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Separation for ordinary differential equation with matrix coefficient

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Abstract

In this paper we give criteria for the separation of the differential operator

$$L[u] = (-1)^m D^{2m} u(x) + q(x)u(x)$$

in the space $L_p(\mathbb{R})^\ell$, ℓ, $m \in \mathbb{N}$ and $p \in (1, \infty)$ where $q(x), x \in \mathbb{R}$, is a $\ell \times \ell$ positive hermitian matrix and prove the existence and uniqueness of the solution for the differential equation

$$(L + \beta E)u(x) = f(x), f(x) \in L_p(\mathbb{R})^\ell$$

where E is the identity operator and $\beta \geq 1$.

1. Introduction

The term "separation" and the first results on the separation of differential expressions are due to W.N. Everitt and M. Giertz [5–7]. They studied the following question, under what conditions on $q(x)$ does $u(x) \in L_2(I)$ and $-u''(x)+q(x)u(x) \in L^2(I)$ imply that $u''(x) \in L_2(I), I = (-\infty, \infty)$. K.KH. Boimatov [1–3], M.O. Otelbaev [10], S.A. Eshakov [4], R.M. Kauffman [8], A.S. Mohamed [9–10] and others have also worked on the problem of separativity.

Now we introduce some definitions that will be used in the subsequent sections: $L_p(\mathbb{R})^\ell,\ p \in (1,\infty),\ \ell \in \mathbb{N}$ denotes the space of vector functions $u(x) = (u_i(x)),\ i = \overline{1,\ell},\ x \in \mathbb{R}$, with the norm:

$$\|u\|_{p,\ell} = \left(\sum_{i=1}^{\ell} \int_{-\infty}^{\infty} |u_i(x)|^p dx \right)^{1/p}.$$

Here, $\| \ \|_{p,\ell}$ means the norm of the vector function in the space $L_p(\mathbb{R})^\ell$. By $W_p^{2m}(\mathbb{R})^\ell$ we mean the space of vector functions $u(x), u \in \mathbb{R}$, that has generalized derivatives $D^\alpha u(x), \alpha < 2m$ in the sense of Sobolev. We say, that the function $u(x) \in W_{p,\mathrm{loc}}^{2m}(\mathbb{R})^\ell$ if for all function $\varphi(x) \in C_o^\infty(\mathbb{R})$ the vector function $\varphi(x)u(x) \in W_p^{2m}(\mathbb{R})^\ell$. We shall consider the following differential expression:

$$L[u] = (-1)^m D^{2m} u(x) + q(x)u(x) \tag{1.1}$$

where $u \in L_p(\mathbb{R})^\ell \cap W_{p,\mathrm{loc}}^{2m}(\mathbb{R})^\ell$ and q is $\ell \times \ell$ hermitian matrix. The differential expression (1.1) has been studied in [2] and [6] when $m = 1,\ p = 2$ the case when $\ell = 1$, $p \in (1,\infty)$ in [3] and [4] in the case of $m = 1$, $p \in (1,\infty), \ell \in \mathbb{N}$ is contained in [9].

In this paper we study the separation of the differential expression (1.1) in the Banach space $L_p(\mathbb{R})^\ell$ for any $p \in (1,\infty)$ and any arbitrary natural numbers m and ℓ.

2. Main results

The differential expression (1.1) is said to be separated in the space $L_p(\mathbb{R})^\ell$ if for all vector function $u \in L_p(\mathbb{R})^\ell \cap W_{p,\mathrm{loc}}^{2m}(\mathbb{R})^\ell$ such that $L[u] \in L_p(\mathbb{R})^\ell$ implies that $D^{2m}u \in L_p(\mathbb{R})^\ell$ and $q\,u \in L_p(\mathbb{R})^\ell$. The above definition equivalent to the coercive estimate

$$\|D^{2m}u\|_{p,\ell} + \|q\,u\|_{p,\ell} \le \delta_1 \Big[\|L[u]\|_{p,\ell} + \|u\|_{p,\ell} \Big]. \tag{2.1}$$

We say that the matrix q belongs to the class $S_{\beta,\ell}, \beta \ge 1$ if the following conditions are satisfied:

(i) $\lambda(x) \ge 1$ for all $x \in \mathbb{R}$ where $\lambda(x)$ is the first eigenvalue of the matrix $q(x)$.
(ii) $\|\big(q(x) - q(y)\big)q^{-1}(y)\| \le \frac{1}{\beta}$ for all $x, y \in \mathbb{R}$ such that $|x - y| \le \beta\lambda^{-1/2m}(x)$.

