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The Poincaré and related groups are algebraically determined Polish groups

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Abstract

The purpose of this paper is to prove a new topological fact about the Poincaré and related groups. If G is a group, say that G is an algebraically determined Polish ($i.e.$, complete separable metric topological) group if, whenever H is a Polish group and $\varphi : H \to G$ is an algebraic isomorphism, then φ is a topological isomorphism. The proper Lorentz group, the proper orthochronous Lorentz group and the Heisenberg group are examples of Polish groups that are not algebraically determined. On the other hand it will be shown that the Lorentz group, the orthochronous Lorentz group and the Poincaré group and the other closely associated semi-direct products are algebraically determined Polish groups.

1. Introducction

The purpose of this paper is to prove a new topological fact about the Poincaré and related groups. If G is group, say that G is an algebraically determined Polish ($i.e.$, complete separable metric topological) group if, whenever H is a Polish group and $\varphi : H \to G$ is an algebraic isomorphism, then φ is a topological isomorphism. This notion is a strengthening of the assertion that every automorphism of G is continuous.

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The proper Lorentz group $\mathcal{L}_s$, the proper orthochronous Lorentz group $\mathcal{L}_e$ and the Heisenberg group are examples of Polish groups that are not algebraically determined. On the other hand it will be shown that the Lorentz group $\mathcal{L}$, the orthochronous Lorentz group $\mathcal{L}_t$ and the Poincaré group $\mathcal{P}$ and the other three closely associated natural semi-direct products ($\mathcal{P}_t$, $\mathcal{P}_s$, $\mathcal{P}_e$) are algebraically determined Polish groups. The main results are contained in the following two theorems.

Theorem 1.1

The groups $\mathcal{P}$, $\mathcal{P}_t$, $\mathcal{P}_s$ and $\mathcal{P}_e$ are all algebraically determined Polish groups.

Theorem 1.2

$\mathcal{L}$ and $\mathcal{L}_t$ are algebraically determined Polish groups.

The methods used are a combination of descriptive set theory, algebra and Lie group theory. Careful definitions and preliminary results, many but not all probably well known, are given in Section 2. The proof of Theorem 1.1 is carried out in a sequence of lemmas in Section 3. Section 4 is devoted to the proof of Theorem 1.2 as well as demonstrating a rather general lemma which has as a consequence that the proper Lorentz group $\mathcal{L}_s$ as well as the proper orthochronous Lorentz group $\mathcal{L}_e$ are not algebraically determined Polish groups. The last Section 5, devoted to the Heisenberg group, may be regarded as a supplement to these results on the Lorentz and Poincare groups. It is simple to show that the Heisenberg group is not an algebraically determined Polish group but it is a little less simple to prove a universal partial continuity result.

The results of this paper can be greatly generalized but the apparently physically relevant groups considered here seem to be of special interest. It would be curious if the theorems presented here had physical implications or interpretations.

Algebraically determined Polish groups have been studied for some time now. The first Polish group shown to be algebraically determined was the infinite symmetric group S_∞ ([9]). Other Polish groups shown to be algebraically determined include compact simple Lie groups ([7]), the p-adic integers ([8]), compact connected metrizable groups with totally disconnected center ([10]), the $ax + b$ group ([11]), the measure preserving transformations of the Lebesgue space $[0, 1]$ ([12]), the group homeomorphisms of Euclidean manifolds and and the group of diffeomorphisms of smooth manifolds([13]) and the group of unitary operators on an infinite dimensional separable Hilbert space ([1]). Very recent work of a related but somewhat different character can be found in [4, 15, 18], and [19].

2. Preliminaries

The Lorentz group, $\mathcal{L}$, is defined to be the collection of 4×4 real matrices which leave the quadratic form $Q((t, x, y, z)) = t^2 - x^2 - y^2 - z^2$ invariant. Note that if $J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ then $J^2 = I$ and $Q(v) = \langle Jv, v\rangle$. The mapping

$$(v, w) \to B(v, w) = [Q(v + w) - Q(v - w)]/4 = \langle J(v), w\rangle$$

is bilinear, $Q(v) = B(v, v)$ and $B(L(v), L(w)) = B(v, w)$ for all v, $w \in \mathbb{R}^4$ and $L \in \mathcal{L}$. A real 4×4 matrix L is an element of $\mathcal{L}$ if and only if $L^T J L = J$. Therefore the elements of $\mathcal{L}$ have determinant ± 1 and are invertible, $\mathcal{L}$ is a group and $L \in \mathcal{L}$ if and only if $L^T \in \mathcal{L}$. If $L = [L_{i,j}]_{1 \le i,j \le 4} \in \mathcal{L}$, then

$$L_{1,1}^2 = 1 + L_{2,1}^2 + L_{3,1}^2 + L_{4,1}^2 = 1 + L_{1,2}^2 + L_{1,3}^2 + L_{1,4}^2 \ge 1.$$

$\mathcal{L}$ is a closed subgroup of $GL(4, \mathbb{R})$ and thus a complete separable metric group. The subgroup of $\mathcal{L}$ which fixes the t-component of every element of $\mathbb{R}^4$ may be identified with $O(3)$. Define $\mathcal{L}_t$, the group of orthochronous transformations, to be the set of elements in $\mathcal{L}$ whose upper-left coordinate is greater than or equal to 1. Then $\mathcal{L}_t$ is closed under the transpose operation and is a closed normal subgroup of $\mathcal{L}$ of index 2. $\mathcal{L}_t$ contains the connected component of the identity in $\mathcal{L}$. Define $\mathcal{L}_s$ as the elements of $\mathcal{L}$ whose determinant is 1. Then $\mathcal{L}_s$ is closed under the transpose operation and is a closed normal subgroup of $\mathcal{L}$ of index two. $\mathcal{L}_s$ is called the group of proper Lorentz transformations. $\mathcal{L}_s$ contains the connected component of the identity in $\mathcal{L}$. Let $\mathcal{L}_e = \mathcal{L}_s \cap \mathcal{L}_t$, the group of orthochronous proper Lorentz transformations. Then $\mathcal{L}_e$ is a closed normal subgroup of $\mathcal{L}$ of index four, is a closed normal subgroup of $\mathcal{L}_s$ of index two and is a closed normal subgroup of $\mathcal{L}_t$ of index two. $\mathcal{L}_e$ contains the connected component of the identity in $\mathcal{L}$ and in fact is the connected component of the identity in $\mathcal{L}$.

