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A polynomial characterization of Hilbert spaces

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Abstract

In this paper, we obtain a new characterization of Hilbert spaces by means of polynomial mappings, extending the linear result of Kwapień.

Introduction

Let X be a Banach space. Kwapień [9] has shown that: X is isomorphic to a Hilbert space if, and only if, for every Banach spaces Y and every absolutely 2-summing operators u from X into Y, the conjugate operator u^* is absolutely 2-summing, i.e., u is strongly 2-summing in the sense of Cohen [4]. This theorem of Kwapień is a response to a question posed by J.S. Cohen [5] who had previously established an isometric version. In this note, we will give a polynomial version of this characterization of Hilbert space. For this, we will introduce and study Cohen strongly summing polynomials, extending the definition given by Cohen for linear operators and by Achour and Mezrag [1] for multilinear mappings. We show, for instance, that a polynomial is Cohen strongly p-summing if, and only if, its associated symmetric multilinear mapping is Cohen strongly p-summing, if and only if, its linearization is strongly p-summing linear operator. As consequence, certain inclusion theorems are given.

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This paper is organized as follows.

In Section 1, we recall some facts about polynomial mappings. In Section 2, we study the space of Cohen strongly p-summing polynomials between Banach spaces. We give a natural analog of Pietsch Domination Theorem similar to the m-linear case. In Section 3, we discuss the relationships between the classes of Cohen strongly p-summing polynomials, multilinear and linear operators. As consequence, we compare this notion with strongly p-summing polynomials (in the sense of Dimant), where the rang space is an $\mathcal{L}_{p^*}$-space. We end this section by comparing Cohen strongly p-summing and p-dominated polynomials when the domain is a Hilbert space.

In Section 4, we give our main result, that is: the Banach space X is isomorphic to a Hilbert space if, and only if, for all Banach spaces Y and for every 2-dominated polynomials P from X into Y, P is Cohen strongy 2-summing polynomial.

1. Definitions and general results

The notation and terminology used in this paper are standard in Banach space theory. However, we shall recall some terminology: Let X be a Banach space and $1 \leq p \leq \infty$, p^* is the conjugate of p, i.e., $\frac{1}{p}+\frac{1}{p^*}=1$. We denote by $l_p^n(X)$ the space of all sequences $\left(x_i\right)_{i=1}^n$ in X equipped with the norm

$$\left\| (x_i)_{1\leq i\leq n} \right\|_{l_p^n(X)} = \Big(\sum_{i=1}^{n} \|x_i\|^p \Big)^{1/p}$$

and by $l_p^{n\ \omega}(X)$ the space of all sequences $(x_i)_{i=1}^n$ in X equipped with the norm

$$\left\| (x_n) \right\|_{l_p^{n\ \omega}(X)} = \sup_{\|x^*\|_{X^*}=1} \Big(\sum_{i=1}^{n} |\langle x_i, x^* \rangle|^p \Big)^{1/p},$$

where X^* denotes the topological dual of X. The closed unit ball of X will be denoted by B_X. The vector space of bounded linear operators from X to Y will be noted by $\mathcal{B}(X,Y)$.

A map $P : X \to Y$ is an m-homogeneous polynomial if there exists a unique symmetric m-linear operator $\hat{P} : X \times ... \times X \to Y$ such that $P(x) = \hat{P}(x, \overset{(m)}{\dots}, x)$ for every $x \in X$. Both are related by the polarization formula [11, Theorem 1.10]

$$\hat{P}\left(x^1, ..., x^m\right) = \frac{1}{m!2^m} \sum_{\substack{\epsilon_i=\pm 1 \\ 1\leq i\leq m}} \epsilon_1 ... \epsilon_m P\Big(\sum_{j=1}^{m} \epsilon_j x^j \Big). \tag{1.1}$$

P is bounded on the unit ball of X if and only if $\hat{P}$ is bounded on the unit ball of $X \times ... \times X$. The norms are related by the inequalities [11, Theorem 1.10]

$$\|P\| \leq \left\| \hat{P} \right\| \leq \frac{m^m}{m!} \|P\|. \tag{1.2}$$

We denote by $\mathcal{P}\left({}^{m}X,Y\right)$, the Banach space of all continuous m-homogeneous polynomials from X into Y endowed with the norm

$$\|P\| = \sup\left\{\, \|P(x)\| : \|x\| \le 1\right\} = \inf\left\{C : \|P(x)\| \le C\,\|x\|^{m},\, x \in X\right\}. \tag{1.3}$$