For example, $q(x) = \begin{bmatrix} M^2\big(1+|x|\big)^2 & 0 \\ 0 & M^4\big(1+|x|\big)^4 \end{bmatrix} \in S_{\beta,2}, M = 24\beta^2$.

Here $\|A\|$ denotes the norm of A considered as a linear operator in $\mathbb{C}^\ell$.

In the following theorems we formulate our main results.

Theorem 2.1

For every $p \in (1,\infty)$ and $m,\ell \in \mathbb{N}$ there exists a number $\beta = \beta(p,\ell) \geq 1$ such that if the matrix $q \in S_{\beta,\ell}$ the differential expression (1.1) is separated in the space $L_p(\mathbb{R})^\ell$.

Theorem 2.2

Consider Theorem 2.1. Then the linear differential equation

$$(-1)^m u^{(2m)}(x) + q(x)u(x) + \beta u(x) = f(x) \tag{2.2}$$

has a unique solution $u \in L_p(\mathbb{R})^\ell \cap W^{2m}_{p,\mathrm{loc}}(\mathbb{R})^\ell$ for all $f(x) \in L_p(\mathbb{R})^\ell$ $x \in \mathbb{R}$, furthermore, we have the coercive estimate

$$\left\|u^{2m}(x)\right\|_{p,\ell} + \left\|q(x)u(x)\right\|_{p,\ell} \leq \delta_2\Big[\left\|u(x)\right\|_{p,\ell} + \left\|f(x)\right\|_{p,\ell}\Big]\,. \tag{2.3}$$

Where δ_1 and δ_2 are constants not depend on $u(x)$.

3. Proof of Theorem 2.1

The proof is somewhat lengthy but straightforward. We subdivide it into four lemmas. Firstly, let us define the real functions $f_0(x)$ and $f_1(x)$ as follows:

$$f_0(x) = \int_{-\infty}^{\infty} \xi\big(\gamma_m \beta^{-1} \lambda^{1/2m}(y) \quad (x-y)\big)\, dy\,;$$
$$f_1(x) = f_0^{-1}(x) \int_{-\infty}^{\infty} \xi\big(\gamma_m \beta^{-1} \lambda^{1/2m}(y) \quad (x-y)\big)\, \lambda^{1/2m}(y)\, dy\,;$$

where $\gamma_m = 3^{2m} - 1$ and

$$\xi(t) = \begin{cases} \cos^4(\pi/2)t & ,\ |t| < 1 \\ 0 & ,\ |t| \geq 1\,. \end{cases}$$

Lemma 3.1

Let $q(x) \in S_{\beta,\ell}, \beta \geq 1$ then the following are valid

$$\frac{1}{3}\lambda^{1/2m}(x) \leq f_1(x) \leq 3\lambda^{1/2m}(x) \tag{3.1}$$

$$\left|\frac{df_1(x)}{dx}\right| \leq C\, \beta^{-1} f_1^2(x) \tag{3.2}$$

where C is a constant depends only on m.

