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Integral representation of linear operators on Orlicz-Bochner spaces

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Abstract

Let (Ω,Σ,μ) be a σ-finite measure space and let $\mathcal{L}(X,Y)$ stand for the space of all bounded linear operators between Banach spaces $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$. We study the problem of integral representation of linear operators from an Orlicz-Bochner space $L^\varphi(\mu,X)$ to Y with respect to operator measures $m:\Sigma\to\mathcal{L}(X,Y)$. It is shown that a linear operator $T:L^\varphi(\mu,X)\to Y$ has the integral representation $T(f)=\int_\Omega f(\omega)dm$ with respect to a φ^*-variationally μ-continuous operator measure m if and only if T is $(\gamma_\varphi,\|\cdot\|_Y)$-continuous, where γ_φ stands for a natural mixed topology on $L^\varphi(\mu,X)$. As an application, we derive Vitali-Hahn-Saks type theorems for families of operator measures.

1. Introduction

Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be real Banach spaces and let B_X stand for the closed unit ball in X. Let $\mathcal{L}(X,Y)$ stand for the space of all bounded linear operators from X to Y. We denote by $\sigma(L,K)$ and $\tau(L,K)$ the weak topology and the Mackey topology on L with respect to a dual pair $\langle L,K\rangle$. Given a Hausdorff locally convex space (L,ξ) by L^*_ξ we will denote its topological dual. Let $\mathbb{N}$ stand for the set of all natural numbers.

Throughout the paper we assume that (Ω,Σ,μ) is a σ-finite, complete measure space. By $\Sigma_f(\mu)$ we denote the δ-ring of the sets $A\in\Sigma$ with $\mu(A)<\infty$. By $\mathcal{S}(\Sigma,X)$ we denote the set of all X-valued Σ-simple functions $s=\Sigma\,(\mathbb{1}_{A_i}\otimes x_i)$, where (A_i) is

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a finite disjoint sequence in Σ, $x_i \in X$, and $(\mathbb{1}_{A_i} \otimes x_i)(\omega) = \mathbb{1}_{A_i}(\omega)x_i$ for $\omega \in \Omega$. By $L^0(\mu, X)$ we denote the set of the μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$. Then $L^0(\mu, X)$ can be provided with the complete metrizable topology $\mathcal{T}_0$ of convergence in measure on all $A \in \Sigma_f(\mu)$. Then $f_n \to 0$ for $\mathcal{T}_0$ if and only if $\mu(\{\omega \in A : \|f_n(\omega)\|_X \geq \varepsilon\}) \longrightarrow 0$ for every $A \in \Sigma_f(\mu)$ and all $\varepsilon > 0$. By $\mathcal{S}(\mu, X)$ we denote the subspace of $L^0(\mu, X)$ consisting of the μ-equivalence classes of all $s \in \mathcal{S}(\Sigma, X)$.

Now we recall terminology concerning Orlicz-Bochner spaces (see [24, 18] for more details). By a Young function we mean here a left continuous convex mapping $\varphi : [0, \infty) \to [0, \infty]$ vanishing and continuous at 0 such that $\varphi(t)/t \to \infty$ as $t \to \infty$. For a Young function φ we denote by φ^* the complementary Young function. The *Orlicz-Bochner space*

$$L^\varphi(\mu, X) = \Big\{ f \in L^0(\mu, X) : \int_\Omega \varphi(\lambda \|f(\omega)\|_X) d\mu < \infty \ \text{ for some } \ \lambda > 0 \Big\}$$

is equipped with the topology $\mathcal{T}_\varphi$ of the norm

$$\|f\|_\varphi = \inf \Big\{ \lambda > 0 : \int_\Omega \varphi(\|f(\omega)\|_X / \lambda) d\mu \leq 1 \Big\}.$$

Then $\|f\|_\varphi \leq 1$ if and only if $\int_\Omega \varphi(\|f(\omega)\|_X) d\mu \leq 1$. For $r > 0$ let

$$B_\varphi(r) = \{ f \in L^\varphi(\mu, X) : \|f\|_\varphi \leq r \}.$$

Note that the space $L^\infty(\mu, X)$ is included but the space $L^1(\mu, X)$ is excluded.

A subset H of $L^\varphi(\mu, X)$ is said to be solid whenever $\|f_1(\omega)\|_X \leq \|f_2(\omega)\|_X$ μ-a.e. and $f_1 \in L^\varphi(\mu, X), f_2 \in H$ imply $f_1 \in H$. A linear topology τ on $L^\varphi(\mu, X)$ is said to be locally solid if it has a local base at zero consisting of solid sets. A linear topology τ on $L^\varphi(\mu, X)$ that is at the same time locally solid and locally convex will be called a locally convex-solid topology on $L^\varphi(\mu, X)$. A seminorm ϱ on $L^\varphi(\mu, X)$ is called solid if $\varrho(f_1) \leq \varphi(f_2)$ whenever $f_1, f_2 \in L^\varphi(\mu, X)$ and $\|f_1(\omega)\|_X \leq \|f_2(\omega)\|_X$ μ-a.e. It is known that a locally convex topology τ on $L^\varphi(\mu, X)$ is locally solid if and only if it is generated by some family of solid seminorms defined on $L^\varphi(\mu, X)$ (see [15]).

In this paper the mixed topology $\gamma[\mathcal{T}_\varphi, \mathcal{T}_0|_{L^\varphi(\mu,X)}]$ (briefly γ_φ) on $L^\varphi(\mu, X)$ is of importance (see [25, 7, 16] for more details). γ_φ is a locally convex-solid topology such that $\mathcal{T}_0|_{L^\varphi(\mu,X)} \subset \gamma_\varphi \subset \mathcal{T}_\varphi$ (see [16, § 3]) and it is the finest locally convex topology on $L^\varphi(\mu, X)$ which agrees with $\mathcal{T}_0$ on $\|\cdot\|_\varphi$-bounded sets (see [25, 2.2.2]). Then a sequence (f_n) in $L^\varphi(\mu, X)$ is γ_φ-convergent to $f \in L^\varphi(\mu, X)$ if and only if $f_n \to f$ for $\mathcal{T}_0$ and $\sup_n \|f_n\|_\varphi < \infty$ (see [25, Theorem 2.6.1], [16, Theorem 3.1]).

