivalent:
(i) $x_n \to 0$ *for* $|\sigma|(L^\varphi, L^{\varphi^*})$.
(ii) $x_n \to 0(\mu)$ *and the set* $\{x_n\}$ *is relatively* $\sigma(L^\varphi, L^{\varphi^*})$*-compact.*
(iii) $x_n \to 0(\mu)$ *and* $\sup_n \|x_n\|_\psi < \infty$ *for some Young function* $\psi \in \Psi_{1i}^\varphi(c)$.

Proof. (i)⇔(ii) See [16, Theorem 2.1].

(ii)⇔(iii) It follows from Theorem 2.5. □

Theorem 2.8

Let $\varphi \in \Phi_{1i}^c$ $(i = 1, 2, 3, 4)$. If $\psi \in \Psi_{1i}^\varphi(c)$ and $Z \subset L^\psi \subset L^\varphi$ and $\sup\{\|x\|_\psi : x \in Z\} < \infty$, then the topologies $\mathcal{T}_0$ and $|\sigma|(L^\varphi, L^{\varphi^*})$ coincide on Z, i.e.,

$$\mathcal{T}_{0_{|Z}} = |\sigma|(L^\varphi, L^{\varphi^*})_{|Z}.$$

Proof. It is well known that $\mathcal{T}_{0_{|Z}} \subset |\sigma|(L^\varphi, L^{\varphi^*})_{|Z}$ (see [7, Ch. X, §5, Lemma 1]). Since $\mathcal{T}_0$ is a linear metrizable topology from Theorem 2.7 it follows that $|\sigma|(L^\varphi, L^{\varphi^*})_{|Z} \subset \mathcal{T}_{0_{|Z}}$. □

Theorem 2.9

Let $\varphi \in \Phi_{1i}^c$ $(i = 1, 2, 3, 4)$. For a subset Z of L^φ the following statements are equivalent:
(i) Z is relatively compact for $|\sigma|(L^\varphi, L^{\varphi^*})$.
(ii) Z is relatively compact for $\mathcal{T}_0$, and there exists a Young function $\psi \in \Psi_{1i}^\varphi(c)$ such that $Z \subset L^\psi$ and $\sup\{\|x\|_\psi : x \in Z\} < \infty$.

Proof. It follows from Theorem 2.5 and [8, Ch. I, §3, Corollary of Lemma 11]. □

3. Applications to the theory of Lebesgue spaces

In this section we will apply Theorem 1.5, Corollary 1.6, Theorem 2.5 and Corollary 2.6 to the theory of Lebesgue spaces. We will assume that (Ω, Σ, μ) is a σ-finite measure space.

A. Let

$$\varphi(u) = \chi_1(u) \vee \chi_\infty(u) \quad \text{for} \quad u \geq 0,$$

where

$$\chi_1(u) = u \quad \text{for} \quad u \geq 0, \quad \chi_\infty = \begin{cases} 0 & \text{for } 0 \leq u \leq 1, \\ \infty & \text{for } u > 1. \end{cases}$$

Then $\varphi \in \Phi_{04}^c$ and $L^\varphi = L^1 \cap L^\infty$. Moreover, by Lemma 2.4, $\varphi^* \in \Phi_{14}^c$ and $\varphi^* = (\chi_1 \vee \chi_\infty)^* \overset{a}{\sim} \chi_1^* \vee \chi_\infty^* = \chi_\infty \vee \chi_1$, so $L^{\varphi^*} = L^1 + L^\infty$ (see [4, Theorem 3]).

Theorem 3.1

The following statements hold:

1^0. *The Mackey topology $\tau(L^1 \cap L^\infty, L^1 + L^\infty)$ is generated by the family of norms:*

$$\{|||\cdot|||_{\psi|L^1\cap L^\infty} : \psi \in \Phi_{03}^c\}.$$

2^0. *For a Riesz seminorm p on $L^1 \cap L^\infty$ the following statements are equivalent:*

(i) p is order continuous.

(ii) There exist $\psi \in \Phi_{03}^c$ and a number $a > 0$ such that

$$p(x) \leq a|||x|||_\psi \quad \text{for} \quad x \in L^1 \cap L^\infty.$$

3^0. *For a subset Z of $L^1 + L^\infty$ the following statements are equivalent*

(i) Z is relatively $\sigma(L^1 + L^\infty, L^1 \cap L^\infty)$-compact.

