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Projective spaces of second order

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Abstract

Grassmannians of higher order appeared for the first time in a paper of A. Szybiak ([3]) in the context of the Cartan method of moving frame. In the present paper we consider a special case of higher order Grassmannian—the projective space of second order. We introduce the projective group of second order acting on this space, derive its Maurer-Cartan equations and show that our generalized projective space is a homogeneous space of this group.

0. Introduction

Grassmannians of higher order appeared for the first time in a paper of A. Szybiak ([3]) in the context of the Cartan method of moving frame (cf. also [2]). Unfortunately, the action of a group on the Stiefel manifold of higher order given in the paper does not factorise to the Grassmann space contrary to the author's claim. This means that the given presentation of the Grassmannian of higher order as a homogeneous space is not correct.

In the present paper we consider a special case of higher order Grassmannian—the projective space of second order. We introduce the projective group of second order which acts on this space, next we derive its Maurer-Cartan equations and show that our generalized projective space is a homogeneous space of this group.

In the same manner one can introduce a homogeneous structure on an arbitrary Grassmann manifold of higher order.

In a forthcoming paper we introduce and examine a theory of G-structures modelled on the projective spaces of second order.

1. The projective space of second order

Let $V_2^1(\mathbb{R}^{n+1}) = \mathrm{Reg}_0^2(\mathbb{R}^1, \mathbb{R}^{n+1})_0$, where $\mathrm{Reg}_0^2(\mathbb{R}^1, \mathbb{R}^{n+1})_0$ denotes the set of all regular jets of second order of mappings $\mathbb{R}^1 \longrightarrow \mathbb{R}^{n+1}$ with the source and target at 0. The manifold $V_2^1(\mathbb{R}^{n+1})$ is called a Stiefel manifold of 1-frames of second order on $\mathbb{R}^{n+1}$ at 0. Note that the Stiefel manifold of 1-frames of first order is an ordinary Stiefel manifold (cf. [3]). If $f : \mathbb{R}^1 \longrightarrow \mathbb{R}^{n+1}, \quad t \longmapsto (f^0(t), \ldots, f^n(t), \quad f(0) = 0$, is a regular mapping then the jet $j_0^2 f$ has the coordinates

$$x^i = \frac{df^i}{dt}(0), \quad y^i = \frac{d^2 f}{dt^2}(0), \quad i = 0, \ldots, n\,.$$

Let L_2^1 denote the group $\mathrm{Reg}_0^2(\mathbb{R}^1, \mathbb{R}^1)_0$ of all regular jets of second order of mappings $\mathbb{R}^1 \longrightarrow \mathbb{R}^1$ with the source and target at 0 and consider a regular map $g : \mathbb{R}^1 \longrightarrow \mathbb{R}^1$, $g(0) = 0$. Then $a = \frac{dg}{dt}(0)$, $b = \frac{d^2 g}{dt^2}(0)$ are the coordinates of the jet $j_0^2 g$. The group multiplication in L_2^1 is the jet composition

$$j_0^2 g \cdot j_0^2 k = j_0^2 (g \circ k)\,.$$

Hence, for $j_0^2 g = (a, b)$, $j_0^2 k = (c, d)$ we have

$$(a, b) \cdot (c, d) = (ac, bc^2 + ad)\,,$$
$$(a, b)^{-1} = \left(\frac{1}{a}, -\frac{b}{a^3}\right).$$

The group L_2^1 acts from the right on $V_2^1(\mathbb{R}^{n+1})$

$$\begin{aligned} V_2^1(\mathbb{R}^{n+1}) \times L_2^1 &\longrightarrow V_2^1(\mathbb{R}^{n+1})\,, \\ \left(j_0^2 f, j_0^2 g\right) &\longmapsto j_0^2 (f \circ g)\,. \end{aligned}$$

This action becomes in coordinates

$$\begin{aligned} &\left((x^0, x^1, \ldots, x^n, y^0, y^1, \ldots, y^n), (a, b)\right) \\ &\longmapsto (x^0 a, x^1 a, \ldots, x^n a, y^0 a^2 + x^0 b, y^1 a^2 + x^1 b, \ldots, y^n a^2 + x^n b)\,. \end{aligned}$$

DEFINITION 1.1. A projective space of second order P_2^n is the set of orbits $V_2^1(\mathbb{R}^{n+1})/L_2^1$.

