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On the positive absolutely summing operators on the space $L_p\,(\mu, X)$

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Abstract

In this paper we obtain a characterization of the positive absolutely summing operators on the space $L_p(\mu, X)$ with $1 \le p < \infty$ and X a Banach lattice, which is analogous with that given by Ch. Swartz in [8] for absolutely summing operators on the space of continuous vector valued functions.

Throughout in this paper X will denote a Banach lattice , $X_+ = \{x \in X \mid x \geq 0\}$ and B is a Banach space. Also, if Y is a normed space and $y_1,\ldots, y_n \in Y$ we write $l_1\big(y_i \mid i = 1, n\big) = \sum_{i=1}^n \|y_i\|$ and $w_1\big(y_i \mid i = 1, n\big) = \sup_{\|y^*\|\le 1} \sum \mid y^*(y_i) \mid$.

Recall that $U \in L(X, B)$ is called a positive absolutely summing operator, if there is a constant $C > 0$ such that for all elements $x_{1,\ldots,}x_n$ in X_+ we have: $l_1(Ux_i \mid i = 1, n) \le Cw_1(x_i \mid i = 1, n)$ and the positive absolutely summing norm of U is $\|U\|_{as+} = \inf C$. Observe that

$$\|U\|_{as+} = \sup\big\{l_1(Ux_i \mid i = 1, n) \mid x_1, \ldots, x_n \in X_+ \; n \geq 1, \\ \text{with } w_1(x_i \mid i = 1, n) \le 1\big\}\,,$$

and $l_1(Ux_i \mid i = 1, n) \le \|U\|_{as+}w_1(x_i \mid i = 1, n)$, for all $x_1, \ldots, x_n \in X_+$.

We denote by $As_+(X, B)$ the space of all positive absolutely summing operators from X into the B. In the sequel we use the following useful relation: $w_1(x_1 \mid i = 1, n) = \| \sum_{i=1}^n x_i\|$, for all $x_1, \ldots, x_n \in X_+$, see [7].

Given $1 \le p < \infty$ we shall always write q for such a number that $1/p + 1/q = 1$. Let (S, Σ, μ) be a finite measure space and $1 \le p < \infty$. We shall denote by $L_p(\mu, X)$ the space of all X-valued measurable functions f such that: $\|f\|_p = (\int_S \|f\|^p d\mu)^{1/p} < \infty$, which is a Banach lattice in a natural way, namely $f \geq 0$ means that, $f(s) \geq 0$, for μ-almost all $s \in S$. Also recall that, if Y is a Banach space, a finitely additive vector measure $G : \Sigma \to Y$, is said to have bounded q-variation if

$$\mid G \mid_q (S) = \sup\left\{ \sum_{E \in \pi} \frac{\|G(E)\|^q}{\mu(E)^{q-1}} \right\} < \infty\,,\ (1 < q < \infty)\,,$$

where the sup is taken over all finite partitions of S and

$$\mid G \mid_\infty (S) = \sup\left\{ \frac{\|G(E)\|}{\mu(E)} \mid E \in \Sigma \right\} < \infty\,,\ (q = \infty)\,.$$

We shall denote by $V^q(\mu, Y)$ the space of such measures with norm given by q-variation (see [6] for details). For $U \in L\big(L_p(\mu, X), B\big)$, $1 \leq p < \infty$, we denote by $G : \Sigma \to L(X, B)$, $G(E)x = U(\chi_E x)\,, E \in \Sigma\,, x \in X$, the representing measure of U. Now we prove the following theorem.