(ii) There exists $\psi \in \Phi_{13}^c$ such that $Z \subset L^\psi$ and

$$\sup\{|||x|||_\psi : x \in Z\} < \infty.$$

4^0. *The following equality holds*

$$L^1 + L^\infty = \bigcup\{L^\psi : \psi \in \Phi_{13}^c\}.$$

B. Let $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let

$$\varphi(u) = \chi_p(u) \vee \chi_\infty(u) \quad \text{for} \quad u \geq 0,$$

where $\chi_p(u) = u^p$ for $u \geq 0$. Then $\varphi \in \Phi_{02}^c$ and $L^\varphi = L^p \cap L^\infty$. Moreover, by Lemma 2.4, $\varphi^* \in \Phi_{12}^c$ and $\varphi^* = (\chi_p \vee \chi_\infty)^* \overset{a}{\sim} \chi_p^* \wedge \chi_\infty^* = \chi_q \wedge \chi_1$; so $L^{\varphi^*} = L^1 + L^q$.

Theorem 3.2.

The following statements hold:

1^0. *The Mackey topology* $\tau(L^p \cap L^\infty, L^q + L^1)$ *is generated by the family of norms:*

$$\Big\{ |||\cdot|||_{\psi | L^p \cap L^\infty} \colon \psi \in \Phi_N \quad \text{and} \ \limsup_{u \to 0} \frac{\psi(u)}{u^p} < \infty \Big\}.$$

2^0. *For a Riesz seminorm* p *on* $L^p \cap L^\infty$ *the following statements are equivalent:*

(i) p *is order continuous.*

(ii) *There exist an* N*-function* ψ *with* $\limsup_{u \to 0} \frac{\psi(u)}{u^p} < \infty$ *and a number* $a > 0$ *such that*

$$p(x) \le a|||x|||_\psi \quad \text{for} \quad x \in L^p \cap L^\infty.$$

3^0. *For a subset* Z *of* $L^q + L^1$ *the following statements are equivalent:*

(i) Z *is relatively* $\sigma(L^q + L^1, L^p \cap L^\infty)$*-compact.*

(ii) *There exists an* N*-function* ψ *with* $\chi_1 \overset{\mathrm{s}}{\prec} \psi$ *such that* $Z \subset L^\psi$ *and*

$$\sup\{|||x|||_\psi \colon x \in Z\} < \infty.$$

4^0. *The following equality holds:*

$$L^q + L^1 = \bigcup \{L^\psi \colon \psi \in \Phi_N \quad \text{and} \quad \chi_q \overset{\mathrm{s}}{\prec} \psi\}.$$

C. Let $p > 1, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Let

$$\varphi(u) = \chi_1(u) \vee \chi_p(u) \quad \text{for} \quad u \ge 0.$$

Then $\varphi \in \Phi_{03}^c$ and $L^\varphi = L^1 \cap L^p$. Moreover, by Lemma 2.4, $\varphi^* \in \Phi_{13}^c$, and $\varphi^* = (\chi_1 \vee \chi_p)^* \overset{\mathrm{a}}{\sim} \chi_1^* \wedge \chi_p^* = \chi_\infty \wedge \chi_q$. Hence $L^{\varphi^*} = L^q + L^\infty$.

Theorem 3.3

The following statements hold:

1^0. *The Mackey topology* $\tau(L^1 \cap L^p, L^q + L^\infty)$ *is generated by the family of norms:*

$$\Big\{ |||\cdot|||_{\psi | L^1 \cap L^p} \colon \ \psi \in \Phi_{03}^c \quad \text{and} \ \limsup_{u \to \infty} \frac{\psi(u)}{u^p} < \infty \Big\}.$$

2^0. *For a Riesz seminorm* p *on* $L^1 \cap L^p$ *the following statements are equivalent:*

(i) p *is order continuous on* $L^1 \cap L^p$.

(ii) There exist $\psi \in \Phi_{03}^c$ *with* $\limsup_{u\to\infty} \frac{\psi(u)}{u^p} < \infty$ *and a number* $a > 0$ *such that*

$$p(x) \le a|||x|||_\psi \quad \text{for} \quad x \in L^1 \cap L^p.$$

3^0. *For a subset* Z *of* $L^q + L^\infty$ *the following statements are equivalent:*

(i) Z *is relatively* $\sigma(L^q + L^\infty, L^1 \cap L^p)$*-compact.*

(ii) There exists $\psi \in \Phi_{13}^c$ *with* $\chi_q \overset{1}{\prec} \psi$ *such that* $Z \subset L^\psi$ *and*

$$\sup\{|||x|||_\psi : x \in Z\} < \infty.$$

4^0. *The following equality holds:*

$$L^q + L^\infty = \bigcup\{L^\psi : \psi \in \Phi_{13}^c \quad \text{and} \quad \chi_q \overset{1}{\prec} \psi\}.$$

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