It is well known that for each strongly Σ-measurable function $f : \Omega \to X$ there exists a sequence (s_n) of X-valued Σ-simple functions such that $\|f(\omega) - s_n(\omega)\|_X \to 0$ for $\omega \in \Omega$ and $\|s_n(\omega)\|_X \leq \|f(\omega)\|_X$ for all $n \in \mathbb{N}$ and $\omega \in \Omega$ (see [14, Theorem 1.6]). It follows that for each $f \in L^\varphi(\mu, X)$ there exists a sequence (s_n) in $\mathcal{S}(\mu, X)$ such that $s_n \to f$ for $\mathcal{T}_0$ and $\|s_n\|_\varphi \leq \|f\|_\varphi$ for all $n \in \mathbb{N}$; that is, $s_n \to f$ for γ_φ. Hence using the Lebesgue dominated convergence theorem we easily derive that $L^\varphi(\mu, X) \cap \mathcal{S}(\Sigma, X)$

is dense in $(E^\varphi(\mu,X),\|\cdot\|_\varphi)$, where

$$E^\varphi(\mu,X)=\Big\{f\in L^0(\mu,X):\int_\Omega\varphi\big(\lambda\|f(\omega)\|_X\big)d\mu<\infty\quad\text{for all}\quad\lambda>0\Big\}.$$

The following characterization of $(\gamma_\varphi,\|\cdot\|_Y)$-continuous linear operators from $L^\varphi(\mu,X)$ to Y will be of importance (see [25, Theorem 2.2.4], [16, Proposition 2.3]).

Proposition 1.1

For a $(\|\cdot\|_\varphi,\|\cdot\|_Y)$-continuous linear operator $T:L^\varphi(\mu,X)\to Y$ the following statements are equivalent:

(i) *T is $(\gamma_\varphi,\|\cdot\|_Y)$-continuous.*

(ii) *$\|T(f_n)\|_Y\to 0$ whenever $f_n\to 0$ for $\mathcal{T}_0$ and $\sup_n\|f_n\|_\varphi<\infty$.*

(iii) *T is $(\mathcal{T}_0|_{B_\varphi(r)},\|\cdot\|_Y)$-continuous for each $r>0$.*

Now we recall basic terminology concerning operator measures (see [8, 12, 13, 14, 4, 19, 20]). A finitely additive mapping $m:\Sigma\to\mathcal{L}(X,Y)$ is called an operator measure. Following [8] for a Young function φ we define a φ^*-semivariation $\widetilde{m}_{\varphi^*}(A)$ of m on $A\in\Sigma$ by

$$\widetilde{m}_{\varphi^*}(A)=\sup\|\Sigma\, a_i\, m(A_i)(x_i)\|_Y,$$

where the supremum is taken over all finite disjoint sequences (A_i) in Σ with $A_i\subset A$ and $a_i\ge 0$, $x_i\in B_X$ for each i with $\Sigma\,\varphi(a_i)\mu(A_i)\le 1$. By $\mathrm{fasv}_{\varphi^*,\mu}(\Sigma,\mathcal{L}(X,Y))$ we denote the set of all operator measures $m:\Sigma\to\mathcal{L}(X,Y)$ with a finite φ^*-semivariation (i.e., $\widetilde{m}_{\varphi^*}(\Omega)<\infty$), which vanish on μ-null sets, i.e., $m(A)=0$ whenever $\mu(A)=0$. Note that if $m\in\mathrm{fasv}_{\varphi^*,\mu}(\Sigma,\mathcal{L}(X,Y))$, then $\widetilde{m}_{\varphi^*}:\Sigma\to[0,\infty)$ is a submeasure i.e., $\widetilde{m}_{\varphi^*}(\emptyset)=0$, $\widetilde{m}_{\varphi^*}(A_1)\le\widetilde{m}_{\varphi^*}(A_2)$ for $A_1,A_2\in\Sigma$ with $A_1\subset A_2$ and $\widetilde{m}_{\varphi^*}(A_1\cup A_2)\le\widetilde{m}_{\varphi^*}(A_1)+\widetilde{m}_{\varphi^*}(A_2)$ for any $A_1,A_2\in\Sigma$.

For a sequence (A_n) in Σ we will write $A_n\searrow_\mu\emptyset$ if $A_n\downarrow$ and $\mu(A_n\cap A)\to 0$ for all $A\in\Sigma_f(\mu)$.

Now we distinguish some class of operator measures.

Definition 1.2 A measure $m\in\mathrm{fasv}_{\varphi^*,\mu}(\Sigma,\mathcal{L}(X,Y))$ is said to be *φ^*-variationally μ-continuous* if $\widetilde{m}_{\varphi^*}(A_n)\to 0$ whenever $A_n\searrow_\mu\emptyset$, $(A_n)\subset\Sigma$.

It is known that if $1\le p<\infty$, $\mu(\Omega)<\infty$ and an operator measure $m:\Sigma\to\mathcal{L}(X,Y)$ vanishes on μ-null sets and has the finite q-semivariation $\widetilde{m}_q(\Omega)$ $(1<q\le\infty$, $\frac{1}{p}+\frac{1}{q}=1)$, then using the fact that $\mathcal{S}(\mu,X)$ is dense in $(L^p(\mu,X),\|\cdot\|_p)$ one can define the integral $\int_\Omega f(\omega)dm$ for all $f\in L^p(\mu,X)$. Moreover, if $T:L^p(\mu,X)\to Y$ is a bounded linear operator, then the associated operator measure $m:\Sigma\to\mathcal{L}(X,Y)$ has the finite q-semivariation $\widetilde{m}_q(\Omega)$ and $T(f)=\int_\Omega f(\omega)dm$ for all $f\in L^p(\mu,X)$ (see [12, § 13, Theorem 3.1], [13, Theorem 4], [14, § 8B]). The relationships of the q-semivariation $\widetilde{m}_q$ to the properties of operators from $L^p(\mu,X)$ to Y were studied in [3]. Diestel found the integral representation of bounded linear operators from an Orlicz-Bochner space $L^\varphi(\mu,X)$ to Banach spaces whenever $\mu(\Omega)<\infty$ and a Young function φ satisfies the Δ_2-condition, i.e., $\limsup\varphi(2t)/\varphi(t)<\infty$ as $t\to\infty$ (see [8, Theorem 2]). Note that $\mathcal{S}(\mu,X)$ is dense in $(L^\varphi(\mu,X),\|\cdot\|_\varphi)$ whenever φ satisfies the

Δ_2-condition and $\mu(\Omega) < \infty$. Dinculeanu [10, 11] studied the integral representation of a certain linear transformation of an Orlicz vector field into a Banach space.

In this paper we study the problem of integral representation of linear operators $T : L^\varphi(\mu, X) \to Y$ (φ is an arbitrary Young function) with respect to an operator measure $m : \Sigma \to \mathcal{L}(X, Y)$. It is shown that T has an integral representation $T(f) = \int_\Omega f(\omega) dm$ for $f \in L^\varphi(\mu, X)$ with respect to a φ^*-variationally μ-continuous measure m if and only if T is $(\gamma_\varphi, \|\cdot\|_Y)$-continuous (see Corollary 2.7 below). As an application, we derive Vitali-Hahn-Saks type theorems for families of operator measures (see Theorem 3.6 and Corollary 3.7 below).