We want to equip P^n_2 with an atlas (U_i, Φ_i), $i = 1, \dots, n$, and, hence, with a manifold structure. Let

$$U_i = \big\{(x^0, x^1, \dots, x^n, y^0, y^1, \dots, y^n) \in P^n_2 : x^i \neq 0\big\}$$

and define mapping $\Phi_i : U_i \longrightarrow \mathbb{R}^{2n}$ by

$$\begin{aligned}&\Phi_i(x^0, x^1, \dots, x^n, y^0, y^1, \dots, y^n)\\ &= \left(\frac{x^0}{x^i}, \frac{x^1}{x^i}, \dots, \widehat{\frac{x^i}{x^i}}, \dots, \frac{x^n}{x^i}, \frac{y^0x^i - x^0y^i}{(x^i)^3}, \dots, \widehat{\frac{y^ix^i - x^iy^i}{(x^i)^3}}, \dots, \frac{y^nx^i - x^ny^i}{(x^i)^3}\right).\end{aligned}$$

Hence we get a structure of smooth manifold on P^n_2.

Let $L^{n+1}_2 = \mathrm{Reg}^2_0(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})_0$ be a manifold of all regular jets of mappings $\mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ with the source and target at 0. The manifold L^{n+1}_2, similarly to L^1_2, is a Lie group. Let us consider a left act ion of the group $L^{n+1}_2 \times L^1_2$ on Stiefel manifold of second order $V^1_2(\mathbb{R}^{n+1})$. If $j^2_0 g \in L^{n+1}_2$, $j^2_0 f \in L^1_2$ and $j^2_0 h \in V^1_2(\mathbb{R}^{n+1})$, then the formula

$$\big((j^2_0 g, j^2_0 f), j^2_0 h\big) \longmapsto j^2_0\big(g \circ h \circ f^{-1}\big)$$

defines a left action of the group $L^{n+1}_2 \times L^1_2$ on $V^1_2(\mathbb{R}^{n+1})$.

In coordinates, if $j^2_0 f = (a, b)$, $j^2_0 h = (x^0, x^1, \dots, x^n, y^0, y^1, \dots, y^n)$, $j^2_0 g = (a^i_j, a^i_{jk})$, then

$$\begin{aligned}&\Big(\big((a^i_j, a^i_{jk}), (a, b)\big), (x^0, x^1, \dots, x^n, y^0, y^1, \dots, y^n)\Big)\\ &\longmapsto \Big(\frac{1}{a}a^i_j x^j, \frac{1}{a^2}a^i_{ks}x^s x^k + \frac{1}{a^2}a^i_k y^k - \frac{b}{a^3}a^i_k x^k\Big).\end{aligned}$$

We are going to show that the action of $L^{n+1}_2 \times L^1_2$ on $V^1_2(\mathbb{R}^{n+1})$ factorises to an action on $P^n_2 = V^1_2(\mathbb{R}^{n+1})/L^1_2$. Let us consider two points lying in the same orbit, for example $(x^0, x^1, \dots, x^n, y^0, y^1, \dots, y^n)$ and

$$\begin{aligned}&\big((x^0, x^1, \dots, x^n, y^0, y^1, \dots, y^n), (\alpha, \beta)\big)\\ &= (x^0\alpha, x^1\alpha, \dots, x^n\alpha, y^0\alpha^2 + x^0\beta, y^1\alpha^2 + x^1\beta, \dots, y^n\alpha^2 + x^n\beta).\end{aligned}$$