If $Y = \mathbb{K}$, we write simply $\mathcal{P}\left({}^{m}X\right)$. For the general theory of polynomials on Banach spaces, we refer to [7] and [11]. By $X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m$ we denote the completed projective tensor product of $X_1, ..., X_m$. If $X = X_1 = ... = X_m$ we write $\widehat{\otimes}_\pi^m X$. By $\otimes_s^m X := X \otimes_s \overset{(m)}{\cdots} \otimes_s X$ we denote the m fold symmetric tensor product of X, that is, the set of all elements $u \in \otimes^m X$ of the form

$$u = \sum_{i=1}^{n} x_i \otimes \overset{(m)}{\cdots} \otimes x_i,\ \ (n \in \mathbb{N}, x_i \in X, 1 \le i \le n).$$

By $\widehat{\otimes}_{\pi,s}^m X$ we denote the closure of $\otimes_s^m X$ in $\widehat{\otimes}_\pi^m X$. For symmetric tensor products, we refer to [8]. If $P \in \mathcal{P}\left({}^{m}X,Y\right)$ we define its linearization $\widetilde{P} : \otimes_s^m X \to Y$ by

$$\widetilde{P}\Big(\sum_{i=1}^{n} x_i \otimes \overset{(m)}{\cdots} \otimes x_i\Big) = \sum_{i=1}^{n} P(x_i)$$

for all $x_i \in X, 1 \le i \le n$. Consider the canonical polynomial $\delta_m : X \to \widehat{\otimes}_{\pi,s}^m X$ define by

$$\delta_m(x) = x \otimes \overset{(m)}{\cdots} \otimes x.$$

We have the next diagram which is commute

$$\begin{array}{ccc} X & \overset{P}{\longrightarrow} & Y \\ & \searrow \delta_m & \uparrow \widetilde{P} \\ & & \widehat{\otimes}_{\pi,s}^m X \end{array}$$

in the other words $P = \widetilde{P} \circ \delta_m$.

A Banach space X is an $\mathcal{L}_{p,\lambda}$-space if every finite dimensional subspace $E \subset X$ is contained in a finite dimensional subspace $F \subset X$ for which there is an isomorphism $u : F \longrightarrow l_p^{\dim F}$ with $\|u\|\left\|u^{-1}\right\| < \lambda$. X is an $\mathcal{L}_p$-space if it is an $\mathcal{L}_{p,\lambda}$-space for some $\lambda > 1$. The space $L_p(\Omega,\mu)$ is an $\mathcal{L}_p$-space for all $\lambda > 1$ $(1 \le p < \infty)$ and every $\mathcal{L}_{p,\lambda}$-space is isomorphic to a subspace of some $L_p(\Omega,\mu)$. If K is a compact Hausdorff space, then $C(K)$ is an $\mathcal{L}_{\infty,\lambda}$-space for every $\lambda > 1$.

The definition of strongly p-summing polynomial was introduced by V. Dimant in [6].

Definition 1.1 Let $1 \le p \le \infty$ and $P \in \mathcal{P}\left({}^{m}X,Y\right)$. The polynomial P is strongly p-summing if there exists a constant $C \ge 0$ such that for, every $x_1, ..., x_n \in X$,

$$\Big(\sum_{i=1}^{n} \|P(x_i)\|^p\Big)^{1/p} \le C \sup_{\Phi \in B_{\mathcal{P}({}^{m}X)}} \Big(\sum_{i=1}^{n} |\Phi(x_i)|^p\Big)^{1/p}. \tag{1.4}$$

The class of strongly p-summing m-homogeneous polynomials from X into Y, which is denoted by $\mathcal{P}_{ss}^{p}({}^{m}X,Y)$ is a Banach space for the norm $\|P\|_{ss,p}$, i.e. the smallest constant C such that inequality (1.4) holds.

We also recall the definition of p-dominated plynomials.

DEFINITION 1.2 [10] Given $1 \leq p < \infty$, a polynomial $P \in \mathcal{P}(^m X, Y)$ is p-dominated if there exists a constant $C > 0$ such that, for all $n \in \mathbb{N}$ and for every $x_1, ..., x_n \in X$,

$$\Big(\sum_{i=1}^{n} \|P(x_i)\|^{p/m} \Big)^{m/p} \leq C \sup_{x^* \in B_{X^*}} \Big(\sum_{i=1}^{n} |x^*(x_i)|^p \Big)^{m/p}.$$

We denote by $\mathcal{P}_d^p(^m X, Y)$ the space of p-dominated polynomials $P : X \to Y$ and by $\delta_p(P)$ the infimum of all C verifying the above inequality. For $p \geq m$, $\delta_p(P)$ is a norm on $\mathcal{P}_d^p(^m X, Y)$, but for $p < m$ it is only a quasinorm. Such polynomials are sometimes called absolutely $(p/m, p)$-summing polynomials.

