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Higher order Hardy inequalities

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Abstract

This note deals with the inequality

$$\left(\int_a^b |u(x)|^q w_0(x)dx\right)^{1/q} \le C\left(\int_a^b |u^{(k)}(x)|^p w_k(x)dx\right)^{1/p}, \tag{1}$$

more precisely, with conditions on the parameters $p > 1, q > 0$ and on the weight functions w_0, w_k (measurable and positive almost everywhere) which ensure that (1) holds for all functions u from a certain class K with a constant $C > 0$ independent of u.

Here $-\infty \le a < b \le \infty$ and $k \in \mathbb{N}$ and we will consider classes K of functions $u = u(x)$ defined on (a, b) whose derivatives of order $k-1$ are absolutely continuous and which satisfy the "boundary conditions"

$$\begin{aligned} u^{(i)}(a) &= 0 \quad \text{for} \quad i \in M_0 \ , \\ u^{(j)}(b) &= 0 \quad \text{for} \quad j \in M_1 \end{aligned} \tag{2}$$

where M_0, M_1 are subsets of the set $M = \{0, 1, \ldots, k-1\}$; we will suppose that the number of conditions in (2) is exactly k. This class will be denoted by

$$AC^{(k-1)}(a, b; M_0, M_1). \tag{3}$$

The conditions (2) are reasonable since they allow to exclude functions like polynomials of order $\le k-1$ for which the right hand side in (1) is zero while the left hand side is positive.

Let us start with some remarks.

(i) We will concentrate on the case

$$k > 1 \tag{4}$$

since for $k = 1$ the problem is completely solved: see, e.g., the book Opic, Kufner [4], Chapter 1. Some particular results concerning the case $k = 2, k = 3$ and - for a special choice of the sets M_0, M_1 - also higher values of k can be found in the paper Kufner, Wannebo [3].

(ii) For $(a,b) = (0,\infty), k \in \mathbb{N}$ arbitrary and $M_0 = M, M_1 = \emptyset$ or $M_0 = \{0, 1, \dots, m-1\}, M_1 = M \setminus M_0, 0 < m < k$, the problem is also solved: see Stepanov [5] or Kufner, Heinig [2], respectively. These results cover all reasonable cases when the interval (a,b) is infinite. Therefore, we will concentrate on the case of a finite interval (a,b). Without loss of generality it can be assumed that

$$(a,b) = (0,1). \tag{5}$$

In the sequel, we will make substantial use of some functions and constants. For $r \neq 1$, we will denote

$$r' = \frac{r}{r-1}, \quad \text{i.e.} \quad \frac{1}{r} + \frac{1}{r'} = 1 \ .$$

Further, let us denote for $i = 1, 2$

$$\begin{aligned} W_{0i}(t) &= w_0(t) t^{\alpha_i q} (1-t)^{\beta_i q}, \\ W_{ki}(t) &= w_k^{1-p'}(t) t^{\gamma_i p'} (1-t)^{\delta_i p'}, \end{aligned} \tag{6}$$

where $w_0(t), w_k(t)$ are the weight functions appearing in (1) and $\alpha_i, \beta_i, \gamma_i, \delta_i$ $(i = 1, 2)$ are certain nonnegative integers, and let us introduce functions

$$B_1(x) = \left(\int_x^1 W_{01}(t)dt \right)^{1/q} \left(\int_0^x W_{k1}(t)dt \right)^{1/p'}, \tag{7}$$

$$B_2(x) = \left(\int_0^x W_{02}(t)dt \right)^{1/q} \left(\int_x^1 W_{k2}(t)dt \right)^{1/p'} \tag{8}$$

and constants

$$A_1 = \left[\int_0^1 \left(\int_x^1 W_{01}(t)dt \right)^{r/q} \left(\int_0^x W_{k1}(t)dt \right)^{r/q'} W_{k1}(x)dx \right]^{1/r}, \tag{9}$$

$$A_2 = \left[\int_0^1 \left(\int_0^x W_{02}(t)dt \right)^{r/q} \left(\int_x^1 W_{k2}(t)dt \right)^{r/q'} W_{k2}(x)dx \right]^{1/r}, \tag{10}$$

where

$$\frac{1}{r} = \frac{1}{q} - \frac{1}{p}\,. \tag{11}$$

We suppose that all expressions appearing in formulas (7) - (10) are well defined. Of course, it also depends on the values $\alpha_i, \beta_i, \gamma_i, \delta_i$ which have not yet been determined. Later, we will show how these integers can be determined by the sets M_0 and M_1 which appear in the conditions (2).