Then

$$\begin{aligned}&\Big(\big((a^i_j, a^i_{jk}), (a, b)\big), (x^0, x^1, \dots, x^n, y^0, y^1, \dots, y^n)\Big)\\ &= \Big(\frac{1}{a}a^i_j x^j, \frac{1}{a^2}a^i_{ks}x^s x^k + \frac{1}{a^2}a^i_k y^k - \frac{b}{a^3}a^i_k x^k\Big)\end{aligned}$$

and

$$\Big(\big((a^i_j, a^i_{jk}),(a,b)\big),(x^0\alpha, x^1\alpha,\dots,x^n\alpha, y^0\alpha^2+x^0\beta y^1\alpha^2+x^1\beta,\dots,y^n\alpha^2+x^n\beta)\Big)$$
$$= \Big(\frac{\alpha}{a}a^i_jx^j, \frac{\alpha}{a^2}a^i_{ks}x^sx^k + \frac{1}{a^2}a^i_k(y^k\alpha^2+x^k\beta) - \frac{\alpha b}{a^3}a^i_kx^k\Big).$$

Now, we should find an element $(\delta,\gamma)\in L^1_2$ such that

$$\Big(\big(\frac{1}{a}a^i_jx^j, \frac{1}{a^2}a^i_{ks}x^sx^k + \frac{1}{a^2}a^i_ky^k - \frac{b}{a^3}a^i_kx^k\big),(\delta,\gamma)\Big)$$
$$= \Big(\frac{\alpha}{a}a^i_jx^j, \frac{\alpha}{a^2}a^i_{ks}x^sx^k + \frac{1}{a^2}a^i_k(y^k\alpha^2+x^k\beta) - \frac{\alpha b}{a^3}a^i_kx^k\Big).$$

It suffices to take $\delta=\frac{1}{\alpha}$ and $\gamma=\frac{b}{a^2\alpha^2}-\frac{\beta}{a\alpha^3}-\frac{a}{\alpha}$. It is easy to check that the group $L^{n+1}_2\times L^1_2$ acts transitively on $V^1_2(\mathbb{R}^{n+1})$. Thus, we have proved

Theorem 1.1

The group $L^{n+1}_2\times L^1_2$ acts transitively on projective space of second order P^n_2.

We now introduce the non-homogeneous coordinates on P^n_2. Let us consider a point $(x^0,x^1,\dots,x^n,y^0,y^1,\dots,y^n)$ with $x^i\neq 0$. Then its non-homogeneous coordinates are defined by

$$\begin{aligned} X^0 &= x^0x^i,\dots,X^{i-1}=\frac{x^{i-1}}{x^i},X^{i+1}=\frac{x^{i+1}}{x^i},\dots,X^n=\frac{x^n}{x^i}\\ Y^0 &= \frac{y^0x^i-x^0y^i}{(x^i)^3},\dots,Y^{i-1}=\frac{y^{i-1}x^i-x^{i-1}y^i}{(x^i)^3},\\ Y^{i+1} &= \frac{y^{i+1}x^i-x^{i+1}y^i}{(x^i)^3},\dots,Y^n=\frac{y^nx^i-x^ny^i}{(x^i)^3}. \end{aligned}$$

We can express the action of the group $L^{n+1}_2\times L^1_2$ on the projective space of second order P^n_2 in the non-homogeneous coordinates.

Let us consider a point $(x^0,\dots,x^n,y^0,\dots,y^n)$ with $x^0\neq 0$. The image of this point under the element $\big((a^i_j,a^i_{jk}),(a,b)\big)\in L^{n+1}_2\times L^1_2$ is the point

$$\Big(\frac{1}{a}a^i_jx^j, \frac{1}{a^2}a^i_{ks}x^sx^k + \frac{1}{a^2}a^i_ky^k - \frac{b}{a^3}a^i_kx^k\Big).$$