2. Cohen strongly p-summing m-homogeneous polynomials

We give in this section a natural generalization of strongly p-summing linear operators. We extend "*Pietsch Domination Theorem*" to the category of polynomials mappings. In the linear case, Cohen easily got it by duality because the adjoint of a strongly p-summing operator is absolutely p^*-summing. In the multilinear case, the proof in [1] is direct induces applying Ky Fan's lemma.

Before giving the definition for polynomials mappings, we start by recalling the definition of Cohen strongly p-summing multilinear operators which is introduced as an extension of strongly p-summing operators.

DEFINITION 2.1 [1] Let $1 \leq p \leq \infty$. An m-linear operator $T : X_1 \times ... \times X_m \longrightarrow Y$ (X_j, Y are arbitrary Banach spaces and $m \in \mathbb{N}$) is Cohen strongly p-summing if and only if there is a constant $C > 0$ such that for any $x_1^j, ..., x_n^j \in X_j$, $(j = 1, ..., m)$ and any $y_1^*, ..., y_n^* \in Y^*$,

$$\sum_{i=1}^{n} \left| \left\langle T\left(x_i^1, ..., x_i^m\right), y_i^* \right\rangle \right| \leq C \Big(\sum_{i=1}^{n} \prod_{j=1}^{m} \left\| x_i^j \right\|_{X_j}^p \Big)^{1/p} \sup_{y \in B_Y} \|(y_i^*(y))\|_{l_{p^*}^n}. \tag{2.1}$$

The class of Cohen strongly p-summing m-linear operators from $X_1 \times ... \times X_m$ into Y, which is denoted by $\mathcal{D}_p^m(X_1, ..., X_m; Y)$, is a Banach space for the norm $d_p^m(T)$, i.e. the smallest constant C such that inequality (2.1) holds.

DEFINITION 2.2 Fix $m \in \mathbb{N}$. Let $1 < p \leq \infty$ and let X, Y be Banach spaces. An m-homogeneous polynomial $P : X \longrightarrow Y$ is Cohen strongly p-summing, if there is a constant $C > 0$ such that for any $x_1, ..., x_n \in X$ and $y_1^*, ..., y_n^* \in Y^*$,

$$\sum_{i=1}^{n} |\langle P(x_i), y_i^* \rangle| \leq C \Big(\sum_{i=1}^{n} \|x_i\|^{mp} \Big)^{1/p} \sup_{y \in B_Y} \left\| (y_i^*(y)) \right\|_{l_{p^*}^n}. \tag{2.2}$$

The class of such polynomials is denoted by $\mathcal{P}_{Coh}^p(^m X, Y)$; it is equipped with the norm $d_p(P)$, i.e. the smallest constant C such that inequality (2.2) holds. For $p = 1$ we have $\mathcal{P}_{Coh}^1(^m X, Y) = \mathcal{P}(^m X, Y)$.

Let us first give an example of a Cohen strongly p-summing polynomial. Let $m \in \mathbb{N}$, $1 \leq p \leq \infty$ and $u : X \to Y$ be a Cohen strongly p-summing linear operator and $\varphi \in X^*$. The polynomial

$$P : X \to Y : P(x) = \varphi^{m-1}(x)\, u(x)$$

is Cohen strongly p-summing. Indeed, for $x_1, ..., x_n \in X$, $y_1^*, ..., y_n^* \in Y^*$,

$$\begin{aligned}\sum_{i=1}^{n}\left|\langle P\left(x_i\right),y_i^*\rangle\right| &= \sum_{i=1}^{n}\left|\left\langle \varphi^{m-1}\left(x_i\right)u\left(x_i\right),y_i^*\right\rangle\right| \\ &= \sum_{i=1}^{n}\left|\left\langle u\left(\varphi^{m-1}\left(x_i\right)x_i\right),y_i^*\right\rangle\right| \\ &\leq d_p(u)\left\|\varphi\right\|^{m-1}\Big(\sum_{i=1}^{n}\left\|x_i\right\|^{mp}\Big)^{1/p}\sup_{y\in B_Y}\Big(\sum_{i=1}^{n}\left|y_i^*(y)\right|^{p^*}\Big)^{1/p^*}.\end{aligned}$$

So, P is Cohen strongly p-summing and $d_p(P) \leq \|\varphi\|^{m-1} d_p(u)$.