2. Integral representation of linear operators on Orlicz-Bochner spaces

From now on we assume that φ is a Young function. Let $m : \Sigma \to \mathcal{L}(X, Y)$ be a finite additive measure. Then for every $s = \sum_{i=1}^n (\mathbb{1}_{A_i} \otimes x_i) \in \mathcal{S}(\Sigma, X)$ we can define the integral with respect to m by

$$\int_\Omega s(\omega) dm := \sum_{i=1}^n m(A_i)(x_i).$$

For $A \in \Sigma$ let

$$\int_A s(\omega) dm := \int_\Omega \mathbb{1}_A(\omega) s(\omega) dm.$$

Proposition 2.1

Assume that $m : \Sigma \to \mathcal{L}(X, Y)$ is a finite additive measure. Then for $A \in \Sigma$ we have

$$\widetilde{m}_{\varphi^*}(A) = \sup\Big\{\Big\| \int_A s(\omega) dm\Big\|_Y : s \in \mathcal{S}(\Sigma, X), \int_\Omega \varphi\big(\|s(\omega)\|_X\big) d\mu \le 1\Big\}.$$

Proof. Let $A \in \Sigma$ be given. Assume that $s = \sum_{i=1}^n (\mathbb{1}_{A_i} \otimes x_i) \in \mathcal{S}(\Sigma, X)$ with $\int_\Omega \varphi(\|s(\omega)\|_X) d\mu \le 1$. Then

$$\sum_{i=1}^n \varphi\big(\|x_i\|_X\big)\mu(A \cap A_i) \le \int_\Omega \varphi\big(\|s(\omega)\|_X\big) d\mu \le 1,$$

so we get

$$\begin{aligned}\Big\| \int_A s(\omega) dm\Big\|_Y &= \Big\| \int_\Omega \mathbb{1}_A(\omega) s(\omega) dm\Big\|_Y \\ &= \Big\| \sum_{i=1}^n \|x_i\|_X \, m(A \cap A_i)\Big(\frac{x_i}{\|x_i\|_X}\Big)\Big\|_Y \le \widetilde{m}_{\varphi^*}(A).\end{aligned}$$

It follows that

$$\sup\Big\{\Big\| \int_A s(\omega) dm\Big\|_Y : s \in \mathcal{S}(\Sigma, X), \int_\Omega \varphi\big(\|s(\omega)\|_X\big) d\mu \le 1\Big\} \le \widetilde{m}_{\varphi^*}(A).$$

For the converse, let $(A_i)_{i=1}^n$ be a disjoint sequence in Σ with $A_i \subset A$ and $x_i \in B_X$, $a_i \geq 0$ for $i = 1, 2, \dots, n$ and $\sum_{i=1}^n \varphi(a_i)\mu(A_i) \leq 1$. Let $s_0 = \sum_{i=1}^n (\mathbb{1}_{A_i} \otimes a_i x_i)$. Then

$$\int_\Omega \varphi(\|s_0(\omega)\|_X) d\mu = \sum_{i=1}^n \varphi(a_i \|x_i\|_X)\mu(A_i) \leq 1.$$

Hence

$$\Big\| \sum_{i=1}^n a_i m(A_i)(x_i) \Big\|_Y = \Big\| \int_\Omega s_0(\omega) dm \Big\|_Y = \Big\| \int_A s_0(\omega) dm \Big\|_Y$$
$$\leq \sup \Big\{ \Big\| \int_A s(\omega) dm \Big\|_Y : s \in \mathcal{S}(\Sigma, X), \int_\Omega \varphi(\|s(\omega)\|_X) d\mu \leq 1 \Big\},$$

which completes the proof. □

Now assume that $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$. Let $s = \sum_{i=1}^n (\mathbb{1}_{A_i} \otimes x_i) \in \mathcal{S}(\Sigma, X)$. Then for every set $A_0 \in \Sigma$ with $\mu(A_0) = 0$, we have $m(A_0) = 0$, so we get

$$\int_\Omega s(\omega) dm = \sum_{i=1}^n m(A_i)(x_i) = \sum_{i=1}^n m\big(A_i \cap (\Omega \setminus A_0)\big)(x_i) = \int_\Omega \mathbb{1}_{\Omega \setminus A_0}(\omega) s(\omega) dm.$$

Hence, in view of Proposition 2.1 we get

Corollary 2.2

Assume that $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$. Then for $A \in \Sigma$ we have

$$\widetilde{m}_{\varphi^*}(A) = \sup \Big\{ \Big\| \int_A s(\omega) dm \Big\|_Y : s \in L^\varphi(\mu, X) \cap \mathcal{S}(\mu, X), \|s\|_\varphi \leq 1 \Big\}.$$

It follows that if $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$, then the integration operator

$$T_m : L^\varphi(\mu, X) \cap \mathcal{S}(\mu, X) \longrightarrow Y$$

defined by $T_m(s) = \int_\Omega s(\omega) dm$ is linear and $(\|\cdot\|_\varphi, \|\cdot\|_Y)$-continuous.

Now we shall show that if $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$ is φ^*-variationally μ-continuous, then T_m can be uniquely extended to a $(\gamma_\varphi, \|\cdot\|_Y)$-continuous linear operator $\overline{T}_m : L^\varphi(\mu, X) \to Y$ with $\|T_m\| = \|\overline{T}_m\|$.

Since for $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$ its semivariation $\widetilde{m}_{\varphi^*} : \Sigma \to [0, \infty)$ is a submeasure, using the standard argument one can obtain the following characterization of φ^*-variationally μ-continuous operator measures.

Proposition 2.3

For $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$ the following statements are equivalent:

(i) *m is φ^*-variationally μ-continuous.*
(ii) *For every $\varepsilon > 0$ there exist $\delta > 0$ and $A_0 \in \Sigma_f(\mu)$ such that $\widetilde{m}_{\varphi^*}(A) \leq \varepsilon$ for all $A \in \Sigma$ with $\mu(A) \leq \delta$ and $\widetilde{m}_{\varphi^*}(\Omega \setminus A_0) \leq \varepsilon$.*

For every $r > 0$ by T_m^r we denote the restriction of $T_m : L^\varphi(\mu, X) \cap \mathcal{S}(\mu, X) \to Y$ to $B_\varphi(r) \cap \mathcal{S}(\mu, X)$.