Let us assume that $\frac{1}{a}a_j^0 x^j \neq 0$. Then

$$\begin{aligned}
\frac{\frac{1}{a}a_0^i x^0 + \ldots + \frac{1}{a}a_n^i x^n}{\frac{1}{a}a_0^0 x^0 + \ldots + \frac{1}{a}a_n^0 x^n} &= \frac{a_0^i + a_1^i \frac{x^1}{x^0} \ldots + a_n^i \frac{x^n}{x^0}}{a_0^0 + a_1^0 \frac{x^1}{x^0} \ldots + a_n^0 \frac{x^n}{x^0}} \\
= \frac{a_0^i + a_1^i X^1 \ldots + a_n^i X^n}{a_0^0 + a_1^0 X^1 \ldots + a_n^0 X^n} &= \frac{\frac{a_0^i}{a_0^0} + \frac{a_1^i}{a_0^0} X^1 \ldots + \frac{a_n^i}{a_0^0} X^n}{1 + \frac{a_1^0}{a_0^0} X^1 \ldots + \frac{a_n^0}{a_0^0} Y^n} \\
= \frac{A^i + A_1^i X^1 \ldots + A_n^i X^n}{1 + A_1 X^1 \ldots + A_n X^n}\,, &
\end{aligned}$$

where we put $A^i = \frac{a_0^i}{a_0^0}, A_i = \frac{a_i^0}{a_0^0}, A_j^i = \frac{a_j^i}{a_0^0}$, for $i, j \geq 1$. Passing to the second group of coordinates we get

$$\begin{aligned}
&\frac{\left(\frac{1}{a^2}a_{jk}^i x^j x^k + \frac{1}{a^2}a_j^i y^j + \frac{b}{a^2}a_j^i x^j\right)\frac{1}{a}a_t^0 x^t - a_t^i x^t\left(\frac{1}{a^2}a_{jk}^0 x^j x^k + \frac{1}{a^2}a_j^0 y^j + \frac{b}{a^2}a_j^0 x^j\right)}{\left(\frac{1}{a}a_t^0 x^t\right)^3} \\
&= \frac{a_{jk}^i a_t^0 - a_{jk}^0 a_t^i)X^j X^k X^t}{(a_0^0 + a_j^0 X^j)^3} + \frac{a_j^i a_t^0 (X^t Y^j - X^j Y^t)}{(a_0^0 + a_j^0 X^j)^3} \\
&= \frac{\left(\frac{a_{jk}^i}{(a_0^0)^2}\frac{a_t^0}{a_0^0} - \frac{a_{jk}^0}{(a_0^0)^2}\frac{a_t^i}{a_0^0}\right)X^j X^k X^t + A^i A_j A Y^j - A^i A_j Y^j + A_j^i A_t A (X^t Y^j - X^j Y^t)}{(1 + A_j x^j)^3}\,,
\end{aligned}$$

where $A = \frac{1}{a_0^0}$ and $i, j, t \geq 1$. Introduce the following notation:

$$\begin{aligned}
B_{tj} = \frac{a_{tj}^0}{(a_0^0)^2}, B^i = \frac{a_{00}^i}{(a_0^0)^2}, B = \frac{a_{00}^0}{(a_0^0)^2}\,, \\
B_j = \frac{a_{0j}^0}{(a_0^0)^2}, B_j^i = \frac{a_{0j}^i}{(a_0^0)^2}\,, B_{tj}^i = \frac{a_{tj}^i}{(a_0^0)^2}\,,
\end{aligned}$$

for $i, j, t \geq 1$. Then

$$\begin{aligned}
&\frac{\left(\frac{1}{a^2}a_{jk}^i x^j x^k + \frac{1}{a^2}a_j^i y^j + \frac{b}{a^2}a_j^i x^j\right)\frac{1}{a}a_t^0 x^t - a_t^i x^t\left(\frac{1}{a^2}a_{jk}^0 x^j x^k + \frac{1}{a^2}a_j^0 y^j + \frac{b}{a^2}a_j^0 x^j\right)}{\left(\frac{1}{a}a_t^0 x^t\right)^3} \\
&= \Big[B^i - BA^i + (B^i A_j - BA_j^i)X^j + 2(B_j^i - B_j A^i)X^j + (B_{tj}^i - B_{tj}A^i)X^t X^j \\
&\quad + 2(B_t^i A_j - B_t A_j^i)X^t X^j + (B_{jk}^i A_t - B_{jk} A_t)X^j X^k X^t + A^i A_j A Y^j \\
&\quad - A_j^i A Y^j + A_j^i A_t A (X^t Y^j - X^j Y^t)\Big](1 + A_j X^j)^{-3}\,.
\end{aligned}$$