For any $t \in \mathbb{R}$, let

$$B_x(t) = \begin{bmatrix} \cosh(t) & \sinh(t) & 0 & 0 \\ \sinh(t) & \tanh(t) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ B_y(t) = \begin{bmatrix} \cosh(t) & 0 & \sinh(t) & 0 \\ 0 & 1 & 0 & 0 \\ \sinh(t) & 0 & \cosh(t) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$B_z(t) = \begin{bmatrix} \cosh(t) & 0 & 0 & \sinh(t) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(t) & 0 & 0 & \cosh(t) \end{bmatrix}, \ R_z(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) & 0 \\ 0 & \sin(t) & \cos(t) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(t) & 0 & -\sin(t) \\ 0 & 0 & 1 & 0 \\ 0 & \sin(t) & 0 & \cos(t) \end{bmatrix} \text{ and } R_x(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(t) & -\sin(t) \\ 0 & 0 & \sin(t) & \cos(t) \end{bmatrix}.$$

Then

$$B_x = B_x(\mathbb{R}),\ B_y = B_y(\mathbb{R}),\ B_z = B_z(\mathbb{R}),\ R_x = R_x(\mathbb{R}),\ R_y = R_y(\mathbb{R}) \text{ and } R_z = R_z(\mathbb{R})$$

are all connected one-parameter subgroups of $\mathcal{L}_e$. The group generated by R_x, R_y and R_z is a connected subgroup of $SO(3) \subset O(3)$. Actually this connected subgroup coincides with $SO(3)$ since simple geometric considerations prove that the group generated by R_x and R_y is transitive on the 2-sphere and the stability subgroup of $SO(3)$ at the north pole is R_z. Here we use the elementary fact that if G is a group, X is a transitive G-space, $H \subset G$ is a subgroup which is transitive on X and contains the stability group of G at some point of X, then $G = H$.

Let $\mathcal{L}_e'$ be the subgroup of $\mathcal{L}_e$ generated by B_x, B_y, B_z, R_x, R_y and R_z (*i.e.*, the group generated by B_x, B_y, B_z and $SO(3)$). Then $\mathcal{L}_e'$ is a connected subgroup of $\mathcal{L}_e$. We will prove that $\mathcal{L}_e = \mathcal{L}_e'$. Let $\mathcal{S}_1 = \{v = (t, x, y, z) \in \mathbb{R}^4 \mid Q(v) = 1\}$ and let $\mathcal{S}_1^+ = \{v = (t, x, y, z) \in \mathbb{R}^4 \mid Q(v) = 1 \text{ and } t > 0\}$. $\mathcal{S}_1$ is invariant under $\mathcal{L}$ and therefore under $\mathcal{L}_e$. We claim that $\mathcal{S}_1^+$ is invariant under $\mathcal{L}_t$ and therefore under $\mathcal{L}_e$. To see this, note that if $L \in \mathcal{L}_t$ and $v = (t, x, y, z) \in \mathcal{S}_1^+$, then the first component of $L(v)$ is

$$\begin{aligned} L_{1,1}t + L_{1,2}x + L_{1,3}y + L_{1,4}z &\geq L_{1,1}t - |L_{1,2}x + L_{1,3}y + L_{1,4}z| \\ &\geq L_{1,1}t - \sqrt{L_{1,2}^2 + L_{1,3}^2 + L_{1,4}^2} \cdot \sqrt{x^2 + y^2 + z^2} \\ &> L_{1,1}t - \sqrt{L_{1,2}^2 + L_{1,3}^2 + L_{1,4}^2 + 1} \cdot \sqrt{x^2 + y^2 + z^2 + 1} = 0. \end{aligned}$$

Next, we claim that $\mathcal{L}_e'$, and therefore $\mathcal{L}_e$, is transitive on $\mathcal{S}_1^+$. To see this, let $(t, x, y, z) \in \mathcal{S}_1^+$. Use an element of $SO(3)$ to map this vector to a vector of the form $(t, x', 0, 0) \in \mathcal{S}_1^+$ and then an element of B_x to map $(t, x', 0, 0)$ to $(1, 0, 0, 0)$. In particular, since $\mathcal{L}_e'$ is connected, we have that $\mathcal{S}_1^+$ is connected.

Finally, let $L \in \mathcal{L}$ fix $(1, 0, 0, 0)$. Then a direct computation shows that $L_{1,1} = 1$ and $L_{2,1} = L_{3,1} = L_{4,1} = 0$. Since $L_{1,1}^2 = 1 + L_{1,2}^2 + L_{1,3}^2 + L_{1,4}^2$, we have that $L_{1,2} = L_{1,3} = L_{1,4} = 0$. Hence, L is a block diagonal matrix with the lower right 3×3 block an element of $O(3)$. Hence, if $L \in \mathcal{L}_s$, then this lower right 3×3 block is an element of $SO(3)$. So the stability subgroup of $\mathcal{L}_e$ at $(1, 0, 0, 0)$ is $SO(3) \subset \mathcal{L}_e'$. Hence, as before, $\mathcal{L}_e = \mathcal{L}_e'$ and $\mathcal{L}_e$ is connected.

Lemma 2.1

$\mathcal{L}_e$ is algebraically generated by $\{x^2 \mid x \in \mathcal{L}_e\}$ and $\{x^2 \mid x \in \mathcal{L}\} \subset \mathcal{L}_e$. Therefore if $\mathcal{L}_e \subset \mathcal{K} \subset \mathcal{L}$, then the group generated by $\{x^2 \mid x \in \mathcal{K}\}$ is $\mathcal{L}_e$.

Proof. Every element of the one-parameter groups B_x, B_y, B_z, R_x, R_y and R_z of course is a square. Since these one-parameter groups algebraically generate $\mathcal{L}_e$, we have that the group algebraically generated by $\{x^2 \mid x \in \mathcal{L}_e\}$ is $\mathcal{L}_e$.

On the other hand, if $L \in \mathcal{L}$ then certainly $L^2 \in \mathcal{L}_s$ and either L or $-L \in \mathcal{L}_t$. Therefore $L^2 = (-L)^2 \in \mathcal{L}_t \cap \mathcal{L}_s = \mathcal{L}_e$. □

The following proposition and its proof are taken from [1].

Proposition 2.2

Let G be a Polish group, $A \subset G$ an analytic subset and $H \subset G$ an analytic subgroup such that A intersects each H-coset in exactly one point and $G = AH$. Then H is closed in G.

Proof. Since the topology on G is Polish, the relative topology on A is second countable and there exists a separating family of relatively open sets $\{C_i\}_{i \geq 1}$ for the topology on A, each of which is an analytic set in G. Let $E_i = C_i H$ for every $i \geq 1$. Each E_i is right-invariant under H and is analytic and hence has the Baire property since each E_i is a product of two analytic sets. The countable collection $\{E_i\}_{i \geq 1}$ separates the H-cosets. Miller's Theorem ([17, Theorem 1]) now implies that H is closed in G. □

The following general proposition ([3, 1.2.6]) is a key tool used in the proofs of the present results.

Proposition 2.3

Let G and H be Polish groups and let $\varphi : G \to H$ be an algebraic isomorphism which is measurable with respect to sets with the Baire property. Then φ is a topological isomorphism. This applies in particular if φ is a Borel mapping.

The next known general result ([3, 1.2.3]) will also be used to study the Heisenberg group.

Proposition 2.4

Let G be a Polish group and let $N \subset G$ be a closed normal subgroup. Then G/N is a Polish group in the quotient topology.

The proof of Theorem 1.2 will require that certain compact connected semisimple Lie groups be algebraically determined Polish groups. A proof that compact connected metrizable groups with totally disconnected center are algebraically determined Polish groups, a much more general result, is in [10], but the proof is somewhat involved. A proof for what is needed here was given in [7], but unfortunately in that paper the theorem was stated only for the restricted class of locally compact groups with a countable basis and not for Polish groups. The following more general result can be found in [1] and is given here for the convenience of the reader.

Proposition 2.5

Let G be a compact connected semisimple Lie group, H a Polish group and $\varphi : H \to G$ a surjective homomorphism such that $\varphi^{-1}(e)$ is an analytic set. Then φ is continuous. In particular if φ is a bijection, then φ is a topological isomorphism and therefore G is an algebraically determined Polish group.