If we suppose for a moment that these integers are known, then the main result can be formulated as follows:

Proposition 1

Let M_0, M_1 be two nonempty subsets of the set $\{0, 1, \dots, k-1\}$ containing together k elements. Let $\alpha_i, \beta_i, \gamma_i, \delta_i,\ \ i = 1, 2,$ be nonnegative integers corresponding to the pair M_0, M_1. Let $w_0(t), w_k(t)$ be weight functions defined on $(0, 1)$ and let

$$1 < p < \infty,\ 0 < q < \infty,\ q \neq 1.$$

Then the (HARDY) inequality

$$\left(\int_0^1 |u(t)|^q w_0(t)dt\right)^{1/q} \leq C \left(\int_0^1 |u^{(k)}(t)|^p w_k(t)dt\right)^{1/p} \tag{12}$$

holds for every function $u \in AC^{(k-1)}(0, 1)$ satisfying the conditions

$$\begin{aligned} u^{(i)}(0) &= 0 \quad \textit{for} \ \ i \in M_0, \\ u^{(j)}(1) &= 0 \quad \textit{for} \ \ j \in M_1 \end{aligned} \tag{13}$$

if and only if

$$\sup_{0<x<1} B_i(x) = B_i < \infty, \quad i = 1, 2 \tag{14}$$

in the case $p \leq q$, and

$$A_i < \infty, \quad i = 1, 2 \tag{15}$$

in the case $p > q$, where $B_i(x)$ and A_i are given by formulas (7) - (11).

Determination of the integers $\alpha_1, \ldots, \delta_2$.

Let us consider a very simple boundary value problem

$$\begin{aligned} u^{(k)} &= f \quad \text{in} \quad (0,1), \\ u^{(i)}(0) &= 0 \quad \text{for} \quad i \in M_0, \\ u^{(j)}(1) &= 0 \quad \text{for} \quad j \in M_1 \end{aligned} \tag{16}$$

where f does not change the sign in $(0,1)$ and M_0, M_1 are the subsets of $M = \{0, 1, ..., k-1\}$ mentioned in Proposition 1.

Suppose that the solution u can be expressed uniquely in the form

$$u(x) = \int_0^x K_1(x,t)f(t)dt + \int_x^1 K_2(x,t)f(t)dt. \tag{17}$$

The kernels $K_1(x,t), K_2(x,t)$ are then polynomials. We will write

$$K_i(x,t) \approx x^{\alpha_i}(1-x)^{\beta_i}t^{\gamma_i}(1-t)^{\delta_i} \tag{18}$$

if there exist positive constants c_1, c_2 such that the estimates

$$c_1 \leq \frac{K_i(x,t)}{x^{\alpha_i}(1-x)^{\beta_i}t^{\gamma_i}(1-t)^{\delta_i}} \leq c_2$$

hold for $0 < t < x < 1$ $(i = 1)$ and $0 < x < t < 1$ $(i = 2)$, respectively.

Now, we will show under what conditions (18) is fulfilled. For this purpose, let us split the set $M = \{0, 1, \ldots, k-1\}$ into s successive groups $G_1, G_2, \ldots, G_s$ $(s \geq 2)$ according to the following scheme:

$$\begin{aligned} G_1 &= \{0, 1, \ldots, m-1\} && (k_1 \ \text{ elements}, \quad k_1 = m), \\ G_2 &= \{m, m+1, \ldots, n-1\} && (k_2 \ \text{ elements}, \quad k_2 = n-m), \\ G_3 &= \{n, n+1, \ldots, r-1\} && (k_3 \ \text{ elements}, \quad k_3 = r-n), \\ & && \cdot \\ & && \cdot \\ G_s &= \{h, h+1, \ldots, k-1\} && (k_s \ \text{ elements}, \quad k_s = k-h), \end{aligned} \tag{19}$$