Theorem 1

$$As_+(L_p(\mu, X), B) = As_+\big(L_p(\mu), As_+(X, B)\big) = V^q(\mu, As_+(X, B))\,.$$

Proof. O. Blasco proved in [3] that $As_+(L_p(\mu), Z) = V^q(\mu, Z)$ for each normed space Z. Hence we have the equality: $As_+\big(L_p(\mu, As_+(X, B)\big) = V^q\big(\mu, As_+(X, B)\big)$. For the equality $As_+\big(L_p(\mu, X), B\big) = V^q\big(\mu, As_+(X, B)\big)$, let $U \in As_+\big(L_p(\mu, X), B\big)$, $E \in \Sigma$, $x_1, ..., x_n \in X_+$. We have:

$$\begin{aligned} l_1\big(G(E)x_i \mid i = 1, n\big) &= l_1\big(U(\chi_E x_i) \mid i = 1, n\big) \leq \|U\|_{as+} w_1\big(\chi_E x_i \mid i = 1, n\big) \\ &= \|U\|_{as+} \Big\| \sum_{i=1}^n \chi_E x_i \Big\| = \|U\|_{as+} \big[\mu(E)\big]^{1/p} \Big\| \sum_{i=1}^n x_i \Big\| \\ &= \|U\|_{as+} \big[\mu(E)\big]^{1/p} w_1\big(x_i \mid i = 1, n\big) \end{aligned}$$

i.e. $G(E) \in As_+(X, Y)$ and $\|G(E)\|_{as+} \leq [\mu(E)]^{1/p} \|U\|_{as+}$, for each $E \in \Sigma$.

If $p = 1$ the above inequality shows that: $G \in V^q(\mu, As_+(X, B))$ and $\mid G \mid_\infty (S) \leq \|U\|_{as+}$.

If $p > 1$, let $\{E_1, ..., E_n\} \subset \Sigma$ be a partition of S, $\alpha_i = [\mu(E_i)]^{-q/p}$, $\beta_i = \|G(E_i)\|_{as+}^{q-1}$ and $\epsilon > 0$.

From the definition of the positive absolutely norm it follows that there exist $(x_{ij})_{j \in \sigma_i} \subset X_+$, σ_i finite $\subset N$ so that:

$$\|G(E_i)\alpha_i\beta_i\|_{as+} - \epsilon/n < l_1(G(E_i)\alpha_i\beta_i x_{ij} \mid j \in \sigma_i)\,,$$

with $w_1(x_{ij} \mid j \in \sigma_i) \leq 1$, for each $i = \overline{1,n}$. From here we obtain:

$$\begin{aligned}\sum_{i=1}^{n} \|G(E_i)\|_{as+}\, \alpha_i\beta_i - \epsilon &< l_1\big(U(\chi_{E_i}\alpha_i\beta_i x_{ij}) \mid j \in \sigma_i, i = \overline{1,n}\big)\\ &\leq \|U\|_{as+} w_1\big(\chi_{E_i}\alpha_i\beta_i x_{ij} \mid j \in \sigma_i, i = \overline{1,n}\big)\\ &= \|U\|_{as+}\Big\|\sum_{i=1}^{n}\sum_{j\in\sigma_i}\chi_{E_i}\alpha_i\beta_i x_{ij}\Big\|\\ &= \|U\|_{as+}\Big(\sum_{i=1}^{n}\alpha_i^p\beta_i^p\mu(E_i)\Big\|\sum_{j\in\sigma_i}x_{ij}\Big\|^p\Big)^{1/p}\end{aligned}$$

As: $\big\|\sum_{j\in\sigma_i} x_{ij}\big\| = w_1(x_{ij} \mid j \in \sigma_i) \leq 1$, for each $i = \overline{1,n}$ we obtain:

$$\sum_{i=1}^{n} \|G(E_i)\|_{as+}\, \alpha_i\beta_i - \epsilon < \|U\|_{as+}\left(\sum_{i=1}^{n}\alpha_i^p\beta_i^p\mu(E_i)\right)^{1/p}$$

from where, $\epsilon > 0$ being arbitrary,

$$\sum_{i=1}^{n} \|G(E_i)\|_{as+}\, \alpha_i\beta_i \leq \|U\|_{as+}\left(\sum_{i=1}^{n}\alpha_i^p\beta_i^p\mu(E_i)\right)^{1/p}.$$