Proof. First note that if $a \in G$ then $\varphi^{-1}(a)$ is an analytic subset of H. To see this, if $w \in H$ satisfies $\varphi(w) = a$, then $\varphi^{-1}(a) = \varphi^{-1}(e) \cdot w$ is certainly an analytic subset of H. Suppose that G is n-dimensional. Then by van der Waerden [20] the sets of the form

$$\Big\{ \prod_{1\le \ell\le n} c_\ell(b_\ell a b_\ell^{-1} a^{-1}) c_\ell^{-1} \mid b_\ell,\, c_\ell \in G \Big\},$$

where a is not in the center of G, are a neighborhood basis at e in G. But then

$$\begin{aligned}\varphi^{-1}\Big(\Big\{ \prod_{1\le \ell\le n} c_\ell(b_\ell a b_\ell^{-1} a^{-1}) c_\ell^{-1} \mid b_\ell,\, c_\ell \in G \Big\}\Big) \\ = \Big\{ \prod_{1\le \ell\le n} c_\ell(b_\ell \varphi^{-1}(a) b_\ell^{-1} \varphi^{-1}(a^{-1})) c_\ell^{-1} \mid b_\ell,\, c_\ell \in H \Big\}\end{aligned}$$

is clearly an analytic set since $\varphi^{-1}(a)$ and $\varphi^{-1}(a^{-1})$ are analytic sets. Therefore φ is continuous by the Banach, Kuratowski and Pettis Theorem ([14, Theorem 9.10, page 61]) since analytic sets are sets with the Baire property. If φ is a bijection, then $\varphi^{-1}(e)$ is a single point and therefore an analytic set. Hence, φ is continuous. φ is a topological isomorphism by Proposition 2.3 and G is an algebraically determined Polish group. □

3. $\mathcal{L} \times_\alpha \mathbb{R}^4$ is algebraically determined

Let $\mathcal{P} = \mathcal{L} \times_\alpha \mathbb{R}^4$ be the Poincare group, *i.e.*, the natural semi-direct product of the Lorentz group $\mathcal{L}$ with $\mathbb{R}^4$. Similarly, we define $\mathcal{P}_t = \mathcal{L}_t \times_\alpha \mathbb{R}^4$, $\mathcal{P}_s = \mathcal{L}_s \times_\alpha \mathbb{R}^4$, and $\mathcal{P}_e = \mathcal{L}_e \times_\alpha \mathbb{R}^4$. All of these groups are Polish groups in their natural topologies. $\{I\} \times_\alpha \mathbb{R}^4$ (respectively, $\mathcal{L} \times_\alpha \{0\}$, $\mathcal{L}_t \times_\alpha \{0\}$, $\mathcal{L}_s \times_\alpha \{0\}$, $\mathcal{L}_e \times_\alpha \{0\}$) is identified algebraically and topologically with $\mathbb{R}^4$ (respectively, $\mathcal{L}$, $\mathcal{L}_t$, $\mathcal{L}_s$, $\mathcal{L}_e$).

Lemma 3.1

$\{I\} \times_\alpha \mathbb{R}^4$ is maximal abelian in $\mathcal{P}$. If $\varphi : H \to \mathcal{P}$ is an algebraic isomorphism of Polish groups, then $\varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4)$ is maximal abelian and hence closed.

Proof. If $(I, v_1), (I, v_2) \in \{I\}\times_\alpha\mathbb{R}^4$, then $(I, v_1)(I, v_2) = (I, v_1+v_2)$ and $(I, v_2)(I, v_1) = (I, v_2 + v_1) = (I, v_1 + v_2)$, so $\{I\} \times_\alpha \mathbb{R}^4$ is an abelian subgroup of $\mathcal{P}$.

If $(L, w) \in \mathcal{P}$ commutes with $\{I\} \times_\alpha \mathbb{R}^4$, then for each $v \in \mathbb{R}^4$, $(I, v)(L, w) = (L, w + v)$ and $(L, w)(I, v) = (L, L(v) + w)$, $L(v) = v$, so $L = I$ and hence $\{I\} \times_\alpha \mathbb{R}^4$ is maximal abelian.

The last statement follows from the general fact that any maximal abelian subgroup of any Hausdorff topological group is closed. □

Lemma 3.2

If $G = \mathcal{L}$, or $\mathcal{L}_s$, or $\mathcal{L}_t$ or $\mathcal{L}_e$, then the centralizers of each of $(B_x \cdot R_x) \times_\alpha \{0\}$, $(B_y \cdot R_y) \times_\alpha \{0\}$ and $(B_z \cdot R_z) \times_\alpha \{0\}$ in $G \times_\alpha \mathbb{R}^4$ are subgroups of $G \times_\alpha \{0\}$ which contain the groups themselves.

Proof. Each of the groups $(B_x \cdot R_x) \times_\alpha \{0\}$, $(B_y \cdot R_y) \times_\alpha \{0\}$ and $(B_z \cdot R_z) \times_\alpha \{0\}$ are abelian and are therefore contained in their centralizers. These centralizers are subgroups of $G \times_\alpha \{0\}$. To see this, it suffices to give a proof for the x case. The y and z cases are similar. Suppose that (L, v) is in the centralizer of $(B_x \cdot R_x) \times_\alpha \{0\}$. Then $B_x(t)R_y(s)(v) = v$ for all s, $t \in \mathbb{R}$ and therefore $v = 0$. □

Corollary 3.3

If $G = \mathcal{L}$, or $\mathcal{L}_s$, or $\mathcal{L}_t$ or $\mathcal{L}_e$ and $\varphi : H \to G \times_\alpha \mathbb{R}^4$ is an algebraic isomorphism, then $\varphi^{-1}(\mathcal{L}_e\times_\alpha\{0\})$ and $\varphi^{-1}(G\times_\alpha\{0\})$ are closed subgroups of H, while $\varphi^{-1}(\mathcal{L}_e\times_\alpha\mathbb{R}^4)$ is an open subgroup of H of finite index less than or equal to 4.

Proof. The subgroup of H algebraically generated by the squares of the centralizers of

$$\varphi^{-1}((B_x \cdot R_x) \times_\alpha \{0\}), \varphi^{-1}((B_y \cdot R_y) \times_\alpha \{0\}) \text{ and } \varphi^{-1}((B_z \cdot R_z) \times_\alpha \{0\})$$

is an analytic subgroup since these centralizers are closed in H, since the set of squares of the elements of a closed subset are an analytic set and since a subgroup of H algebraically generated by a sequence of analytic subsets of H is analytic. This subgroup is contained in $\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})$ by Lemma 3.2 and Lemma 2.1. On the other hand $\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})$ is contained in this subgroup since $B_x \cdot R_x$, $B_y \cdot R_y$ and $B_z \cdot R_z$ are contained in their own centralizers and algebraically generate $\mathcal{L}_e$ by the discussion in Section 2. Hence, $\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})$ is an analytic subgroup of H. Furthermore, $\varphi^{-1}(G \times_\alpha \{0\})$ is an analytic subgroup of H since it consists of a finite number of cosets of $\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})$. Now $\varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4)$ is a maximal abelian and therefore closed subgroup of H by Lemma 3.1. Furthermore

$$H = \varphi^{-1}\big(\{I\} \times_\alpha \mathbb{R}^4\big) \cdot \varphi^{-1}\big(G \times_\alpha \{0\}\big)$$

since

$$\big(G \times_\alpha \mathbb{R}^4\big) = \big(\{I\} \times_\alpha \mathbb{R}^4\big) \cdot \big(G \times_\alpha \{0\}\big)$$

and

$$\varphi^{-1}\big(\{I\} \times_\alpha \mathbb{R}^4\big) \cap \varphi^{-1}\big(G \times_\alpha \{0\}\big) = \varphi^{-1}\big((I,0)\big)$$

since

$$\big(\{I\} \times_\alpha \mathbb{R}^4\big) \cap \big(G \times_\alpha \{0\}\big) = \big\{(I,0)\big\}\,.$$

$\varphi^{-1}\big(\{I\} \times_\alpha \mathbb{R}^4\big)$ hits each $\varphi^{-1}(G \times_\alpha \{0\})$ coset in one and only one point and therefore $\varphi^{-1}(G \times_\alpha \{0\})$ is closed in H by Proposition 2.2.