(i.e. G_i has k_i elements, $k_i > 0$, $i = 1, 2, \ldots, s$, and $k_1 + k_2 + \ldots + k_s = k$), and suppose that the sets M_0 and M_1 appearing in the boundary conditions in (16) are defined as follows:

$$M_0 = G_1 \cup G_2 \cup \ldots \cup G_{s-1}, \quad M_1 = G_2 \cup G_4 \cup \ldots \cup G_s \ \text{ for } s \text{ even}. \tag{20}$$

$$M_0 = G_1 \cup G_2 \cup \ldots \cup G_s, \quad M_1 = G_2 \cup G_4 \cup \ldots \cup G_{s-1} \ \text{ for } s \text{ odd}. \tag{21}$$

Then we have

Proposition 2

If the set $M = \{0, 1, \dots, k-1\}$ is splitted into s groups according to (19), the sets M_0 and M_1 are defined by (20) and (21) and the solution u to the boundary value problem (16) can be expressed in the form (17), then

$$\begin{aligned}
&K_1(x,t) \approx x^{k_1-1}t^{k_2}, \quad K_2(x,t) \approx x^{k_1}t^{k_2-1} \text{ for } s=2, \\
&K_i(x,t) \approx x^{k_1}(1-t)^{k_s}, \quad i=1,2, \text{ for } s \text{ odd}, \\
&K_i(x,t) \approx x^{k_1}t^{k_s}, \quad i=1,2, \text{ for } s>2 \text{ even}.
\end{aligned} \tag{22}$$

Remarks. (i) The proof of Proposition 2 is elementary but cumbersome. It is based on the fact that the solution u to the boundary value problem (16) can be expressed in the form

$$\begin{aligned}
u(x) = c_o \int_0^x (x-t_1)^{k_1-1} \int_{t_1}^1 (t_2-t_1)^{k_2-1} \int_0^{t_2} (t_2-t_3)^{k_3-1} \dots \\
\dots F(t_{s-1})dt_{s-1}\dots dt_2 dt_1
\end{aligned}$$

where $c_0 = [(k_1-1)!(k_2-1)!\dots(k_s-1)!]^{-1}$ and $F(t_{s-1})$ is either

$$\int_{t_{s-1}}^1 (t_s-t_{s-1})^{k_s-1} f(t_s)dt_s \quad \text{for } s \text{ even}$$

or

$$\int_0^{t_{s-1}} (t_{s-1}-t_s)^{k_s-1} f(t_s)dt_s \quad \text{for } s \text{ odd} .$$

For $s=2$, it can be found in the paper [3], for $s>2$ in the preprint [1].

(ii) In (20), (21) we have always assumed that the first group G_1 belongs to M_0 so that we start with the boundary condition $u(0)=0, 0 \in M_0$. If we suppose that $0 \in M_1$, i.e. that the boundary condition $u(1)=0$ appears in (16), and have

$$M_0 = G_2 \cup G_4 \cup \dots, \quad M_1 = G_1 \cup G_3 \cup \dots,$$

then we simply exchange the role of the sets M_0 and M_1, i.e. of the endpoints $x=0$ and $x=1$, and a corresponding assertion holds again, if we replace in (22) x by $1-x$ and t by $1-t$.

(iii) In the foregoing cases, we have assumed that

$$M_0 \cup M_1 = M, \quad \text{i.e.} \quad M_0 \cap M_1 = \emptyset.$$

If the sets M_0 and M_1 again have together k elements, but have a nonempty intersection, then the method described above cannot be used. Nonetheless, many examples allow to expect that - provided there is a unique representation of the solution u of (16) in the form (17) - the kernels $K_i(x,t)$ again behave according to (18). Therefore, let us formulate the following conjecture:

Suppose that $M_0 \cap M_1 \neq \emptyset$.