Proof. The proof is similar to the proof of the Lemma 3.2 in [8]. □

Concerning the next lemma, let us define the real valued function

$$f(x) = \beta^{1/4m} + \beta^{-1/4m} f_1(x) .$$

For sufficiently large value of β and from Lemma 3.1 the function $f(x)$ satisfies the inequalities $1 \le f(x)$ and $\left|\frac{df(x)}{dx}\right| \le f^2(x)$ for all $x \in \mathbb{R}$ and using [Lemma 2.1; 3] there exists a partition of unity $\sum_{j=1}^{\infty} \varphi_j(x) \equiv 1,\ x \in \mathbb{R}$, of multiplicity less than a constant Γ having the following properties:

(i) $\varphi_j(x) \in \mathbb{C}_0^{\infty}(\mathbb{R})\,, \quad j = 1, 2, \ldots$

(ii) $|D_x^{\alpha} \varphi_j(x)| \le M_{\alpha} f^{\alpha}(x)\,,$ for all $x \in \mathbb{R}$ and $\alpha \in \mathbb{N}$.

(iii) $|x-y| f(x) \le 1,$ for all $x, y \in \operatorname{supp} \varphi_j$.

Let ϕ_j be an operator multiplied by the function φ_j on the space $L_p(\mathbb{R})^{\ell}$ that is, $\phi_j u(x) = \varphi_j(x) u(x)$ and R_j is an integral operator on $L_p(\mathbb{R})^{\ell}$ with kernel $R_j(x,y) = (2\pi)^{-1} \int_{-\infty}^{\infty} \frac{e^{is(x-y)}}{|s|^{2m} I + q(x_j) + \beta I}\, ds$ where $x_j \in \operatorname{supp} \varphi_j$ is a fixed point and I is a unit matrix of order ℓ. Consider the operator $F = \sum_{j=1}^{\infty} \phi_j R_j \phi_j$ in the space $L_p(\mathbb{R})^{\ell}$ it is clear that $F : C_0^{\infty}(\mathbb{R})^{\ell} \longrightarrow C_0^{\infty}(\mathbb{R})^{\ell}$.

Lemma 3.2

For all $u(x) \in C_0^{\infty}(\mathbb{R})^{\ell}$ and $\beta \ge 1$ the following is valid

$$\big(L + \beta E\big) F\, u(x) = \big(E + G\big)\, u(x)$$

where $G = H_0 + H$ and

$$H_0 = \sum_{j=1}^{\infty} \phi_j \big[q(x) - q(x_j)\big] R_j \phi_j\,;$$

$$H = (-1)^m \sum_{j=1}^{\infty} \sum_{k=1}^{2m} \binom{2m}{k} \phi_j^{(k)} R_j^{(2m-k)} \phi_j\,;$$

where $\phi_j^{(k)}$ is the operator multiplied by the function $\frac{d^k}{dx^k}\varphi_j$ and $R_j^{(2m-k)}$ is the operator $D^{2m-k} R_j$, where $D = \frac{d}{dx}$.

Proof. Assuming that $L_j = (-1)^m D^{2m} + q(x_j)$. Since $(L_j + \beta E)R_j = E$, then $(L+\beta E)F\,u(x) = (E+G)\,u(x)$, where

$$G = \sum_{j=1}^{\infty} \phi_j (L - L_j) R_j \phi_j + \sum_{j=1}^{\infty} \big[L + \beta E\,,\, \phi_j\big] R_j \phi_j \tag{3.3}$$

In the second term on the right hand side the symbol [,] means the commutator that is, $[T_1, T_2] = T_1T_2 - T_2T_1$ where T_1 and T_2 are two operators. From the definition of L and L_j we have

$$\sum_{j=1}^{\infty} \phi_j (L - L_j) R_j \phi_j = \sum_{j=1}^{\infty} \phi_j \big(q(x) - q(x_j)\big) R_j \phi_j = H_0\,. \tag{3.4}$$

It is easy to see that $\big[L + \beta E\,,\, \phi_j\big] = \big[L\,,\, \phi_j\big]$.