Proposition 2.4

Assume that $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$ *is* φ^**-variationally* μ*-continuous. Then for each* $r > 0$ *the mapping*

$$T_m^r : B_\varphi(r) \cap \mathcal{S}(\mu, X) \longrightarrow Y$$

is $(\mathcal{T}_0|_{B_\varphi(r)\cap\mathcal{S}(\mu,X)}, \|\cdot\|_Y)$*-uniformly continuous.*

Proof. For $\eta > 0$ and $A \in \Sigma_f(\mu)$ let

$$V(A,\eta) = \big\{ f \in L^0(\mu, X) : \mu\big(\{\omega \in A : \|f(\omega)\|_X \geq \eta\}\big) < \eta \big\}.$$

Then the family $\mathcal{B}_0 = \{V(A,\eta) : \eta > 0, A \in \Sigma_f(\mu)\}$ is a local base at 0 for $\mathcal{T}_0$.

Let us fix $r > 0$ and let $\varepsilon > 0$ be given. Then in view of Proposition 2.3 there exist $A_0 \in \Sigma_f(\mu)$ and $\delta > 0$ such that $\widetilde{m}_{\varphi^*}(\Omega \setminus A_0) \leq \frac{\varepsilon}{8r}$ and $\widetilde{m}_{\varphi^*}(A) \leq \frac{\varepsilon}{8r}$ for every $A \in \Sigma$ with $\mu(A) \leq \delta$. Choose $\eta \in (0, \delta)$ such that $\varphi(\sqrt{\eta})\mu(A_0) \leq 1$ and $\sqrt{\eta} \leq \frac{\varepsilon}{2\widetilde{m}_{\varphi^*}(A_0)+1}$.

Take $s_1, s_2 \in B_\varphi(r) \cap \mathcal{S}(\mu, X)$ such that $s_1 - s_2 \in V(A_0, \eta)$. Then $\|\frac{s_1-s_2}{2r}\|_\varphi \leq 1$. Let $A(\eta) = \{\omega \in \Omega : \|s_1(\omega) - s_2(\omega)\|_X \geq \eta\}$. Since $\mu(A(\eta) \cap A_0) < \eta$, we have $\widetilde{m}_{\varphi^*}(A(\eta)\cap A_0) \leq \frac{\varepsilon}{8r}$. We have $s_1 - s_2 = \sum_{i=1}^k (\mathbb{1}_{A_i} \otimes x_i)$, where $(A_i)_{i=1}^k$ is a disjoint sequence in Σ and $x_i \in X$ for $1 \leq i \leq k$. Let

$$I = \big\{ i \in \{1, \ldots, k\} : \|x_i\|_X \geq \eta \big\} \quad \text{and} \quad J = \big\{ i \in \{1, \ldots, k\} : \|x_i\|_X < \eta \big\}.$$

Then

$$\begin{aligned}\sum_{i\in I} \varphi\Big(\frac{\|x_i\|_X}{2r}\Big)\mu(A_i) &\leq \sum_{i=1}^k \varphi\Big(\frac{\|x_i\|_X}{2r}\Big)\mu(A_i) \\ &= \int_\Omega \varphi\Big(\frac{\|s_1(\omega) - s_2(\omega)\|_X}{2r}\Big) d\mu \leq 1,\end{aligned}$$

so

$$\Big\| \sum_{i\in I} \frac{\|x_i\|_X}{2r} m(A_i \cap A_0)\Big(\frac{x_i}{\|x_i\|_X}\Big)\Big\|_Y \leq \widetilde{m}_{\varphi^*}(A(\eta)\cap A_0),$$

and

$$\Big\| \sum_{i=1}^k \frac{\|x_i\|_X}{2r} m(A_i \cap (\Omega \setminus A_0))\Big(\frac{x_i}{\|x_i\|_X}\Big)\Big\|_Y \leq \widetilde{m}_{\varphi^*}(\Omega \setminus A_0).$$

Moreover,

$$\Big\| \sum_{i\in J} \frac{\|x_i\|_X}{\sqrt{\eta}} m(A_i \cap A_0)\left(\frac{x_i}{\|x_i\|_X}\right)\Big\|_Y \leq \widetilde{m}_{\varphi^*}(A_0),$$

because

$$\begin{aligned}\sum_{i\in J} \varphi\frac{\|x_i\|_X}{\sqrt{\eta}}\, \mu(A_i \cap A_0) &\leq \sum_{i\in J} \varphi(\sqrt{\eta})\, \mu(A_i \cap A_0) \\ &= \varphi(\sqrt{\eta})\, \mu\Big(A_0 \cap \bigcup_{i\in J} A_i\Big) \leq \varphi(\sqrt{\eta})\, \mu(A_0) \leq 1.\end{aligned}$$

Hence we have

$$
\begin{aligned}
\|T_m^r(s_1) - T_m^r(s_2)\|_Y &= \Big\| \int_\Omega s_1(\omega)dm - \int_\Omega s_2(\omega)dm \Big\|_Y \\
&= \Big\| \int_\Omega (s_1(\omega) - s_2(\omega))\, dm \Big\|_Y = \Big\| \sum_{i=1}^k m(A_i)(x_i) \Big\|_Y \\
&= \Big\| \sum_{i\in I} m(A_i \cap A_0)(x_i) + \sum_{i\in J} m(A_i \cap A_0)(x_i) \\
&\quad + \sum_{i=1}^k m(A_i \cap (\Omega \setminus A_0))(x_i) \Big\|_Y \\
&\le 2r \Big\| \sum_{i\in I} \frac{\|x_i\|_X}{2r}\, m(A_i \cap A_0)\Big(\frac{x_i}{\|x_i\|_X}\Big) \Big\|_Y \\
&\quad + \sqrt{\eta}\, \Big\| \sum_{i\in J} \frac{\|x_i\|_X}{\sqrt{\eta}}\, m(A_i \cap A_0)\Big(\frac{x_i}{\|x_i\|_X}\Big) \Big\|_Y \\
&\quad + 2r \Big\| \sum_{i=1}^k \frac{\|x_i\|_X}{2r}\, m(A_i \cap (\Omega \setminus A_0))\Big(\frac{x_i}{\|x_i\|_X}\Big) \Big\|_Y \\
&\le 2r\, \widetilde{m}_{\varphi^*}\big(A(\eta) \cap A_0\big) + \sqrt{\eta}\, \widetilde{m}_{\varphi^*}(A_0) + 2r\, \widetilde{m}_{\varphi^*}(\Omega \setminus A_0) \\
&\le \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.
\end{aligned}
$$

This means that T_m^r is $(\mathcal{T}_0|_{B_\varphi(r)\cap \mathcal{S}(\mu,X)}, \|\cdot\|_Y)$-uniformly continuous. □

Now we are in position to state our main result.