The polynomial version of "*Pietsch Domination Theorem*" goes as follows. Its proof is an adaptation of the proof for the multilinear case (see [1]).

Theorem 2.3

Let $m \in \mathbb{N}$. An m-homogeneous polynomial $P \in \mathcal{P}(^mX,Y)$ is Cohen strongly p-summing $(1 < p \leq \infty)$ if and only if there is a Radon probability measure μ on $B_{Y^{**}}$ and $C > 0$ such that, for all$x \in X$ and $y^* \in Y^*$

$$\left|\langle P\left(x\right),y^*\rangle\right| \leq C\left\|x\right\|^m\Big(\int_{B_{Y^{**}}}\left|y^*(y^{**})\right|^{p^*}d\mu(y^{**})\Big)^{1/p^*}. \tag{2.3}$$

Moreover, in this case $d_p(P) = \min\{C : C$ verifies $(2.3)\}$.

An immediate consequence of Theorem 2.3 is the following.

Corollary 2.4

Let $1 \leq p_1 \leq p_2 < \infty$. If $P \in \mathcal{P}^{p_2}_{Coh}(^mX,Y)$ then $P \in \mathcal{P}^{p_1}_{Coh}(^mX,Y)$ and $d_{p_1}(P) \leq d_{p_2}(P)$.

3. Characterization and inclusion theorems

In this section, we investigate connections between the class of Cohen strongly summing polynomials and other classes of polynomials mappings, such as p-dominated and strongly summing polynomials (in the sense of Dimant). First, we give the relation between P and its associated symmetric m-linear operator $\hat{P}$ concerning the notion of Cohen strongly p-summing. A similar characterization holds for p-dominated polynomials (see [10]).

Theorem 3.1

The polynomial $P \in \mathcal{P}(^mX,Y)$ is Cohen strongly p-summing if, and only if, its associated symmetric m-linear operator $\hat{P} \in \mathcal{L}(^mX,Y)$ is Cohen strongly p-summing.

Proof. Let us first assume that $\hat{P}$ is Cohen strongly p-summing. Let $x_1,...,x_n$ be in X and $y_1^*,...,y_n^*$ be in Y^*; then

$$\begin{aligned}\sum_{i=1}^{n}\left|\left\langle P\left(x_i\right),y_i^*\right\rangle\right| &= \sum_{i=1}^{n}\left|\left\langle \hat{P}\left(x_i,...,x_i\right),y_i^*\right\rangle\right| \\ &\leq d_p^m(\hat{P})\Big(\sum_{i=1}^{n}\left\|x_i\right\|^{mp}\Big)^{1/p}\sup_{y\in B_Y}\left\|\left(y_i^*(y)\right)\right\|_{l_{p^*}^n}.\end{aligned}$$

Hence, P is Cohen strongly p-summing and $d_p(P) \leq d_p^m(\hat{P})$.

Conversely, let $P \in \mathcal{P}^p_{Coh}(^m X, Y)$. Let $x^j \in X$ such that $\left\| x^j \right\| \leq 1$ $(1 \leq j \leq m)$ and $y^* \in Y^*$. Using the polarization formula (1.1), we obtain

$$\begin{aligned}
\left| \left\langle \hat{P}\left(x^1, ..., x^m\right), y^* \right\rangle \right| &= \Big| < \frac{1}{m!2^m} \sum_{\epsilon_1,...,\epsilon_m = \pm 1} \epsilon_1 ... \epsilon_m P\Big(\sum_{j=1}^m \epsilon_j x^j\Big), y^* > \Big| \\
&\leq \frac{1}{m!2^m} \sum_{\epsilon_1,...,\epsilon_m = \pm 1} \Big| < P\Big(\sum_{j=1}^m \epsilon_j x^j\Big), y^* > \Big| \\
&\leq \frac{1}{m!2^m} \sum_{\epsilon_1,...,\epsilon_m = \pm 1} d_p(P) \Big\| \sum_{j=1}^m \epsilon_j x^j \Big\|^m \\
&\quad \times \Big(\int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} \, d\mu(y^{**}) \Big)^{1/p^*} \\
&\leq \frac{1}{m!2^m} d_p(P) \sum_{\epsilon_1,...,\epsilon_m = \pm 1} \Big(\sum_{j=1}^m \left\| x^j \right\| \Big)^m \\
&\quad \times \Big(\int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} \, d\mu(y^{**}) \Big)^{1/p^*} \\
&\leq \frac{1}{m!2^m} d_p(P) 2^m m^m \Big(\int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} \, d\mu(y^{**}) \Big)^{1/p^*} \\
&\leq \frac{m^m}{m!} d_p(P) \Big(\int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} \, d\mu(y^{**}) \Big)^{1/p^*}.
\end{aligned}$$