Hence

$$\begin{aligned}\sum_{j=1}^{\infty} \big[L + \beta E\,,\, \phi_j\big] R_j \phi_j &= \sum_{j=1}^{\infty} \big[L\,,\, \phi_j\big] R_j \phi_j \\ &= \sum_{j=1}^{\infty} \big(L\phi_j R_j \phi_j - \phi_j L R_j \phi_j\big) \\ &= (-1)^m \sum_{j=1}^{\infty} \big(D^{2m}\phi_j R_j \phi_j - \phi_j D^{2m} R_j \phi_j\big)\,.\end{aligned}$$

By using Leibniz formula for differentiation we get

$$\begin{aligned}&\sum_{j=1}^{\infty} \big[L + \beta E\,,\, \phi_j\big] R_j \phi_j \\ &= (-1)^m \sum_{j=1}^{\infty} \left[\left(\sum_{k=0}^{2m} \binom{2m}{k} \phi_j^{(k)} R_j^{(2m-k)} \phi_j\right) - \phi_j R_j^{(2m)} \phi_j\right] \\ &= (-1)^m \sum_{j=1}^{\infty} \sum_{k=1}^{2m} \binom{2m}{k} \phi_j^{(k)} R_j^{2m-k} \phi_j = H\,. \end{aligned} \tag{3.5}$$

From (3.3), (3.4) and (3.5) we get the proof of Lemma 3.2. □

Lemma 3.3

There exist numbers $\mu_1(p)$ and $\mu_2(p)$ such that if $q(x) \in S_{\beta,\ell}, \beta \geq 1$ the following inequalities are valid

$$\|q(x_j) R_j\| \leq \mu_1(p) \tag{3.6}$$

$$\|R_j^{(2m-k)}\| \leq \mu_2(p)\big(\lambda(x_j) + \beta\big)^{-k/2m}\,,\; k = \overline{0, 2m} \tag{3.7}$$

where $\|\ \|$ means the norm of operator in the space $L_p(\mathbb{R})^\ell$.

Proof. The operator $q(x_j)R_j$ is an integral operator with the kernel

$$(2\pi)^{-1}q(x_j)\int_{-\infty}^{\infty}\frac{e^{is(x-y)}}{|s|^{2m}I+q(x_j)+\beta I}ds\,.$$

Since $q(x)$ is a hermitian matrix then the operator $q(x_j)R_j$ is unitary equivalent to

$$\operatorname{diag}\left\{\frac{\lambda_1(x_j)}{|s|^{2m}+\lambda_1^{2m}(x_j)+\beta},\ldots,\frac{\lambda_\ell(x_j)}{|s|^{2m}+\lambda_\ell^{2m}(x_j)+\beta}\right\}$$

using [Lemma 2.3; 3] we get

$$\|q(x_j)R_j\| = C_1(p)\max_{1\le r\le \ell}\sup_{s\in\mathbb{R}}\frac{\lambda_r(x_j)}{|s|^{2m}+\lambda_r(x_j)+\beta}$$

hence (3.6) follows. The integral operator $R_j^{(2m-k)}$ is unitarily equivalent to

$$\operatorname{diag}\Big\{R_{1,j}^{(2m-k)},\ R_{2,j}^{(2m-k)},\ldots,R_{\ell,j}^{(2m-k)}\Big\}$$

where $R_{r,j}, r=\overline{1,\ell}$ are operators in the space $L_p(\mathbb{R})$ with kernel

$$(2\pi)^{-1}\int_{-\infty}^{\infty}\frac{e^{is(x-y)}}{|s|^{2m}+\lambda_r(x_j)+\beta}\,ds$$

by using [Lemma 2.3; 3] we get

$$\begin{aligned}\|R_j^{(2m-k)}\| &= C_2(p)\max_{1\le r\le\ell}\|R_{r,j}^{(2m-k)}\|\\ &\le C_2(p)\max_{1\le r\le\ell}\sup_{s\in\mathbb{R}}\frac{|is|^{2m-k}}{|s|^{2m}+\lambda_r(x_j)+\beta}\,.\end{aligned}$$

For any two positive numbers A and B the following is true $A^{2\alpha}B^{2(1-\alpha)}\le A^2+B^2, \alpha\in[0,1]$ therefore, by taking $A=|s|^m$,

$$B=\big(\lambda_r(x_j)+\beta\big)^{1/2}\quad\text{and}\quad \alpha=1-\frac{k}{2m}\quad\text{we get (3.7).}\ \square$$