$\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})$ is an analytic subgroup of $\varphi^{-1}(G \times_\alpha \{0\})$ and is of second category in $\varphi^{-1}(G \times_\alpha \{0\})$ since $\varphi^{-1}(G \times_\alpha \{0\})$ is a finite union of $\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})$-cosets. Hence $\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})$ is closed in $\varphi^{-1}(G \times_\alpha \{0\})$ and therefore in H by either another use of Proposition 2.2 or by [2, Théorème 1, page 21]. Furthermore

$$\varphi^{-1}(\mathcal{L}_e \times_\alpha \mathbb{R}^4) = \varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4) \cdot \varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})$$

is an analytic subgroup of H that is of finite index in H and therefore is both open and closed by still another application of Proposition 2.2 or by [2, Théorème 1, page 21]. □

Proposition 3.4

Let $\mathcal{S}_0^+ = \{v = (t,x,y,z) \in \mathbb{R}^4 \mid Q(v) = 0$ and $t > 0\}$. $\mathcal{S}_0^+$ is invariant under $\mathcal{L}_e$, $\mathcal{L}_e$ acts transitively on $\mathcal{S}_0^+$ and $(\mathcal{S}_0^+ + \mathcal{S}_0^+) = \mathcal{C}_0^+$, where $\mathcal{C}_0^+ = \{(t,x,y,z) \mid t^2 \geq x^2 + y^2 + z^2$ and $t > 0\}$.

Proof. If $L \in \mathcal{L}_e$ and $v = (t,x,y,z) \in \mathcal{S}_0^+$, we have that the first component of $L(v)$ is

$$\begin{aligned} L_{1,1}t + L_{1,2}x + L_{1,3}y + L_{1,4}z &\geq L_{1,1}t - \sqrt{L_{1,2}^2 + L_{1,3}^2 + L_{1,4}^2}\sqrt{x^2+y^2+z^2} \\ &= L_{1,1}t - \sqrt{L_{1,1}^2 - 1} \cdot t > 0. \end{aligned}$$

Hence, $\mathcal{S}_0^+$ is invariant under $\mathcal{L}_e$. To show that $\mathcal{L}_e$ is transitive on $\mathcal{S}_0^+$, first choose an element $R \in SO(3) \subset \mathcal{L}_e$ such that $R^{-1}(v) = (t, x', 0, 0)$, where $t^2 - x'^2 = 0$ and then choose an element $B \in B_x$ such that $B(1,1,0,0) = (t,x',0,0)$. Then $RB \in \mathcal{L}_e$ and $(RB)(1,1,0,0) = v$ and $\mathcal{L}_e$ acts transitively on $\mathcal{S}_0^+$.

The Schwarz inequality implies that

$$\mathcal{S}_0^+ + \mathcal{S}_0^+ \subset \{(t,x,y,z) \mid t^2 \geq x^2 + y^2 + z^2 \text{ and } t > 0\}.$$

Conversely, suppose that $t > 0$ and $t^2 > x^2$. Then

$$(t,x,0,0) = \Big(\frac{t+x}{2}, \frac{x+t}{2}, 0, 0\Big) + \Big(\frac{t-x}{2}, \frac{x-t}{2}, 0, 0\Big),$$

where
$$\Big(\frac{t+x}{2}, \frac{x+t}{2}, 0, 0\Big), \Big(\frac{t-x}{2}, \frac{x-t}{2}, 0, 0\Big) \in \mathcal{S}_0^+.$$
If $t > 0$ and $t^2 = x^2$, then
$$(t, x, 0, 0) = \Big(\frac{t}{2}, \frac{x}{2}, 0, 0\Big) + \Big(\frac{t}{2}, \frac{x}{2}, 0, 0\Big),$$
where
$$\Big(\frac{t}{2}, \frac{x}{2}, 0, 0\Big) \in \mathcal{S}_0^+.$$
Therefore
$$\big\{(t, x, 0, 0) \mid t > 0 \text{ and } t^2 \geq x^2\big\} \subset \mathcal{S}_0^+ + \mathcal{S}_0^+.$$
The right hand side of this containment is invariant under $SO(3) \subset \mathcal{L}_e$ and therefore
$$\begin{aligned}\big\{(t, x, y, z) \mid t^2 \geq x^2 + y^2 + z^2 \text{ and } t > 0\big\}\\ = SO(3)\Big(\big\{(t, x, 0, 0) \mid t > 0 \text{ and } t^2 \geq x^2\big\}\Big) \subset \mathcal{S}_0^+ + \mathcal{S}_0^+.\end{aligned}$$
□

Lemma 3.5

Let $\{v_\ell\}_{\ell \geq 1} \subset \mathbb{R}^4$ be dense. Then the σ-algebra of subsets of $\mathbb{R}^4$ generated by $\{v_\ell + \mathcal{C}_0^+\}_{\ell \geq 1}$ is $\mathcal{B}(\mathbb{R}^4)$, the set of all Borel subsets of $\mathbb{R}^4$.

Proof. Note that $\mathrm{Int}(\mathcal{C}_0^+) = \{(t, x, y, z) \mid t^2 > x^2 + y^2 + z^2 \text{ and } t > 0\}$ and that $\mathrm{Cl}(\mathcal{C}_0^+) = \{(t, x, y, z) \mid t^2 \geq x^2 + y^2 + z^2 \text{ and } t \geq 0\} = \mathcal{C}_0^+ \cup \{(0, 0, 0, 0)\}$. Note also that $\mathrm{Int}(v + \mathcal{C}_0^+) = v + \mathrm{Int}(\mathcal{C}_0^+)$ and $\mathrm{Cl}(v + \mathcal{C}_0^+) = v + \mathrm{Cl}(\mathcal{C}_0^+)$ for all $v \in \mathbb{R}^4$ since translation is a homeomorphism.