So, for every $x^j \in B_X$ $(1 \leq j \leq m)$, we have

$$\left| \left\langle \hat{P}\left(x^1, ..., x^m\right), y^* \right\rangle \right| \leq \frac{m^m}{m!} d_p(P) \Big(\int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} \, d\mu(y^{**}) \Big)^{1/p^*}$$

and for $x^j \in X$ $(x^j \neq 0)$,

$$\left| \left\langle \hat{P}\left(\frac{x^1}{\|x^1\|}, ..., \frac{x^m}{\|x^m\|}\right), y^* \right\rangle \right| \leq \frac{m^m}{m!} d_p(P) \Big(\int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} \, d\mu(y^{**}) \Big)^{1/p^*}.$$

Thus

$$\left| \left\langle \hat{P}\left(x^1, ..., x^m\right), y^* \right\rangle \right| \leq \frac{m^m}{m!} d_p(P) \prod_{j=1}^m \left\| x^j \right\| \Big(\int_{B_{Y^{**}}} |y^*(y^{**})|^{p^*} \, d\mu(y^{**}) \Big)^{1/p^*}.$$

Therefore, by [1, Theorem 2.4], $\hat{P}$ is Cohen strongly p-summing and $d^m_p\big(\hat{P}\big) \leq \frac{m^m}{m!} d_p(P)$. $\square$

The following characterization of Cohen strongly p-summing polynomials is useful and will be used later.

Proposition 3.2

Let $1 < p \leq \infty$. Let $P : X \to Y$ be a m-homogeneous polynomial and $\widetilde{P}$ its linearization. The following properties are equivalent

(a) *The polynomial P belongs to $\mathcal{P}_{Coh}^{p}(^{m}X,Y)$.*
(b) *The operator $\widetilde{P}$ belongs to $\mathcal{D}_p\left(\widehat{\otimes}_{\pi,s}^{m}X,Y\right)$.*

Proof. Suppose that $\widetilde{P}\in\mathcal{D}_p\left(\widehat{\otimes}_{\pi,s}^{m}X,Y\right)$. For $x_1,...,x_n\in X$ and $y_1^*,...,y_n^*\in Y^*$, we have

$$\begin{aligned}\sum_{i=1}^{n}\left|\left\langle P\left(x_i\right),y_i^*\right\rangle\right| &= \sum_{i=1}^{n}\left|\left\langle \widetilde{P}\left(x_i\otimes...\otimes x_i\right),y_i^*\right\rangle\right|\\ &\leq d_p\left(\widetilde{P}\right)\Big(\sum_{i=1}^{n}\|x_i\otimes...\otimes x_i\|^p\Big)^{1/p}\sup_{y\in B_Y}\|(y_i^*(y))\|_{l_{p^*}^n}\\ &\leq d_p\left(\widetilde{P}\right)\Big(\sum_{i=1}^{n}\|x_i\|^{mp}\Big)^{1/p}\sup_{y\in B_Y}\|(y_i^*(y))\|_{l_{p^*}^n}.\end{aligned}$$

Conversely, suppose that P is Cohen strongly p-summing. Let $v\in\widehat{\otimes}_{\pi,s}^{m}X$ such that $v\neq 0$ and $y^*\in Y^*$. Suppose that $v=\sum_{i=1}^{n}x_i\otimes...\otimes x_i$. Then

$$\begin{aligned}\left|\left\langle \widetilde{P}\left(v\right),y^*\right\rangle\right| &\leq \sum_{i=1}^{n}\left|\left\langle P\left(x_i\right),y^*\right\rangle\right|\\ &\leq \sum_{i=1}^{n}d_p^m\left(P\right)\|x_i\|^m\Big(\int_{B_{Y^{**}}}|y^{**}(y^*)|^{p^*}\,d\mu(y^{**})\Big)^{1/p^*}\\ &= d_p^m\left(P\right)\Big(\sum_{i=1}^{n}\|x_i\|^m\Big)\Big(\int_{B_{Y^{**}}}|y^{**}(y^*)|^{p^*}\,d\mu(y^{**})\Big)^{1/p^*}.\end{aligned}$$