DEFINITION 1.2. The coordinates on the generalized projective group $L_2^{n+1} \times L_2^1$ given by

$$\begin{aligned} A^i &= \frac{a_0^i}{a_0^0}, A_i = \frac{a_i^0}{a_0^0}, A_j^i = \frac{a_j^i}{a_0^0}, A = \frac{1}{a_0^0}, \\ B_{tj} &= \frac{a_{tj}^0}{(a_0^0)^2}, B^i = \frac{a_{00}^i}{(a_0^0)^2}, B = \frac{a_{00}^0}{(a_0^0)^2}, \\ B_j &= \frac{a_{0j}^0}{(a_0^0)^2}, B_j^i = \frac{a_{0j}^i}{(a_0^0)^2}, B_{tj}^i = \frac{a_{tj}^i}{(a_0^0)^2}, \end{aligned}$$

will be called the non-homogeneous coordinates on this group.

Consider the isotropy subgroup of the point $(0, \dots, 0)$. This group is described by the equations

$$A^i = 0, \quad B^i - BA^i = 0\,.$$

Corollary 1.1

The isotropy group is given by the equations $A^i = B^i = 0$. The dimension of the generalized projective group $L_2^{n+1} \times L_2^1$ is equal to $\frac{5n^2+7n+4}{2}$, the dimension of the isotropy group is equal to $\frac{5n^2+3n+4}{2}$, and the dimension of the projective space of second order P_n^2 is $2n$.

2. Maurer–Cartan equations of generalized projective group

We proceed to find the Maurer-Cartan equations of projective group of second order for forms $\omega^i, \omega_j^i, \omega_j, \omega, \beta_{tj}, \beta^i, \beta, \beta_j^i, \beta_j, \beta_{jk}^i$, which coincide in the neutral element of the group with the differentials $dA^i, dA_j^i, dA_j, dA, dB_{tj}, dB^i, dB, dB_j^i, dB_j, dB_{jk}^i$. Let $\alpha, \beta, \gamma = 0, 1, \dots, n$. We put

$$\left(\bar\omega_\beta^\alpha, \bar\omega_{\beta\gamma}^\alpha\right) = \left(a_\beta^\alpha, a_{\beta\gamma}^\alpha\right)^{-1} \cdot \left(da_\beta^\alpha, da_{\beta\gamma}^\alpha\right).$$

Some standard calculations lead to

$$\left(\bar\omega_\beta^\alpha, \bar\omega_{\beta\gamma}^\alpha\right) = \left(\tilde a_\delta^\alpha da_\beta^\delta, -\tilde a_\delta^\alpha a_{\varepsilon\gamma}^\delta \tilde a_\sigma^\varepsilon da_\beta^\sigma - \tilde a_\delta^\alpha a_{\beta\sigma}^\delta \tilde a_\varepsilon^\sigma da_\gamma^\varepsilon + \tilde a_\sigma^\alpha da_{\beta\gamma}^\sigma\right).$$

Thus,

$$\left(d\bar\omega_\beta^\alpha, d\bar\omega_{\beta\gamma}^\alpha\right) = \left(-\bar\omega_\sigma^\alpha \wedge \bar\omega_\beta^\sigma, -\bar\omega_{\sigma\beta}^\alpha \wedge \bar\omega_\gamma^\sigma - \bar\omega_\sigma^\alpha \wedge \bar\omega_{\beta\gamma}^\sigma\right) \quad \alpha, \beta, \gamma = 0, 1, \dots, n\,.$$