The σ-algebra of subsets of $\mathbb{R}^4$ generated by $\{v_\ell + \mathcal{C}_0^+\}_{\ell \geq 1}$ will be $\mathcal{B}(\mathbb{R}^4)$ if the sequence $\{v_\ell + \mathcal{C}_0^+\}_{\ell \geq 1}$ separates the points of $\mathbb{R}^4$ by Mackey [16]. Suppose that (t_1, x_1, y_1, z_1), $(t_2, x_2, y_2, z_2) \in \mathbb{R}^4$ are distinct points. If one can prove that there exists $v \in \mathbb{R}^4$ such that $(t_2, x_2, y_2, z_2) - v \in \mathrm{Int}(\mathcal{C}_0^+)$ and $(t_1, x_1, y_1, z_1) - v \notin \mathrm{Cl}(\mathcal{C}_0^+)$, then, by continuity of addition, the same will be true with v replaced with some v_ℓ since the collection $\{v_\ell\}_{\ell \geq 1}$ is dense in $\mathbb{R}^4$. But then $(t_2, x_2, y_2, z_2) \in v_\ell + \mathcal{C}_0^+$ and $(t_1, x_1, y_1, z_1) \notin v_\ell + \mathcal{C}_0^+$.

First suppose that $t_1 < t_2$. If $v = \Big(\frac{t_1+t_2}{2}, x_2, y_2, z_2\Big)$, then
$$\big(t_2, x_2, y_2, z_2\big) - v = \Big(\frac{t_2 - t_1}{2}, 0, 0, 0\Big) \in \mathrm{Int}(\mathcal{C}_0^+)$$
and
$$(t_1, x_1, y_1, z_1) - v = \Big(-\frac{t_2 - t_1}{2}, x_1 - x_2, y_1 - y_2, z_1 - z_2\Big) \notin \mathrm{Cl}(\mathcal{C}_0^+).$$
Next, suppose that $t_1 = t_2 = t$. If
$$v = \Big(t - \frac{\sqrt{(x_2 - x_1)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}}{2}, x_2, y_2, z_2\Big),$$

then

$$(t_2, x_2, y_2, z_2) - v = \Big(\frac{\sqrt{(x_2-x_1)^2+(y_1-y_2)^2+(z_1-z_2)^2}}{2}, 0, 0, 0\Big) \in \mathrm{Int}(\mathcal{C}_0^+)$$

and

$$\begin{aligned}&(t_1, x_1, y_1, z_1) - v \\ &\quad = \Big(\frac{\sqrt{(x_2-x_1)^2+(y_1-y_2)^2+(z_1-z_2)^2}}{2}, x_1 - x_2, y_1 - y_2, z_1 - z_2\Big) \notin \mathrm{Cl}(\mathcal{C}_0^+).\end{aligned}$$

□

Proposition 3.6

If $\varphi : H \to \mathcal{L}_e \times_\alpha \mathbb{R}^4$ is an algebraic isomorphism of Polish groups, then $\varphi|\varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4) : \varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4) \to \{I\} \times_\alpha \mathbb{R}^4$ is a topological isomorphism of Polish groups.

Proof. Lemma 3.1 implies that $\varphi^{-1}(\{I\}\times_\alpha\mathbb{R}^4)$ is a maximal abelian subgroup of H and therefore is a closed and hence Polish subgroup. $\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})$ is a closed subgroup of H by Corollary 3.3.

$$\mathcal{C}_0^+ = \big\{L((1,1,0,0)) \mid L \in \mathcal{L}_e\big\} + \big\{M((1,1,0,0)) \mid M \in \mathcal{L}_e\big\}$$

and therefore

$$\begin{aligned}\{I\} \times_\alpha \mathcal{C}_0^+ = &\big\{(L,0)(I,(1,1,0,0))(L,0)^{-1} \mid L \in \mathcal{L}_e\big\} \\ &\times \big\{(M,0)(I,(1,1,0,0))(M,0)^{-1} \mid M \in \mathcal{L}_e\big\}\end{aligned}$$

by Proposition 3.4. Hence,

$$\begin{aligned}\varphi^{-1}\big(\{I\} \times_\alpha \mathcal{C}_0^+\big) &= \big\{\varphi^{-1}((L,0))\varphi^{-1}((I,(1,1,0,0)))\varphi^{-1}((L,0))^{-1} \mid L \in \mathcal{L}_e\big\} \\ &\quad \times \big\{\varphi^{-1}((M,0))\varphi^{-1}((I,(1,1,0,0)))\varphi^{-1}((M,0))^{-1} \mid M \in \mathcal{L}_e\big\} \\ &= \big\{L\varphi^{-1}((I,(1,1,0,0)))L^{-1} \mid L \in \varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})\big\} \\ &\quad \times \big\{M\varphi^{-1}((I,(1,1,0,0)))M^{-1} \mid M \in \varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})\big\}\end{aligned}$$

is an analytic subset of $\varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4)$. Furthermore,

$$\varphi^{-1}\big(\{I\} \times_\alpha (v + \mathcal{C}_0^+)\big) = \varphi^{-1}((I,v)) \cdot \varphi^{-1}\big(\{I\} \times_\alpha \mathcal{C}_0^+\big)$$

is an analytic subset of $\varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4)$ for every $v \in \mathbb{R}^4$. Therefore $\varphi^{-1}(\{I\} \times_\alpha B)$ is in the σ-algebra generated by the analytic sets for every Borel subset $B \subset \mathbb{R}^4$ by Lemma 3.5. Hence, $\varphi|\varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4)$ is measurable with respect to the sets with the Baire property and therefore is a topological isomorphism by Proposition 2.3. □

Proposition 3.7

If $\varphi : H \to \mathcal{L}_e \times_\alpha \mathbb{R}^4$ is an isomorphism of Polish groups, then $\varphi|\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\}) : \varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\}) \to \mathcal{L}_e \times_\alpha \{0\}$ is a topological isomorphism.

Proof. Let $\{v_\ell\}_{\ell\geq 1} \subset \mathbb{R}^4$ be dense in $\mathbb{R}^4$ and define

$$\varphi_1 : \varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\}) \to \prod_{\ell\geq 1} \varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4)$$

by

$$\varphi_1(z) = \prod_{\ell\geq 1} z\varphi^{-1}((I, v_\ell))z^{-1},$$

$$\varphi_2 : \prod_{\ell\geq 1} \varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4) \to \prod_{\ell\geq 1} \{I\} \times_\alpha \mathbb{R}^4$$

by

$$\varphi_2\Big(\prod_{\ell\geq 1} z_\ell\Big) = \prod_{\ell\geq 1} \varphi(z_\ell)$$

and

$$\varphi_3 : \mathcal{L}_e \times_\alpha \{0\} \to \prod_{\ell\geq 1} \{I\} \times_\alpha \mathbb{R}^4$$

by

$$\varphi_3((L,0)) = \prod_{\ell\geq 1}(L,0)(I,v_\ell)(L,0)^{-1} = \prod_{\ell\geq 1}(I, L(v_\ell)).$$

First, notice that φ_1 maps $\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})$ into $\prod_{\ell\geq 1} \varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4)$, for if $L \in \mathcal{L}_e$ then

$$\begin{aligned}\varphi_1(\varphi^{-1}((L,0))) &= \prod_{\ell\geq 1} \varphi^{-1}((L,0))\varphi^{-1}((I,v_\ell))\varphi^{-1}((L,0))^{-1} \\ &= \prod_{\ell\geq 1} \varphi^{-1}((L,0)(I,v_\ell)(L^{-1},0)) = \prod_{\ell\geq 1} \varphi^{-1}((I, L(v_\ell))).\end{aligned}$$

φ_1 is continuous since $z \mapsto z\varphi^{-1}((0, v_\ell))z^{-1}$ is continuous for each v_ℓ and the product of continuous maps is also continuous.