Taking the infimum over all represents of v we get

$$\left|\left\langle \widetilde{P}\left(v\right),y^*\right\rangle\right|\leq d_p^m\left(P\right)\|v\|\Big(\int_{B_{Y^{**}}}|y^{**}(y^*)|^{p^*}\,d\mu(y^{**})\Big)^{1/p^*}.$$

Therefore, by [4, Theorem 2.3.1], $\widetilde{P}$ is Cohen strongly p-summing and $d_p^m\left(P\right)=d_p\left(\widetilde{P}\right)$. □

Corollary 3.3

The following are equivalent for $P\in\mathcal{P}(^{m}X,Y)$.

(1) *The polynomial P is Cohen strongly p-summing.*
(2) *The operator $\widetilde{P}$ is Cohen strongly p-summing from $\widehat{\otimes}_{\pi,s}^{m}X$ into Y.*
(3) *There exist a Cohen strongly p-summing operator u and a polynomial Q such that $P=u\circ Q$.*

Proof. (1) $\Leftrightarrow$ (2) Proposition 3.2.

(2) $\Rightarrow$ (3) We have the result directly from the factorization $P=\widetilde{P}\circ\delta_m$.

(3) $\Rightarrow$ (1) There is a Banach space Z, a linear operator u in $\mathcal{D}_p\left(Z,Y\right)$ and a polynomial Q in $\mathcal{P}(^{m}X,Z)$ such that $P=u\circ Q$. For $x_1,...,x_n\in X$ and $y_1^*,...,y_n^*\in Y^*$, we have

$$\begin{aligned}\sum_{i=1}^{n}\left|\left\langle P\left(x_i\right),y_i^*\right\rangle\right| &= \sum_{i=1}^{n}\left|\left\langle u\circ Q\left(x_i\right),y_i^*\right\rangle\right| \\ &\leq d_p\left(u\right)\Big(\sum_{i=1}^{n}\left\|Q\left(x_i\right)\right\|^p\Big)^{1/p}\sup_{y\in B_Y}\left\|\left(y_i^*(y)\right)\right\|_{l_{p^*}^n} \\ &\leq d_p\left(u\right)\left\|Q\right\|\Big(\sum_{i=1}^{n}\left\|x_i\right\|^{mp}\Big)^{1/p}\sup_{y\in B_Y}\left\|\left(y_i^*(y)\right)\right\|_{l_{p^*}^n}.\end{aligned}$$

Hence, P is Cohen strongly p-summing. □

If Y is an $\mathcal{L}_{p^*}$-space, we have the following inclusion.

Corollary 3.4

Let $1 < p < \infty$. If Y is an $\mathcal{L}_{p^*}$-space then

$$\mathcal{P}_{Coh}^{p}({}^{m}X,Y)\subset\mathcal{P}_{ss}^{p^*}({}^{m}X,Y).$$

Proof. Let $P\in\mathcal{P}_{Coh}^{p}({}^{m}X,Y)$. By Corollary 3.3, there exist a Cohen strongly p-summing operator $u: Z\to Y$ and a polynomial $Q: X\to Z$ such that $P = u\circ Q$. Since Y is an $\mathcal{L}_{p^*}$-space the operator $u: Z\to Y$ is p^*-summing by [4, Theorem 3.2.3]. Now, for $x_i\in X$ $(1\leq i\leq n)$

$$\begin{aligned}\Big(\sum_{i=1}^{n}\left\|P\left(x_i\right)\right\|^{p^*}\Big)^{1/p^*} &= \Big(\sum_{i=1}^{n}\left\|u\circ Q\left(x_i\right)\right\|^{p^*}\Big)^{1/p^*} \\ &\leq \pi_{p^*}\left(u\right)\sup_{z^*\in B_{Z^*}}\Big(\sum_{i=1}^{n}\left|z^*\left(Q\left(x_i\right)\right)\right|^{p^*}\Big)^{1/p^*}\end{aligned}$$

since the polynomial $z^*\circ Q$ belongs to $\mathcal{P}\left({}^{m}X\right)$ we have

$$\Big(\sum_{i=1}^{n}\left\|P\left(x_i\right)\right\|^{p^*}\Big)^{1/p^*}\leq\frac{\pi_{p^*}\left(u\right)}{\left\|Q\right\|}\sup_{\Phi\in B_{\mathcal{P}({}^{m}X)}}\Big(\sum_{i=1}^{n}\left|\Phi\left(x_i\right)\right|^{p^*}\Big)^{1/p^*}$$

i.e., P is strongly p^*-summing polynomial. □

Next, we give an inclusion between the class of p-dominated and Cohen strongly q-summing polynomials. The linear version of this inclusion is due to Cohen [4] for $p = q = 2$ and to Bu [3] for all p and q. The reader can see [1] for more details about the multilinear version.