Using the definition of α_i, β_i and $1/p + 1/q = 1$ the above relation give:

$$\sum_{i=1}^{n}\frac{\|G(E_i)\|_{as+}^q}{\mu(E_i)^{q-1}} \leq \|U\|_{as+}\left(\sum_{i=1}^{n}\frac{\|G(E_i)\|_{as+}^q}{\mu(E_i)^{q-1}}\right)^{1/p}$$

from where:

$$\left(\sum_{i=1}^{n}\frac{\|G(E_i)\|_{as+}^q}{\mu(E_i)^{q-1}}\right)^{1/q} \leq \|U\|_{as+}\,,$$

i.e. $G \in V^q\big(\mu, As_+(X,B)\big)$ and $|G|_q(S) \leq \|U\|_{as+}$.

Conversely if $G \in V^q\big(\mu, As_+(X,B)\big)$, then $U \in As_+\big(L_p(\mu,X),B\big)$. For $f \in L_p(\mu,X)$, $f \geq 0$ and $\epsilon > 0$ there exist a simple function g that takes its values in the range of f, hence $g \geq 0$, such that $\|f-g\|_p < \epsilon$ (see [5] the proof of Theorem 2 p. 45 and the Pettis measurability Theorem [5] p. 42). This shows that it suffices to prove that the restriction of U to the subspace of all simple functions from $L_p(\mu,X)$ is a positive absolutely summing operator. Let $f_1,...,f_n \geq 0$ simple functions. Then there exist a partition $\{E_1,...,E_k\} \subset \Sigma$ and $x_{ij} \geq 0$ such that $f_i = \sum_{j=1}^{k} \chi_{E_j} x_{ij}$ for each $i = 1, n$. Then:

$$l_1(Uf_i \mid i = 1, n) = l_1\left(\sum_{j=1}^{k} G(E_j) x_{ij} \mid i = 1, n\right) \leq \sum_{j=1}^{k} l_1\big(G(E_j) x_{ij} \mid i = 1, n\big)$$

$$\leq \sum_{j=1}^{k} \|G(E_j)\|_{as+} w_1(x_{ij} \mid i = 1, n)$$

because G takes its values in $As_+(X, B)$. Using the Holder inequality we obtain:

$$l_1(Uf_i \mid i = 1, n) \leq \left(\sum_{j=i}^{k} \frac{\|G(E_j)\|_{as+}^q}{\mu(E_j)^{q-1}}\right)^{1/q} \left(\sum_{j=1}^{k} \mu(E_j) \big[w_1(x_{ij} \mid i = 1, n)\big]^p\right)^{1/p}$$

$$\leq |G|_q (S) \left(\sum_{j=1}^{k} \mu(E_j) \| \sum_{i=1}^{n} x_{ij} \|^p\right)^{1/p} = |G|_q (S) \left\| \sum_{i=1}^{n} f_i \right\|$$

$$= |G|_q (S) w_1\big(f_i \mid i = 1, n\big)$$

i.e. U is positive absolutely summing and $\|U\|_{as+} \leq |G|_q (S)$ and the Theorem is proved. □

We are indebted to the referee for the next Corollary which is a version of Fubini Theorem.

Corollary 2

$$V^p\big([0,1]^2, B\big) = V^p\big([0,1], V^p([0,1], B)\big).$$

Proof. Using the Theorem 1 and the well known fact that $L_q\big([0,1]^2\big) = L_q\big([0,1], L_q[0,1]\big)$ we have the equalities:

$$V^p\big([0,1]^2, B\big) = As_+\big(L_q([0,1]^2), B\big) = As_+\big(L_q([0,1], L_q[0,1]), B\big)$$

$$= As_+\big(L_q([0,1], As_+(L_q[0,1], B)\big) = V^p\big([0,1], V^p([0,1], B)\big) . \ \square$$

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