φ_2 is continuous by Proposition 3.6.

φ_3 is continuous by the same mode of reasoning used to prove that φ_1 is continuous. φ_3 is injective since $\{v_\ell\}_{\ell\geq 1}$ is dense in $\mathbb{R}^4$ and each $L : \mathbb{R}^4 \to \mathbb{R}^4$ is continuous. Thus the Lusin-Souslin Theorem (page 89, [14]) gives us that $\varphi_3(\mathcal{L}_e \times_\alpha \{0\})$ is a Borel subset of $\prod_{\ell\geq 1}\{I\} \times_\alpha \mathbb{R}^4$ and that $\varphi_3^{-1} : \varphi_3(\mathcal{L}_e \times_\alpha \{0\}) \to \mathcal{L}_e \times_\alpha \{0\}$ is a Borel mapping.

Finally, notice that $\varphi_3^{-1} \circ \varphi_2 \circ \varphi_1$ is well-defined and

$$\begin{aligned}(\varphi_3^{-1} \circ \varphi_2 \circ \varphi_1)\big(\varphi^{-1}((L,0))\big) &= (\varphi_3^{-1} \circ \varphi_2)\Big(\prod_{\ell\geq 1} \varphi^{-1}\big((I, L(v_\ell))\big)\Big) \\ &= \varphi_3^{-1}\Big(\prod_{\ell\geq 1}(I, L(v_\ell))\Big) = (L,0) = \varphi\big(\varphi^{-1}((L,0))\big).\end{aligned}$$

Hence $\varphi_3^{-1} \circ \varphi_2 \circ \varphi_1 = \varphi|\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\})$. This yields that

$$\varphi|\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\}) : \varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\}) \to \mathcal{L}_e \times_\alpha \{0\}$$

is a Borel isomorphism of Polish groups and hence is a topological isomorphism by Proposition 2.3. □

Proposition 3.8

If $\varphi : H \to \mathcal{L}_e \times_\alpha \mathbb{R}^4$ is an algebraic isomorphism of Polish groups, then φ is a topological isomorphism.

Proof. $\Psi : H \times H \to H$ defined by $\Psi(a, b) = ba$ is continuous. $\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\}) \times \varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4)$ is closed in $H \times H$ and hence is itself Polish and

$$\Psi_0 = \Psi|\varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\}) \times \varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4)$$

is a continuous bijection since

$$\Psi_0(\varphi^{-1}((L, 0)), \varphi^{-1}((I, v))) = \varphi^{-1}((L, v)).$$

So by the Lusin-Souslin Theorem ([14, page 89]) Ψ_0^{-1} is a Borel mapping. Also notice that if we define $\Phi_1 : \varphi^{-1}(\mathcal{L}_e \times_\alpha \{0\}) \times \varphi^{-1}(\{I\} \times_\alpha \mathbb{R}^4)$ by $\Phi_1(a, b) = (\varphi(a), \varphi(b))$, then Φ_1 is continuous by Propositions 3.6 and 3.7. Moreover, if we define

$$\Phi_2 : (\mathcal{L}_e \times_\alpha \mathbb{R}^4) \times (\mathcal{L}_e \times_\alpha \mathbb{R}^4) \to \mathcal{L}_e \times_\alpha \mathbb{R}^4$$

by $\Phi_2(a, b) = ba$, then Φ_2 is continuous and so $\varphi = \Phi_2 \circ \Phi_1 \circ \Psi_0^{-1}$ is a Borel mapping. Therefore φ is a topological isomorphism by Proposition 2.3. □

Proof of Theorem 1.1. We have already proven that $\mathcal{P}_e$ is an algebraically determined Polish group in Proposition 3.8. We prove that $\mathcal{P}$ is an algebraically determined Polish group. The proofs that $\mathcal{P}_t$ and $\mathcal{P}_s$ are algebraically determined Polish groups are similar. Let H be a Polish group and let $\varphi : H \to \mathcal{P}$ be an algebraic isomorphism. Then $\varphi|\varphi^{-1}(\mathcal{L}_e \times_\alpha \mathbb{R}^4) : \varphi^{-1}(\mathcal{L}_e \times_\alpha \mathbb{R}^4) \to \mathcal{L}_e \times_\alpha \mathbb{R}^4$ is an algebraic isomorphism of Polish groups by Corollary 3.3. Hence, $\varphi|\varphi^{-1}(\mathcal{L}_e \times_\alpha \mathbb{R}^4)$ is continuous since $\mathcal{P}_e$ is an algebraically determined Polish group and therefore φ itself is continuous since $\varphi^{-1}(\mathcal{L}_e \times_\alpha \mathbb{R}^4)$ is an open subgroup of H by Corollary 3.3. Therefore φ is a topological isomorphism by Proposition 2.3.

4. The Lorentz Group and Related Groups

The purpose of this section is to prove that $\mathcal{L}$ and $\mathcal{L}_t$ are algebraically determined Polish groups but that $\mathcal{L}_s$ and $\mathcal{L}_e$ are not.

Lemma 4.1

Let G be a second countable Lie group with Lie algebra $\mathfrak{g}$ and $X \in \mathfrak{g}$ such that $\text{span}(\{Ad(g)(X) \mid g \in G\}) = \mathfrak{g}$ *and let $G_X = \{\exp(tX) \mid t \in \mathbb{R}\}$. Suppose that H is a Polish group, $\varphi : H \to G$ is an algebraic isomorphism, $\varphi^{-1}(G_X)$ is a Borel subgroup of H and $\varphi|\varphi^{-1}(G_X) : \varphi^{-1}(G_X) \to G_X$ is a Borel mapping. Then φ is a topological isomorphism.*

Proof. Let $n = \dim(\mathfrak{g})$ and let $e = a_1, \ldots, a_n \in G$ such that $X_\ell = Ad(a_\ell)(X)$ $(1 \le \ell \le n)$ forms a basis for $\mathfrak{g}$. Let

$$G_\ell = \left\{ \exp(tX_\ell) \mid t \in \mathbb{R} \right\} = a_\ell G_X a_\ell^{-1} . \varphi^{-1}(G_\ell) = \varphi^{-1}(a_\ell)\varphi^{-1}(G_X)\varphi^{-1}(a_\ell)^{-1}$$

is a Borel subgroup of H and $\varphi|\varphi^{-1}(G_\ell) : \varphi^{-1}(G_\ell) \to G_\ell$ is a Borel mapping since

$$\varphi^{-1}\big(a_\ell \exp(tX) a_\ell^{-1}\big) \to \varphi^{-1}\big(\exp(tX)\big) \to \exp(tX) \to a_\ell \exp(tX) a_\ell^{-1}$$

is a composition of Borel mappings. For every $\epsilon > 0$ which is sufficiently small, the set

$$N_\epsilon = \big\{ \exp(t_1 X_1) \cdots \exp(t_n X_n) \mid t_\ell \in <-\epsilon, \epsilon> \big\}$$

is a subbasic open neighborhood of $e \in G$ ([6, Lemma 2.4, page 105]). Furthermore

$$\varphi^{-1}(N_\epsilon) = \varphi^{-1}\big(\{\exp(t_1 X_1) \mid t_1 \in <-\epsilon, \epsilon>\}\big) \cdots \varphi^{-1}\big(\{\exp(t_n X_n) \mid t_n \in <-\epsilon, \epsilon>\}\big)$$

is an analytic subset of H since each $\varphi^{-1}(\{\exp(t_\ell X_\ell) \mid t_\ell \in <-\epsilon, \epsilon>\})$ is a Borel subset of H. Next, if $U \subset G$ is open, then $U = \cup_{m \geq 1} g_m N_{\epsilon_m}$ for suitable $g_m \in G$ and $\epsilon_m > 0$. Hence, $\varphi^{-1}(U) = \cup_{m \geq 1} \varphi^{-1}(g_m) \varphi^{-1}(N_{\epsilon_m})$ is an analytic subset of H. Therefore, if $B \subset G$ is a Borel set, then $\varphi^{-1}(B)$ is in the σ-algebra generated by the analytic subsets of H and therefore is a set with the Baire property. Hence, φ is a topological isomorphism by Proposition 2.3. $\square$