Theorem 2.5

Assume that $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$ *is* φ^**-variationally* μ*-continuous. Then* $T_m : L^\varphi(\mu,X) \cap \mathcal{S}(\mu,X) \to Y$ *has a unique* $(\gamma_\varphi, \|\cdot\|_Y)$*-continuous linear extension* $\overline{T}_m : L^\varphi(\mu,X) \to Y$ *and* $\|\overline{T}_m\| = \widetilde{m}_{\varphi^*}(\Omega)$.

Proof. From Proposition 2.4 it follows that for every $r > 0$ there exists a unique $(\mathcal{T}_0|_{cl(B_\varphi(r)\cap\mathcal{S}(\mu,X))}, \|\cdot\|_Y)$-uniformly continuous extension

$$\overline{T}_m^r : cl\big(B_\varphi(r) \cap \mathcal{S}(\mu,X)\big) \to Y$$

of the mapping $T_m^r : B_\varphi(r) \cap \mathcal{S}(\mu,X) \to Y$ (the closure is taken in $\mathcal{T}_0|_{L^\varphi(\mu,X)}$) (see [2, Theorem 2.6]). Since $B_\varphi(r) \subset cl(B_\varphi(r) \cap \mathcal{S}(\mu,X))$ for $r > 0$, the restricted mapping $\overline{T}_m^r|_{B_\varphi(r)} : B_\varphi(r) \to Y$ is $(\mathcal{T}_0|_{B_\varphi(r)}, \|\cdot\|_Y)$-uniformly continuous. Then for $f \in B_\varphi(r)$ we have

$$\overline{T}_m^r(f) = \lim T_m^r(s_n) = \lim \int_\Omega s_n(\omega)dm,$$

where (s_n) is a sequence in $B_\varphi(r) \cap \mathcal{S}(\mu,X)$ such that $s_n \to f$ for $\mathcal{T}_0$. Note that for $0 < r_1 < r_2$ we have $\overline{T}_m^{r_2}|_{B_\varphi(r_1)} = \overline{T}_m^{r_1}|_{B_\varphi(r_1)}$. Define the linear operator $\overline{T}_m : L^\varphi(\mu,X) \to Y$ by

$$\overline{T}_m(f) = \overline{T}_m^r(f) \quad \text{for} \quad f \in B_\varphi(r),\ r > 0.$$

Then $\overline{T}_m|_{L^\varphi(\mu,X)\cap\mathcal{S}(\mu,X)} = T_m$, and in view of Proposition 1.1 $\overline{T}_m$ is $(\gamma_\varphi, \|\cdot\|_Y)$-continuous.

Note that by Corollary 2.2

$$\begin{aligned}\widetilde{m}_{\varphi^*}(\Omega) &= \sup\left\{\|T_m(s)\|_Y : s \in L^\varphi(\mu,X)\cap\mathcal{S}(\mu,X), \|s\|_\varphi \le 1\right\}\\ &\le \sup\left\{\|\overline{T}_m(f)\|_Y : f \in L^\varphi(\mu,X),\ \|f\|_\varphi \le 1\right\} = \|\overline{T}_m\|.\end{aligned}$$

Now let $f \in L^\varphi(\mu,X)$ and $\|f\|_\varphi \le 1$. Then there exists a sequence (s_n) in $L^\varphi(\mu,X)\cap\mathcal{S}(\mu,X)$ such that $s_n \to f$ for $\mathcal{T}_0$ and $\|s_n\|_\varphi \le \|f\|_\varphi \le 1$ for $n \in \mathbb{N}$. Hence $\overline{T}_m(f) = \lim T_m(s_n)$. Let $\varepsilon > 0$ be given. Then there exists $n_\varepsilon \in \mathbb{N}$ such $\|\overline{T}_m(f) - T_m(s_{n_\varepsilon})\|_Y \le \varepsilon$. Hence

$$\begin{aligned}\|\overline{T}_m(f)\|_Y &\le \|\overline{T}_m(f) - T_m(s_{n_\varepsilon})\|_Y + \|T_m(s_{n_\varepsilon})\|_Y\\ &\le \varepsilon + \left\|\int_\Omega s_{n_\varepsilon}(\omega)dm\right\|_Y \le \varepsilon + \widetilde{m}_{\varphi^*}(\Omega).\end{aligned}$$

It follows that $\|\overline{T}_m\| \le \widetilde{m}_{\varphi^*}(\Omega)$. □

In view of Theorems 2.4 and 2.5 we have

DEFINITION 2.6 Assume that $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$ is φ^*-variationally μ-continuous. For every $f \in L^\varphi(\mu,X)$ we define the integral $\int_\Omega f(\omega)dm$ by the equality:

$$\int_\Omega f(\omega)dm := \overline{T}_m(f).$$

Assume now that $T : L^\varphi(\mu,X) \to Y$ is a $(\|\cdot\|_\varphi, \|\cdot\|_Y)$-continuous linear operator and let $m : \Sigma \to \mathcal{L}(X,Y)$ be its representing measure defined by

$$m(A)(x) := T(\mathbb{1}_A \otimes x) \quad \text{for} \quad A \in \Sigma \quad \text{and} \quad x \in X.$$

Then by Corollary 2.2 we get

$$\begin{aligned}\widetilde{m}_{\varphi^*}(\Omega) &= \sup\left\{\|T(s)\|_Y : s \in L^\varphi(\mu,X)\cap\mathcal{S}(\mu,X),\ \|s\|_\varphi \le 1\right\}\\ &\le \|T\| < \infty,\end{aligned}$$

and $m(A) = 0$ whenever $\mu(A) = 0$. This means that $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$.

Theorem 2.7

Let $L^\varphi(\mu,X) \to Y$ be a $(\gamma_\varphi, \|\cdot\|_Y)$-continuous linear operator. Then its representing measure $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$ is φ^*-variationally μ-continuous and

$$T(f) = \overline{T}_m(f) = \int_\Omega f(\omega)dm \quad \textit{for all} \quad f \in L^\varphi(\mu,X).$$

Proof. Note that T is $(\|\cdot\|_\varphi, \|\cdot\|)$-continuous because $\gamma_\varphi \subset \mathcal{T}_\varphi$. Hence $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$. To show that m is φ^*-variationally μ-continuous, assume that $A_n \searrow_\mu \emptyset, (A_n) \subset \Sigma$. Then for each $n \in \mathbb{N}$ there exist a sequence $(A_{n,i})_{i=1}^{k_n}$ in Σ with $A_{n,i} \subset A_n$ and $a_{n,i} \geq 0$, $x_{n,i} \in B_X$ for $1 \leq i \leq k_n$ with $\sum_{i=1}^{k_n} \varphi(a_{n,i})\mu(A_{n,i}) \leq 1$ such that

$$\widetilde{m}_{\varphi^*}(A_n) \leq \Big\| \sum_{i=1}^{k_n} a_{n,i}\, m(A_{n,i})(x_{n,i}) \Big\|_Y + \frac{1}{n}.$$