Corollary 3.5

Let $m\in\mathbb{N}$ and $1 < p,q < \infty$. Let H be a Hilbert space and Y be a Banach space. Let $P\in\mathcal{P}\left({}^{m}H,Y\right)$. If P is p-dominated polynomial, then P is Cohen strongly q-summing polynomial.

Proof. Fix $1 < p,q < \infty$. Let $P\in\mathcal{P}\left({}^{m}H,Y\right)$ and $\hat{P}\in\mathcal{L}\left({}^{m}H,Y\right)$ the associated symmetric m-linear. Assume that P is p-dominated polynomial. By [10, Theorem 6], $\hat{P}$ is p-dominated and by [1, Theorem 3.2], $\hat{P}$ is a Cohen strongly q-summing m-linear operator. So, Theorem 3.1. concludes the proof. □

4. Main result

Given $m \in \mathbb{N}$ and X be a Banach space. In this section, we show that: X is isomorphic to a Hilbert space if, and only if, for every Banach space Y and every m-homogeneous 2-dominated polynomial from X into Y, u is Cohen strongly 2-summing. We start by a preparatory result.

Theorem 4.1

Let $m \in \mathbb{N}, 1 < p, q < \infty$ and X, Y be Banach spaces such that $\mathcal{P}_d^p(^m X, Y) \subseteq \mathcal{P}_{Coh}^q(^m X, Y)$. Then $\Pi_p(X,Y) \subseteq \mathcal{D}_q(X,Y)$.

Proof. Let $u \in \Pi_p(X,Y)$, we will show that $u \in \mathcal{D}_q(X,Y)$. Fix $x_0 \in B_X$ and $x_0^* \in B_{X^*}$ such that $x_0^*(x_0) = 1$. Define the operator $\pi_j : \widehat{\otimes}_{\pi,s}^{j+1} X \to \widehat{\otimes}_{\pi,s}^{j} X$ $(1 \leq j \leq m-1)$ by

$$\pi_j\Big(\sum_{i=1}^n x_i \otimes \overset{(j+1)}{\cdots} \otimes x_i\Big) = \sum_{i=1}^n x_0^*(x_i) x_i \otimes \overset{(j)}{\cdots} \otimes x_i.$$

Let $\delta_m : X \to \widehat{\otimes}_{\pi,s}^m X$ be the canonical polynomial. We show that the polynomial $P := u \circ \pi_1 \circ ... \circ \pi_{m-1} \circ \delta_m : X \to Y$ is p-dominated. We reason by induction on m. For $m = 1$, the statement is trivial. We suppose now that

$$u \circ \pi_1 \circ ... \circ \pi_{m-2} \circ \delta_{m-1} : X \to Y$$

is p-dominated. Let $x_i \in X$ $(1 \leq i \leq n)$. Then

$$\begin{aligned}\sum_{i=1}^n \|P(x_i)\|^{p/m} &= \sum_{i=1}^n \|u \circ \pi_1 \circ ... \circ \pi_{m-1} \circ \delta_m(x_i)\|^{p/m} \\ &= \sum_{i=1}^n \big\|u \circ \pi_1 \circ ... \circ \pi_{m-1}\big(x_i \otimes \overset{(m)}{\cdots} \otimes x_i\big)\big\|^{p/m} \\ &= \sum_{i=1}^n |x_0^*(x_i)|^{p/m} \big\|u \circ \pi_1 \circ ... \circ \pi_{m-2}\big(x_i \otimes \overset{(m-1)}{\cdots} \otimes x_i\big)\big\|^{p/m}.\end{aligned}$$

By Hölder's inequality

$$\begin{aligned}&\sum_{i=1}^n \|P(x_i)\|^{p/m} \\ &\leq \Big(\sum_{i=1}^n |x_0^*(x_i)|^p\Big)^{1/m} \Big(\sum_{i=1}^n \|u \circ \pi_1 \circ ... \circ \pi_{m-2} \circ \delta_{m-1}(x_i)\|^{p/(m-1)}\Big)^{(m-1)/m}.\end{aligned}$$