Proof of Theorem 1.2. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, an involution in $\mathcal{L}_t$. Let H be a Polish group and let $\varphi : H \to \mathcal{L}_t$ be an algebraic isomorphism. Centralizer$(\{A\}) = O(3)$ and therefore $\varphi^{-1}(O(3)) =$ Centralizer$(\varphi^{-1}(\{A\}))$ is a closed subgroup of H. The commutator subgroup of $O(3)$ is $SO(3)$ and therefore the commutator subgroup of $\varphi^{-1}(O(3))$, viz., $\varphi^{-1}(SO(3))$, is an analytic subgroup of $\varphi^{-1}(O(3))$ of index two. Hence, $\varphi^{-1}(SO(3))$ is a closed subgroup of $\varphi^{-1}(O(3))$ and therefore a closed subgroup of H by [2, Théorème 1, page 21]. $\varphi|\varphi^{-1}(SO(3)) : \varphi^{-1}(SO(3)) \to SO(3)$ is an algebraic isomorphism of Polish groups and is therefore a topological isomorphism by Proposition 2.5 since $SO(3)$ is a compact connected simple Lie group. Therefore there certainly is a compact one-parameter subgroup of T of $SO(3)$ such that $\varphi^{-1}(T)$ is compact and $\varphi|\varphi^{-1}(T) : \varphi^{-1}(T) \to T$ is a topological isomorphism. Let $\mathfrak{l}$ be the Lie algebra of $\mathcal{L}_t$ (the same as the Lie algebras of $\mathcal{L}_e$ or $\mathcal{L}_s$ or $\mathcal{L}$). The adjoint representation of $\mathcal{L}_e$ (and therefore of $\mathcal{L}_t$ or $\mathcal{L}_s$ or $\mathcal{L}$) acts irreducibly on $\mathfrak{l}$ since $\mathfrak{l}$ is a simple Lie algebra. Now use Lemma 4.1 to conclude that φ is a topological isomorphism and that $\mathcal{L}_t$ is an algebraically determined Polish group.

An almost word-for-word repetition of this argument proves that $\mathcal{L}$ is an algebraically determined Polish group. The only change needed is that the centralizer of A in $\mathcal{L}$ is $(\pm I) \cdot O(3)$ and not just $O(3)$. But the commutator subgroup of $(\pm I) \cdot O(3)$ is $SO(3)$ just as it was for $O(3)$ and $\varphi^{-1}(SO(3))$ is again closed in Centralizer$(\varphi^{-1}(\{A\}))$ and therefore in H since $\varphi^{-1}(SO(3))$ is of index four in Centralizer$(\varphi^{-1}(\{A\}))$. $\square$

Lemma 4.2

Let G be a topological group and and let $Z \subset$ Center(G) be a subgroup such that $z \in Z$ implies $z^2 = e$. Suppose that each of the sets $\{w^2 \mid w \in W\}$, where W is an open neighborhood of $e \in G$, contains a neighborhood of $e \in G$ (which is true if G is *a Lie group). Suppose that φ is an automorphism of G such that $\varphi(Z) = Z$ and such that the induced automorphism $\tilde{\varphi} : G/Z \to G/Z$ is continuous. Then φ is continuous.*

Proof. Let $\pi : G \to G/Z$ be the quotient mapping. $\tilde{\varphi}(\pi(a)) = \pi(\varphi(a))$ for all $a \in G$. Let U be an open neighborhood of $e \in G$ and choose an open neighborhood V of $e \in G$ such that $V^2 \subset U$. Since $\tilde{\varphi}$ is continuous there is an open neighborhood W of $e \in G$ such that $\tilde{\varphi}(\pi(W)) \subset \pi(V)$. Therefore $\pi(\varphi(W)) \subset \pi(V)$, $Z\varphi(W) \subset ZV$ and

$$\begin{aligned}
&\varphi\big(\mathrm{Int}(\{w^2 \mid w \in W\})\big) \subset \varphi\big(\{w^2 \mid w \in W\}\big) \\
&\quad = \big\{(z\varphi(w))^2 \mid w \in W \text{ and } z \in Z\big\} \subset \big\{(zv)^2 \mid v \in V \text{ and } z \in Z\big\} \\
&\quad = \big\{v^2 \mid v \in V\big\} \subset V^2 \subset U.
\end{aligned}$$

Since U is an arbitrary open neighborhood of $e \in G$ and $\mathrm{Int}(\{w^2 \mid w \in W\})$ is a neighborhood of $e \in G$, we have that φ is continuous at $e \in G$ and therefore is continuous.

If G is a Lie group, U is an open neighborhood of $e \in G$ and $\mathfrak{g}$ is its Lie algebra, then there is an open convex neighborhood C of $0 \in \mathfrak{g}$ such that the restriction of the exponential mapping $\exp : \mathfrak{g} \to G$ to C is a homeomorphism onto an open neighborhood $V \subset U$ of $e \in G$ ([6, Proposition 1.6, page 94]). If $v \in V$ and $v = \exp(X)$ for $X \in C$, then $X/2 \in C$, $\exp(X/2) \in V$ and $v = \exp(X/2)^2$ and thus every element of V is a square. □

Proposition 4.3

If G is either $\mathcal{L}_s$ or $\mathcal{L}_e$, then G is not an algebraically determined Polish group.

Proof. In order to show that a Polish group is not algebraically determined it suffices to show that it has a discontinuous automorphism. $\mathcal{L}_e$ is a connected Lie group with $SL(2, \mathbb{C})$ as its simply connected covering group. If ψ is a discontinuous automorphism of $\mathbb{C}$, then von Neumann pointed out that the mapping $\varphi : SL(2, \mathbb{C}) \to SL(2, \mathbb{C})$ given by $\varphi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} \psi(a) & \psi(b) \\ \psi(c) & \psi(d) \end{bmatrix}$ is a discontinuous automorphism of $SL(2, \mathbb{C})$. Since $\mathrm{Center}(SL(2, \mathbb{C})) = \{\pm I\}$, φ induces an automorphism $\tilde{\varphi}$ of $\mathcal{L}_e = SL(2, \mathbb{C})/\{\pm I\}$ which must be discontinuous by Lemma 4.2.