Let $s_n = \sum_{i=1}^{k_n} (\mathbb{1}_{A_{n,i}} \otimes a_{n,i}\, x_{n,i})$ for $n \in \mathbb{N}$. Then

$$\int_\Omega \varphi\big(\|s_n(\omega)\|_X\big) d\mu = \sum_{i=1}^{k_n} \varphi\big(\|a_{n,i}\, x_{n,i}\|_X\big)\mu(A_{n,i}) \leq \sum_{i=1}^{k_n} \varphi(a_{n,i})\mu(A_{n,i}) \leq 1,$$

so $\|s_n\|_\varphi \leq 1$ for $n \in \mathbb{N}$. Moreover, for every $A \in \Sigma_f(\mu)$ and $\varepsilon > 0$ we have $\{\omega \in A : \|s_n(\omega)\|_X \geq \varepsilon\} \subset A_n \cap A$ for all $n \in \mathbb{N}$. Thus $\mu(\{\omega \in A : \|s_n(\omega)\|_X \geq \varepsilon\}) \to 0$, i.e., $s_n \to 0$ for $\mathcal{T}_0$. Hence $s_n \to 0$ for γ_φ, so

$$\Big\| \sum_{i=1}^{k_n} a_{n,i}\; m(A_{n,i})(x_{n,i}) \Big\|_Y = \|T(s_n)\|_Y \longrightarrow 0.$$

It follows that $\widetilde{m}_{\varphi^*}(A_n) \to 0$, i.e., m is φ^*-variationally μ-continuous.

Now let $f \in L^\varphi(\mu, X)$. Then there exists a sequence (s_n) in $L^\varphi(\mu,X) \cap \mathcal{S}(\mu, X)$ such that $s_n \to f$ for γ_φ. Since T and the integration operator $\overline{T}_m$ are $(\gamma_\varphi, \|\cdot\|_Y)$-continuous (see Theorem 2.5) we have

$$T(f) = \lim T(s_n) = \lim T_m(s_n) = \overline{T}_m(f) = \int_\Omega f(\omega) dm.$$

$\square$

As a consequence of Theorems 2.5 and 2.7 we get

Corollary 2.8

Let $T : L^\varphi(\mu, X) \to Y$ be a $(\|\cdot\|_\varphi, \|\cdot\|_Y)$-continuous linear operator and let $m \in \mathrm{fasv}_{\varphi^*,\mu}(\Sigma, \mathcal{L}(X,Y))$ be its representing measure. Then the following statements are equivalent:

(i) *T is $(\gamma_\varphi, \|\cdot\|_Y)$-continuous.*

(ii) *m is φ^*-variationally μ-continuous.*

Note that the Lebesgue-Bochner space $L^\infty(\mu, X)$ is equal to the Orlicz-Bochner space $L^{\varphi_\infty}(\mu, X)$, where

$$\varphi_\infty(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ \infty & \text{if } t > 1. \end{cases}$$

The space $L^\infty(\mu, X)$ is provided with the norm $\|f\|_\infty = \operatorname{ess\,sup}_{\omega\in\Omega} \|f(\omega)\|_X$. Then $\varphi^*_\infty(t) = t^1$ for all $t \geq 0$. Hence for a measure $m : \Sigma \to \mathcal{L}(X,Y)$ and $A \in \Sigma$ we have

$$\widetilde{m}(A) = \widetilde{m}_{\varphi^*_\infty}(A) = \sup \| \, \Sigma\, m(A_i)(x_i) \, \|_Y,$$

where the supremum is taken over all finite disjoint sequences (A_i) in Σ with $A_i \subset A$ and $x_i \in B_X$ for each i. We will briefly write $\mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X,Y))$ instead of $\mathrm{fasv}_{\varphi^*_\infty,\mu}(\Sigma, \mathcal{L}(X,Y))$. We will say that $\mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X,Y))$ is variationally μ-continuous if $\widetilde{m}(A_n) \to 0$ as $A_n \searrow_\mu \emptyset$, $(A_n) \subset \Sigma$. By γ_∞ $(= \gamma_{\varphi_\infty})$ we will denote the mixed topology on $L^\infty(\mu, X)$.

For each $y^* \in Y^*$ let $m_{y^*} : \Sigma \to X^*$ be a measure defined by

$$m_{y^*}(A)(x) = \langle m(A)(x), y^* \rangle \text{ for every } A \in \Sigma \text{ and } x \in X.$$

It is well known that for $A \in \Sigma$,

$$\widetilde{m}(A) = \sup \left\{ |m_{y^*}|(A) : y^* \in B_{Y^*} \right\},$$

where $|m_{y^*}|$ stands for the variation of a measure m_{y^*} (see [4, Theorem 5]). It follows that $m \in \mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X,Y))$ is variationally μ-continuous if and only if the family $\{|m_{y^*}| : y^* \in B_{Y^*}\}$ is uniformly μ-continuous.

Recall that a continuous finite valued Young function φ is said to be an N-function if φ vanishes only at 0 and $\varphi(t)/t \to 0$ as $t \to 0$, $\varphi(t)/t \to \infty$ as $t \to \infty$ (see [18]). Note that if $\mu(\Omega) < \infty$, then $\mathcal{S}(\mu, X) \subset L^\infty(\mu, X) \subset E^\varphi(\mu, X) \subset L^\varphi(\mu, X)$ for any N-function φ. The following characterization of γ_∞ will be useful (see [21, Theorem 4.5]).

Theorem 2.9

Assume that (Ω, Σ, μ) is a finite measure space. Then γ_∞ is generated by a family of norms $\|\cdot\|_\varphi|_{L^\infty(\mu,X)}$, where φ runs over the family of all N-functions.

Now we are ready to present a characterization of variationally μ-continuous measures $m \in \mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X,Y))$.