From the properties of $\varphi_j(x)$, $x\in\operatorname{supp}\varphi_j$ we have $|x-x_j|f(x_j)\le 1$, for all $x,x_j\in\operatorname{supp}\varphi_j$. By using Lemma 3.1 we get

$$f(x_j)=\beta^{1/4m}+\beta^{-1/4m}f_1(x_j)\ge\beta^{1/4m}+\frac{1}{3}\beta^{-1/4m}\lambda^{1/2m}(x_j)\,.$$

Hence

$$|x - x_j| \leq \frac{1}{\beta^{1/4m} + \frac{1}{3}\beta^{-1/4m}\lambda^{1/2m}(x_j)} \leq 3\beta^{1/4m}\lambda^{-1/2m}(x_j)\,.$$

Since $q \in S_{\beta,\ell}$ then

$$\|\big(q(x) - q(x_j)\big)q^{-1}(x_j)\| \leq \beta^{-1/4m}\,. \tag{3.8}$$

Now we estimate the norm of the operator H_0 :

$$\begin{aligned}\|H_0\| &\leq \sigma_1(p)\sup_j \|\phi_j\big(q(x) - q(x_j)\big)R_j\phi_j\| \\ &\leq \sigma_1(p)\sup_j \|\phi_j\|\ \|\big(q(x) - q(x_j)\big)q^{-1}(x_j)\|\ \|q(x_j)R_j\|\ \|\phi_j\|\,.\end{aligned}$$

By using Lemma 3.3 and (3.8) where $\|\phi_j\| \leq 1$ we have

$$\|H_0\| \leq \mu_3(p)\beta^{-1/4m} \tag{3.9}$$

by using the property (ii) of the partition and Lemma 3.1 we get

$$\|\phi_j^{(k)}\| \leq \mu_4\beta^{-k/4m}\big(\beta + \lambda(x_j)\big)^{k/2m}\,, \quad k = 0, 1, 2, \ldots 2m \tag{3.10}$$

using Lemma 3.3 and (3.10) to estimate the norm of operator H

$$\begin{aligned}\|H\| &\leq \sigma_2(p)\sup_j \max_{1\leq k\leq 2m} \frac{(2m)!2m}{k!(2m-k)!}\Big[\|\phi_j^{(k)}\|\ \|R_j^{(2m-k)}\|\ \|\phi_j\|\Big] \\ &\leq \mu_5(p, m)\beta^{-1/4m}\,.\end{aligned} \tag{3.11}$$

From (3.9) and (3.11) and since $\beta \geq 1$ then $\|G\| \leq \mu_6(p)\beta^{-1/4m}$ and for a suitable large value of β we can write $\|G\| \leq 1/2$ and from the operator theory, see [12] page 140, the operator $(E + G)^{-1}$ exists and bounded; furthermore $(E + G)^{-1} = \sum_{n=0}^{\infty} G^n$ and $\|(E + G)^{-1}\| \leq 2$.

Now from Lemma 3.2 we get

$$\big(L + \beta E\big)^{-1} = F(E + G)^{-1} = F(E - \tilde{G}) = F\sum_{n=0}^{\infty} G^n \tag{3.12}$$

where $\tilde{G} = E - (E + G)^{-1}$ and $\|\tilde{G}\| \leq 3$.