$\tilde{\varphi}$ may be extended to an automorphism of $\mathcal{L}_s$ by defining $\tilde{\varphi}(L) = -\tilde{\varphi}(-L)$ if $L \in \mathcal{L}_s \setminus \mathcal{L}_e$. This extended automorphism is not continuous on $\mathcal{L}_s$ since it is not continuous on the open subgroup $\mathcal{L}_e$. □

5. The Heisenberg Group

The purpose of this section is to point out that the Heisenberg group is not an algebraically determined Polish group but that it does possess partial automatic continuity.

Proposition 5.1

If G is the Heisenberg group, then G is not an algebraically determined Polish group. However, if H is a Polish group, G is the Heisenberg group and $\varphi : H \to G$ is an *algebraic isomorphism, then* $\varphi|\text{Center}(H) : \text{Center}(H) \to \text{Center}(G)$ *is a topological isomorphism and the induced quotient isomorphism* $\tilde{\varphi} : H/\text{Center}(H) \to G/\text{Center}(G)$ *is a topological isomorphism of Polish groups.*

Proof. Recall that if G is the Heisenberg group, then G consists of all 3×3 strictly upper triangular real matrices with 1's along the diagonal, so G is a Polish group in its natural topology. The connection between G and the canonical commutation relations is well known. A typical element of G is of the form $g(a,b,c) = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$, where a, b, $c \in \mathbb{R}$, we have the multiplication $g(a,b,c)g(e,f,g) = g(a+e, ag+b+f, c+g)$, the identity $g(0,0,0)$ and $g(a,b,c,)^{-1} = g(-a, ac{-}b, -c)$. If ψ is any discontinuous additive mapping of $\mathbb{R} \to \mathbb{R}$, then check that $g(a,b,c) \mapsto g(a, b+\psi(a), c)$ is a discontinuous automorphism of G. Hence, G is not an algebraically determined Polish group.

Suppose that H is a Polish group and $\varphi : H \to G$ is an algebraic isomorphism. It is simple to check that $\text{Center}(G) = \{g(0,b,0) \mid b \in \mathbb{R}\}$ and that $A_1 = \{g(a,b,0) \mid a,\, b \in \mathbb{R}\}$ is a maximal abelian subgroup of G. Therefore $\text{Center}(H) = \varphi^{-1}(\text{Center}(G))$ and $\varphi^{-1}(A_1)$ are closed subgroups of H and therefore Polish groups since the center and any maximal abelian subgroup of a Hausdorff topological group are closed. Notice that

$$g(a,b,c)g(1,0,\lambda) = g(1+a, \lambda a+b, \lambda+c) \text{ and } g(1,0,\lambda)g(a,b,c) = g(a+1, b+c, c+\lambda).$$

So for fixed $\lambda \in \mathbb{R}$,

$$\text{Centralizer}_G\big(g(1,0,\lambda)\big) = \big\{g(a,b,\lambda a) \mid a,\, b \in \mathbb{R}\big\}.$$

Therefore both

$$\varphi^{-1}\big(\{g(a,b,a) \mid a,\, b \in \mathbb{R}\}\big) \text{ and } \varphi^{-1}\big(\{g(a',b',-a') \mid a',\, b' \in \mathbb{R}\}\big)$$

are closed subgroups of H since

$$\varphi^{-1}\big(\text{Centralizer}_G(g(1,0,\lambda))\big) = \text{Centralizer}_H\big(\varphi^{-1}(g(1,0,\lambda))\big)$$

is a closed subgroup of H. Hence, the set of pairs in

$$\varphi^{-1}\big(\text{Centralizer}(g(1,0,1))\big) \times \varphi^{-1}\big(\text{Centralizer}(g(1,0,-1))\big)$$

whose product lies in $\varphi^{-1}(A_1)$ is a closed subset of $H \times H$ since multiplication is continuous. This set is

$$\big\{(\varphi^{-1}(g(a,b,a)), \varphi^{-1}(g(a,b',-a)) \mid a,\, b \text{ and } b' \in \mathbb{R}\big\}.$$

The set of commutators

$$\varphi^{-1}\big(g(a,b,a)\big)\varphi^{-1}\big(g(a,b',-a)\big)\varphi^{-1}\big(g(a,b,a)\big)^{-1}\varphi^{-1}\big(g(a,b',-a)\big)^{-1}$$

of such pairs, viz., $\{\varphi^{-1}(g(0,-2a^2,0)) \mid a \in \mathbb{R}\}$, is therefore an analytic set in H. By taking inverses and reparameterizing we see that $\{\varphi^{-1}(g(0,x^2,0)) \mid x \in \mathbb{R}\}$, is an analytic subset of H. From this it follows that

$$\{\varphi^{-1}(g(0,b+x^2,0)) \mid x \in \mathbb{R}\} = \varphi^{-1}(g(0,b,0)) \cdot \{\varphi^{-1}(g(0,x^2,0)) \mid x \in \mathbb{R}\},$$

for fixed $b \in \mathbb{R}$, is an analytic subset of Center$(H) \subset H$. Since sets of the form

$$\{y \in \mathbb{R} \mid y \geq b\} = \{y \mid y = b + x^2,\, x \in \mathbb{R}\}$$

generate the Borel subsets of $\mathbb{R}$, we have that $\{\varphi^{-1}(g(0,y,0)) \mid y \in B\}$, where $B \subset \mathbb{R}$ is a Borel set, lies in the σ-algebra generated by the analytic subsets of H. Hence, $\varphi|\text{Center}(H) : \text{Center}(H) \to \text{Center}(G)$ is measurable with respect to the sets with the Baire property and therefore is a topological isomorphism by Proposition 2.3.

Next, the mapping $H \to \mathbb{R}$, $\varphi^{-1}(g(a,b,c)) \to a$ is continuous since it is the composition of the continuous mappings

$$\begin{aligned}\varphi^{-1}\big(g(a,b,c)\big) &\to \varphi^{-1}\big(g(a,b,c)\big)\varphi^{-1}\big(g(0,0,1)\big)\varphi^{-1}\big(g(a,b,c)\big)^{-1}\varphi^{-1}\big(g(0,0,1)\big)^{-1}\\ &= \varphi^{-1}\big(g(0,a,0)\big) \to g(0,a,0) \to a.\end{aligned}$$

Similarly, the mapping $H \to \mathbb{R}$, $\varphi^{-1}(g(a,b,c)) \to c$ is continuous since it is the composition of the continuous mappings

$$\begin{aligned}\varphi^{-1}\big(g(a,b,c)\big) &\to \varphi^{-1}\big(g(a,b,c)\big)\varphi^{-1}\big(g(-1,0,0)\big)\varphi^{-1}\big(g(a,b,c)\big)^{-1}\varphi^{-1}\big(g(-1,0,0)\big)^{-1}\\ &= \varphi^{-1}\big(g(0,c,0)\big) \to g(0,c,0) \to c.\end{aligned}$$

Let $\pi : G \to G/\text{Center}(G)$ be the natural quotient mapping. The natural mapping

$$H \to \mathbb{R}^2 \to G \to G/\text{Center}(G),\; \varphi^{-1}(g(a,b,c)) \to (a,c) \to g(a,0,c) \to \pi(g(a,0,c))$$

is therefore a continuous homomorphism with kernel Center(H). $H/\text{Center}(H)$ is a Polish group in the quotient topology by Proposition 2.4 and the natural induced mapping $\tilde{\varphi} : H/\text{Center}(H) \to G/\text{Center}(G)$ is a continuous algebraic isomorphism and therefore a topological isomorphism by Proposition 2.3. □

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