Corollary 2.10

Assume that (Ω, Σ, μ) is a finite measure space. Then for $m \in \mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X,Y))$ the following statements are equivalent:

(i) *m is variationally μ-continuous.*

(ii) *There exists an N-function φ such that $\widetilde{m}_{\varphi^*}(\Omega) < \infty$.*

Proof. (i)$\Longrightarrow$(ii) Assume that m is variationally μ-continuous. Then in view of Theorem 2.5 the integration operator $\overline{T}_m : L^\infty(\mu, X) \to Y$ is $(\gamma_\infty, \|\cdot\|_Y)$-continuous. Hence by Theorem 2.9 there exists an N-function φ such that $\overline{T}_m$ is $(\|\cdot\|_\varphi|_{L^\infty(\mu,X)}, \|\cdot\|_Y)$-continuous. Hence in view of Corollary 2.2 we get

$$\widetilde{m}_{\varphi^*}(\Omega) = \sup \left\{ \|\overline{T}_m(s)\|_Y : s \in \mathcal{S}(\mu, X), \|s\|_\varphi \le 1 \right\} < \infty.$$

(ii)$\Longrightarrow$(i) Assume that φ is an N-function such that $\widetilde{m}_{\varphi^*}(\Omega) < \infty$. Let $T_m(s) = \int_\Omega s(\omega)dm$ for $s \in \mathcal{S}(\mu, X)$. Then by Corollary 2.2 for every $s \in \mathcal{S}(\mu, X)$ we have $\|T_m(s)\|_Y \le \widetilde{m}_{\varphi^*}(\Omega) \cdot \|s\|_\varphi$. This means that the operator $T_m : \mathcal{S}(\mu, X) \to Y$ is $(\|\cdot\|_\varphi, \|\cdot\|_Y)$-continuous. Since $\mathcal{S}(\mu, X) \subset L^\infty(\mu, X) \subset E^\varphi(\mu, X)$ and $\mathcal{S}(\mu, X)$ is dense in $(E^\varphi(\mu, X), \|\cdot\|_\varphi)$, we see that $\mathcal{S}(\mu, X)$ is dense in $(L^\infty(\mu, X), \|\cdot\|_\varphi|_{L^\infty(\mu,X)})$. Hence there exists a $(\|\cdot\|_\varphi|_{L^\infty(\mu,X)}, \|\cdot\|_Y)$-continuous extension $\overline{T}_m : L^\infty(\mu, X) \to Y$ of T_m. In view of Theorem 2.9 $\overline{T}_m$ is $(\gamma_\infty, \|\cdot\|_Y)$-continuous, and by Theorem 2.7 m is variationally μ-continuous. $\square$

3. Vitali-Hahn-Saks type theorems for operator measures

It is known that if the Banach dual X^* of X has the Radon-Nikodym property (i.e., X is an Asplund space; see [9, p. 213]), then

$$L^\infty(\mu, X)^*_{\gamma_\infty} = \{F_g : g \in L^1(\mu, X^*)\},$$

where $F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle d\mu$ for all $f \in L^\infty(\mu, X)$ (see [5, Theorem 4.1] [6, Theorem 4.3]). The following characterization of γ_∞ will be needed (see [22, Corollary 4.3]).

Theorem 3.1

Assume that X^* has the Radon-Nikodym property. Then γ_∞ is a Mackey topology, i.e.,

$$\gamma_\infty = \tau\left(L^\infty(\mu, X),\ L^\infty(\mu, X)^*_{\gamma_\infty}\right) = \tau(L^\infty(\mu, X), L^1(\mu, X^*)).$$

We will need the following general result.

Proposition 3.2

Assume the X^* has the Radon-Nikodym property. Let $T : L^\infty(\mu, X) \to Y$ be a $(\|\cdot\|_\infty, \|\cdot\|_Y)$-continuous linear operator and $m \in \mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X, Y))$ be its representing measure. Then the following statements are equivalent:

(i) *m is variationally μ-continuous.*

(ii) *T is $(\tau(L^\infty(\mu, X), L^1(\mu, X^*)), \|\cdot\|_Y)$-continuous.*

(iii) *T is $(\sigma(L^\infty(\mu, X), L^1(\mu, X^*)), \sigma(Y, Y^*))$-continuous.*

(iv) *$y^* \circ T \in L^\infty(\mu, X)^*_{\gamma_\infty}$ for each $y^* \in Y^*$.*

Proof. (i)$\Longleftrightarrow$(ii) It follows from Corollary 2.8 and Theorem 3.1.

(ii)$\Longleftrightarrow$(iii) See [1, Example 11, p. 149].

(iii)$\Longleftrightarrow$(iv) See [1, Theorem 9.26]. □

Let $\mathcal{L}(L^\infty(\mu, X), Y)$ stand for the space of all bounded linear operators from $L^\infty(\mu, X)$ to a Banach space Y. The *strong operator topology* (briefly SOT) is a locally convex topology on $\mathcal{L}(L^\infty(\mu, X), Y)$ defined by the family of seminorms $\{p_f : f \in L^\infty(\mu, X)\}$, where $p_f(T) = \|T(f)\|_Y$ for all $T \in \mathcal{L}(L^\infty(\mu, X), Y)$. The *weak operator topology* (briefly WOT) is a locally convex topology on $\mathcal{L}(L^\infty(\mu, X), Y)$ defined by the family of seminorms $\{p_{f,y^*} : f \in L^\infty(\mu, X), y^* \in Y^*\}$, where $p_{f,y^*}(T) = |\langle T(f), y^*\rangle|$ for all $T \in \mathcal{L}(L^\infty(\mu, X), Y)$. In view of the Banach-Steinhaus theorem the space $\mathcal{L}(L^\infty(\mu, X), Y)$ provided with SOT is sequentially complete. By $\mathcal{L}_{\gamma_\infty}(L^\infty(\mu, X), Y)$ we denote the subspace of $\mathcal{L}(L^\infty(\mu, X), Y)$ consisting of all those $T \in \mathcal{L}(L^\infty(\mu, X), Y)$ which are $(\gamma_\infty, \|\cdot\|_Y)$-continuous.

Proposition 3.3

Assume that X^* has the Radon-Nikodym property. Then $\mathcal{L}_{\gamma_\infty}(L^\infty(\mu, X), Y)$ is a sequentially closed subspace of $\mathcal{L}(L^\infty(\mu, X), Y)$ for WOT.

Proof. Let (T_n) be a sequence in $\mathcal{L}_{\gamma_\infty}(L^\infty(\mu,X),Y)$ such that $T_n \to T$ for WOT, where $T \in \mathcal{L}(L^\infty(\mu,X),Y)$. Let $y_0^* \in Y^*$ be given. Then for every $f \in L^\infty(\mu,X)$ we have $(y_0^* \circ T)(f) = \lim(y_0^* \circ T_n)(f)$, where $y_0^* \circ T_n \in L^\infty(\mu,X)^*_{\gamma_\infty}$ for $n \in \mathbb{N}$ (see Theorem 3.1 and Proposition 3.2) and $y_0^* \circ T \in L^\infty(\mu,X)^*$ (= the Banach dual of $L^\infty(\mu,X)$). It follows that $(y_0^* \circ T_n)$ is a $\sigma(L^\infty(\mu,X)^*_{\gamma_\infty}, L^\infty(\mu,X))$-Cauchy sequence in $L^\infty(\mu,X)^*_{\gamma_\infty}$. Since the space $(L^\infty(\mu,X)^*_{\gamma_\infty}, \sigma(L^\infty(\mu,X)^*_{\gamma_\infty}, L^\infty(\mu,X)))$ is sequentially complete (see [22, Corollary 4.3]), there exists $F_0 \in L^\infty(\mu,X)^*_{\gamma_\infty}$ such that $F_0(f) = \lim(y_0^* \circ T_n)(f)$ for each $f \in L^\infty(\mu,X)$, so $F_0 = y_0^* \circ T \in L^\infty(\mu,X)^*_{\gamma_\infty}$. Making use of Proposition 3.2 and Theorem 3.1 we derive that $T \in \mathcal{L}_{\gamma_\infty}(L^\infty(\mu,X),Y)$. □