(a) Define $\widetilde{M_1}$ by

$$\widetilde{M_1} = M \setminus M_0.$$

Then the pair $M_0, \widetilde{M_1}$ satisfies the conditions of either Proposition 2 (if $G_1 \subset M_0$) or of part (ii) of this Remark (if $G_1 \subset \widetilde{M_1}$), and consequently, the kernels $K_i^{(a)}(x,t)$ corresponding to the pair $M_0, \widetilde{M_1}$ satisfy (18): There are positive integers $\alpha_i(a), \beta_i(a), \gamma_i(a), \delta_i(a)$ such that

$$K_i^{(a)}(x,t) \approx x^{\alpha_i(a)}(1-x)^{\beta_i(a)} t^{\gamma_i(a)}(1-t)^{\delta_i(a)}, \quad i = 1, 2.$$

(b) Define $\widetilde{M_0}$ by

$$\widetilde{M_0} = M \setminus M_1.$$

Then the pair $\widetilde{M_0}, M_1$ again satisfies the conditions which allow to state that for the corresponding kernels $K_i^{(b)}(x,t)$ we have

$$K_i^{(b)}(x,t) \approx x^{\alpha_i(b)}(1-x)^{\beta_i(b)} t^{\gamma_i(b)}(1-t)^{\delta_i(b)}, \quad i = 1, 2.$$

(c) For the kernels $K_i(x,t)$ corresponding to the initial pair M_0, M_1 we have (18) with

$$\alpha_i = \alpha_i(a), \beta_i = \beta_i(b), \gamma_i = \gamma_i(a), \delta_i = \delta_i(b).$$

Idea of the proof of Proposition 1

We consider the Hardy inequality (12) on the class $AC^{(k-1)}(0,1; M_0, M_1)$, i.e., for functions u satisfying (13). Therefore, let us consider the boundary value problem (16) and denote by T the operator defined by formula (17):

$$(Tf)(x) = \int_0^x K_1(x,t) f(t) dt + \int_x^1 K_2(x,t) f(t) dt.$$

Since the function $u = Tf$ satisfies conditions (13) and $u^{(k)} = f$, we can instead of the inequality (12) investigate the inequality

$$\left(\int_0^1 |(Tf)(x)|^q w_0(x) dx \right)^{1/q} \leq C \left(\int_0^1 f^p(x) w_k(x) dx \right)^{1/p} \tag{23}$$

for functions $f \geq 0$.

Now, it can be shown that the validity of (23) for $f \geq 0$ is equivalent to the validity of the inequalities

$$\left(\int_0^1 |(J_i f)(x)|^q w_0(x)dx\right)^{1/q} \leq C_i \left(\int_0^1 f^p(x) w_k(x)dx\right)^{1/p}, \quad i = 1, 2, \tag{24}$$

where

$$(J_1 f)(x) = \int_0^x K_1(x,t) f(t)dt, \quad (J_2 f)(x) = \int_x^1 K_2(x,t) f(t)dt.$$

But due to (18), the inequalities (24) are equivalent to the inequalities

$$\left(\int_0^1 \left[x^{\alpha_1}(1-x)^{\beta_1} \int_0^x t^{\gamma_1}(1-t)^{\delta_1} f(t)dt\right]^q w_0(x)dx\right)^{1/q} \leq \widetilde{C}_1 \left(\int_0^1 f^p(x) w_k(x)dx\right)^{1/p}$$

and

$$\left(\int_0^1 \left[x^{\alpha_2}(1-x)^{\beta_2} \int_x^1 t^{\gamma_2}(1-t)^{\delta_2} f(t)dt\right]^q w_0(x)dx\right)^{1/q} \leq \widetilde{C}_2 \left(\int_0^1 f^p(x) w_k(x)dx\right)^{1/p}$$

respectively, and these last two inequalities can be easily rewritten into the form

$$\left(\int_0^1 |(Hg)(x)|^q w(x)dx\right)^{1/q} \leq \widetilde{C} \left(\int_0^1 g^p(x) v(x)dx\right)^{1/p}, \tag{25}$$

where H is the Hardy operator,

$$(Hg)(x) = \int_0^x g(t)dt \quad \text{or} \quad (Hg)(x) = \int_x^1 g(t)dt\,.$$

Finally, necessary and sufficient conditions for the validity of (25) (see, e.g., [4]) lead to the conditions (14) (if $p \leq q$) or (15) (if $p > q$).

Consequently, the integers $\alpha_1, \ldots, \delta_2$ which appear in (6) can be determined from the behavior of the kernels $K_1(x,t), K_2(x,t)$ described by (18). $\square$

References

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