Lemma 3.4

There exist numbers $\mu_7(p)$ *and* $\mu_8(p,m)$, $p \in (1,\infty)$ *such that if* $q \in S_{\beta,\ell}, \beta$ *sufficiently large the following are valid*

$$\|qF\| \leq \mu_7(p)\,;$$
$$\|D^{2m}F\| \leq \mu_8(p,m)$$

Proof. From (3.8) we have

$$\|q(x)q^{-1}(x_j)\| \leq \beta^{-1} + 1 = \sigma_3\,. \tag{3.13}$$

Then one gets

$$\begin{aligned}\|qF\| &\leq \sigma_4(p)\sup_j \|q(x)\phi_j R_j \phi_j\| \\ &\leq \sigma_4(p)\sup_j \sup_{x\in \operatorname{supp}\varphi_j} \|q(x)q^{-1}(x_j)\|\ \|q(x_j)R_j\|\ \|\phi_j\|\,.\end{aligned}$$

Using [Lemma 2.2; 3], Lemma 3.3 and (3.13) the first inequality is proved. And similarly we can prove the second part of the lemma. □

Now we can estimate $\|q(L+\beta E)^{-1}\|$ by using (3.12) and Lemma 3.3

$$\|q(L+\beta E)^{-1}\| = \|qF(E-\tilde{G})\| \leq \|qF\|\ \|(E-\tilde{G}) = 2\mu_7(p) = \mu_9(p)\,.$$

By using the above estimate we have the following

$$\|q(L+\beta E)^{-1}v\|_{p,\ell} \leq \|q(L+\beta E)^{-1}\|\ \|v\|_{p,\ell} \leq \mu_9(p)\|v\|_{p,\ell}$$

where $v \in L_p(\mathbb{R})^\ell \cap W^{2m}_{p,\mathrm{loc}}(\mathbb{R})^\ell$. Put $(L+\beta E)^{-1}v = u$ to get

$$\begin{aligned}\|q\,u\|_{p,\ell} &\leq \mu_9(p)\|(L+\beta E)u\|_{p,\ell} \\ &\leq \mu_{10}(p)\Big[\|L[u]\|_{p,\ell} + \|u\|_{p,\ell}\Big] < \infty\end{aligned} \tag{3.14}$$

that is $q\,u \in L_p(\mathbb{R})^\ell$ and similarly we can obtain

$$\|D^{2m}u\|_{p,\ell} \leq \mu_{11}(p,m)\Big[\|L[u]\|_{p,\ell} + \|u\|_{p,\ell}\Big] < \infty \tag{3.15}$$

that is $D^{2m}u \in L_p(\mathbb{R})^\ell$.

Finally we conclude that the differentiable expression (1.1) is separated in the space $L_p(\mathbb{R})^\ell$ and the coercive estimate (2.1) is obtained from (3.14) and (3.15) which complete the proof of Theorem (2.1). □

4. Proof of Theorem 2.2

For a given $f_0(x) \in L_p(\mathbb{R})^\ell$ the differential equation (2.2) takes the form

$$(L + \beta E)u(x) = f_0(x) \tag{4.1}$$

where L is the differential expression (1.1).

Hence from (3.12) the solution of the differential equation (2.2) exists. From (3.7), β is sufficiently large, we have $\|R_j\| \leq \mu_{12}(p)$.

Using [Lemma 2.2; 3] and (3.12) to obtain

$$\begin{aligned}\|(L+\beta E)^{-1}\| \leq \|F\| \; \|(E+G)^{-1}\| &\leq 2\sigma_5(p) \sup_j \|\phi_j R_j \phi_j\| \\ &\leq 2\sigma_5(p)\mu_{12}(p) = \mu_{13}(p)\,.\end{aligned}$$

Hence for all $f_0(x) \in L_p(\mathbb{R})^\ell$ we have

$$\|(L+\beta E)^{-1} f_0(x)\|_{p,\ell} \leq \mu_{13}(p) \|f_0(x)\|_{p,\ell}\,. \tag{4.2}$$

For a given $f_0(x) \in L_p(\mathbb{R})^\ell$ suppose u_1 is another solution of the differential equation (4.1) then

$$(L+\beta E)\;(u-u_1) = 0\,.$$

From (4.2) if $f_0 = 0$ then

$$(L+\beta E)^{-1} f_0(x) = 0\,.$$

Therefore, $u_1 = u$ and the uniqueness is proved.

By substituting from (2.2) into (2.1) we get the coercive estimate (2.3) which complete the proof. □

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