The definition of local coordinate system on the generalized projective group implies that

$$dA^i = \frac{da_0^i a_0^0 - da_0^0 a_0^i}{(a_0^0)^2},$$
$$dA_j^i = \frac{da_j^i a_0^0 + da_0^0 a_j^i}{(a_0^0)^2}.$$

Thus, at the identity of the projective group

$$\omega^i = dA^i = da_0^i = \bar{\omega}_0^i, \omega_j^i = \bar{\omega}_j^i + \delta_j^i \bar{\omega}_0^0, \omega_j = \bar{\omega}_j^0$$

and, similarly,

$$\omega = -\bar{\omega}_0^0, \beta^i = \bar{\omega}_{00}^i, \beta_j = \bar{\omega}_{0j}^0,$$
$$\beta = \bar{\omega}_{00}^0, \beta_{jk}^i = \bar{\omega}_{jk}^i, \beta_{tj} = \bar{\omega}_{tj}^0, \beta_j^i = \bar{\omega}_{0j}^i.$$

Using the above notation the following theorem is true.

Theorem 2.1

Let $(\omega^i, \omega_j^i, \omega_j; \omega, \beta^i, \beta_j, \beta, \beta_{jk}^i, \beta_{kj}, \beta_j^i)$, $i, j, k = 1, \ldots, n$, *be the left invariant forms on the projective group of second order* $L_2^{n+1} \times L_2^1$ *which coincide with* $dA^i, dA_j^i, dA_j, dA, dB_{tj}, dB^i, dB, dB_j^i, dB_j, dB_{jk}^i$ *at the identity. Then the Maurer-Cartan equations of* $L_2^{n+1} \times L_2^1$ *are given by*

$$(a) \qquad \begin{cases} d\omega^i = -\omega_k^i \wedge \omega^k, \\ d\omega_j^i = -\omega_k^i \wedge \omega_j^k - \omega^i \wedge \omega_j + \delta_j^i \omega_k \wedge \omega^k, \\ d\omega^i = -\omega_k \wedge \omega_j^k. \end{cases}$$

$$(b) \quad \begin{cases} d\beta = \omega \wedge \beta - 2\beta_i \wedge \omega^i - \omega_i \wedge \beta^i, \\ d\beta_k^i = \beta_k^i \wedge \omega - \beta_{tk}^i \wedge \omega^t - \omega^i \wedge \beta_k - \omega_t^i \wedge \beta_k^t - \beta^i \wedge \omega_k - \beta_t^i \wedge \omega_k^t, \\ d\beta_{jk}^i = -\beta_k^i \wedge \omega_j - \beta_{tk}^i \wedge \omega_j^t - \beta_j^i \wedge \omega_k - \beta_{tj}^i \wedge \omega_k^t + \beta_{kj}^i \wedge \omega - \omega^i \wedge \beta_{jk} - \omega_t^i \wedge \beta_{jk}^t, \\ d\beta_{kj} = -\beta_j \wedge \omega_k - \beta_{tj} \wedge \omega_k^t + \beta_{kj} \wedge \omega - \beta_{tk} \wedge \omega_j^t - \omega_t \wedge \beta_{kj}^t, \\ d\beta^i = \beta^i \wedge \omega - 2\beta_t^i \wedge \omega^t - \beta^i \wedge \beta - \omega_t^i \wedge \beta^t, \\ d\beta_j = \beta_j \wedge \omega - \beta_{tj} \wedge \omega^t - \beta \wedge \omega_j - \beta_t \wedge \omega_j^t - \omega_t \wedge \beta_j^t, \\ d\omega = \omega_i \wedge \omega^i. \end{cases}$$

Part (a) of the above Maurer-Cartan equations coincides exactly with the equations given in [1] for the classical projective space. Thus, our space essentially generalizes the ordinary projective space. In a forthcoming paper we will examine the G-structures modelled on this space.

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