We obtain by the induction hypothesis and the fact that $x_0^* \in B_{X^*}$

$$\begin{aligned}\sum_{i=1}^{n}\|P(x_i)\|^{p/m} &\leq \Big(\sum_{i=1}^{n}\big|x_0^*(x_i)\big|^p\Big)^{1/m}\Big(C\sup_{x^*\in B_{X^*}}\big(\sum_{i=1}^{n}\big|x^*(x_i\big|^p\big)^{(m-1)/p}\Big)^{p/m}\\ &= C^{p/m}\Big(\sum_{i=1}^{n}\big|x_0^*(x_i)\big|^p\Big)^{1/m}\sup_{x^*\in B_{X^*}}\Big(\sum_{i=1}^{n}\big|x^*\big(x_i\big)\big|^p\Big)^{(m-1)/m}\\ &\leq C^{p/m}\sup_{x^*\in B_{X^*}}\sum_{i=1}^{n}\big|x^*\big(x_i\big)\big|^p.\end{aligned}$$

Therefore, P is p-dominated and hence Cohen strongly q-summing. By the decomposition $P = \widetilde{P}\circ\delta_m$ we have $\widetilde{P} = u\circ\pi_1\circ\ldots\circ\pi_{m-1}$ which is Cohen strongly q-summing by the Proposition 3.2. Now, as it has been shown in the proof of [2, Theorem 3], there are operators $k_j : \widehat{\otimes}^j_{\pi,s}X \to \widehat{\otimes}^{j+1}_{\pi,s}X$ $(1\leq j\leq m-1)$ defined in terms of x_0^* and x_0 such that $\pi_j\circ k_j$ is the identity map on $\widehat{\otimes}^j_{\pi,s}X$. We have

$$u = u\circ\pi_1\circ\ldots\circ\pi_{m-1}\circ k_{j-1}\circ\ldots\circ k_1 : X\to Y$$

which is, by the ideal property, Cohen strongly q-summing. □

Theorem 4.2

Let X be a Banach space. The following properties are equivalent.

(1) *The space X is isomorphic to a Hilbert space.*
(2) *For all $m\in\mathbb{N}$, $1 < p, q < \infty$, and every Banach space Y.*

$$\mathcal{P}_d^p(^mX, Y)\subseteq\mathcal{P}_{Coh}^q(^mX, Y)$$

(3) *For all $m\in\mathbb{N}$ and every Banach space Y.*

$$\mathcal{P}_d^2(^mX, Y)\subseteq\mathcal{P}_{Coh}^2(^mX, Y)$$

Proof. (1) ⇒ (2) Immediate by Corollary 3.5.
(2) ⇒ (3) Obviously.
(3) ⇒ (1) It is enough to apply Theorem 4.1 and Kwapień's theorem. □

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References

1. D. Achour and L. Mezrag, On the Cohen strongly p-summing multilinear operators, *J. Math. Anal. Appl.* **327** (2007), 550–563.
2. F. Blasco, Complementation in spaces of symmetric tensor products and polynomials, *Studia Math.* **123** (1997), 165–173.
3. Q. Bu, Some mapping properties of p-summing operators with hilbertian domain, *Contemp. Math.* **328** (2003), 145–149.

4. J.S. Cohen, Absolutely p-summing, p-nuclear operators and their conjugates, *Math. Ann.* **201** (1973), 177–200.
5. J.S. Cohen, A characterization of inner product spaces using absolutely 2-summing operators, *Studia Math.* **38** (1970), 271–276.
6. V. Dimant, Strongly p-summing multilinear mappings, *J. Math. Anal. Appl.* **278** (2003), 182–193.
7. S. Dineen, *Complex Analysis on Infinite Dimensional Spaces,* Springer Monographs in Mathematics, Springer-Verlag, London, London, 1999.
8. K. Floret, Natural norms on symmetric tensor-products of normed spaces, *Note Mat.* **17** (1997), 153–188.
9. S. Kwapień, A linear topological characterization of inner product space, *Studia Math.* **38** (1970), 277–278.
10. Y. Meléndez and A. Tonge, Polynomials and the Pietsch domination theorem, *Math. Proc. R. Ir. Acad.* **99A** (1999), 195–212.
11. J. Mujica, *Complex Analysis in Banach Spaces, Holomorphic Functions and Domains of holomorphy in Finite and Infinite Dimensions,* North-Holland Mathematics Studies **120**, Amsterdam, 1986.