Corollary 3.4

Assume that X^* has the Radon-Nikodym property. Then

(i) $\mathcal{L}_{\gamma_\infty}(L^\infty(\mu,X),Y)$ *is a sequentially closed subspace of* $\mathcal{L}(L^\infty(\mu,X),Y)$ *for* SOT.
(ii) *The space* $(\mathcal{L}_{\gamma_\infty}(L^\infty(\mu,X),Y), \mathrm{SOT})$ *is sequentially complete*

Proof. (i) It follows from Proposition 3.3 because WOT $\subset$ SOT.

(ii) It follows from (i) because the space $(\mathcal{L}(L^\infty(\mu,X),Y), \mathrm{SOT})$ is sequentially complete. □

The following result will be of importance (see [22, Theorem 5.5]).

Theorem 3.5

Assume that X^* has the Radon-Nikodym property. Let $\mathcal{K}$ be a SOT-*compact subset of $\mathcal{L}_{\gamma_\infty}(L^\infty(\mu,X),Y)$. Then $\mathcal{K}$ is $(\gamma_\infty, \|\cdot\|_Y)$-equicontinuous.*

Now we are in position to prove a modification and correction of [23, Theorems 4.3 and 4.4] concerning Vitali-Hahn-Saks type theorems for families of operator measures.

Theorem 3.6

Assume that X^* has the Radon-Nikodym property. Let $\mathcal{M}$ be a subset of $\mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X,Y))$ *consisting of variationally μ-continuous measures such that the set of the corresponding integration operators $\{\overline{T}_m : m \in \mathcal{M}\}$ is a* SOT-*compact subset of $\mathcal{L}_{\gamma_\infty}(L^\infty(\mu,X),Y)$. Then the set $\mathcal{M}$ is uniformly variationally μ-continuous, i.e.,* $\sup_{m\in\mathcal{M}} \widetilde{m}(A_n) \to 0$ *whenever* $A_n \searrow_\mu \emptyset$, $(A_n) \subset \Sigma$.

Proof. In view of Theorem 3.5 the family $\{\overline{T}_m : m \in \mathcal{M}\}$ is $(\gamma_\infty, \|\cdot\|_Y)$-equicontinuous. We know that γ_∞ is generated by a family $\{\varrho_\alpha : \alpha \in \mathcal{A}\}$ of solid seminorms on $L^\infty(\mu,X)$.

Let $\varepsilon > 0$ be given. Then there exist $\alpha_i \in \mathcal{A}$ for $i = 1,\ldots,i_0 \in \mathbb{N}$ and $\delta > 0$ such that

$$\sup_{m\in\mathcal{M}} \|\overline{T}_m(f)\|_Y \le \frac{\varepsilon}{2} \quad \text{whenever} \quad \max_{1\le i\le i_0} \varrho_{\alpha_i}(f) \le \delta. \tag{1}$$

Assume now that $A_n \searrow_\mu \emptyset, (A_n) \subset \Sigma$. For a fixed $x_0 \in X$ with $\|x_0\|_X = 1$ let $f_n = \mathbb{1}_{A_n} \otimes x_0$ for $n \in \mathbb{N}$. Then for $A \in \Sigma_f(\mu)$ we have $\{\omega \in A_n \cap A : \|f_n(\omega)\|_X \ge \varepsilon\} \subset A_n \cap A$ for $n \in \mathbb{N}$ and $\mu(A_n \cap A) \to 0$. Hence $f_n \to 0$ for $\mathcal{T}_0$. Since $\sup_n \|f_n\|_\infty \le 1$ we have $f_n \to 0$ for γ_∞. It follows that there exists $n_0 \in \mathbb{N}$ such that $\max_{1\le i\le i_0} \varrho_{\alpha_i}(f_n) \le \delta$ for $n \ge n_0$.

Now let $n \in \mathbb{N}$ be fixed. Then for every $m \in \mathcal{M}$ there exist a finite sequence $(A_{n,j}^m)_{j=1}^{k_{m,n}}$ in Σ with $A_{n,j}^{k_{m,n}} \subset A_n$ and $x_{n,j}^m \in B_X$ for $1 \le i \le k_{m,n}$ such that

$$\widetilde{m}(A_n) \le \Big\| \sum_{j=1}^{k_{m,n}} m(A_{n,j}^m)(x_{n,j}^m) \Big\|_Y + \frac{\varepsilon}{2}. \tag{2}$$

Let $s_n^m = \sum_{j=1}^{k_{m,n}} (\mathbb{1}_{A_{n,j}^m} \otimes x_{n,j}^m)$ for $m \in \mathcal{M}$. Then $\|s_n^m(\omega)\|_X \le \|f_n(\omega)\|_X$ for $\omega \in \Omega$ and every $m \in \mathcal{M}$, and hence $\max_{1\le i\le i_0} \varrho_{\alpha_i}(s_n^m) \le \max_{1\le i\le i_0} \varrho_{\alpha_i}(f_n)$ for every $m \in \mathcal{M}$. Hence by (1) and (2) for $n \ge n_0$ we get

$$\sup_{m\in\mathcal{M}} \widetilde{m}(A_n) \le \sup_{m\in\mathcal{M}} \big\|T_m(s_n^m)\big\|_Y + \frac{\varepsilon}{2} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This means that $\sup_{m\in\mathcal{M}} \widetilde{m}(A_n) \underset{n}{\longrightarrow} 0$, as desired. □

As a consequence of Theorem 3.6 we have

Corollary 3.7

Assume that X^* has the Radon-Nikodym property. Let $m_k \in \mathrm{fasv}_\mu(\Sigma, \mathcal{L}(X,Y))$ be variationally μ-continuous measures for $k \in \mathbb{N}$. Assume that for every $f \in L^\infty(\mu, X)$,

$$T(f) := \lim_k \overline{T}_{m_k}(f) = \lim_k \int_\Omega f(\omega)dm_k$$

exists in $(Y, \|\cdot\|_Y)$. Then the operator $T : L^\infty(\mu, X) \to Y$ is $(\gamma_\infty, \|\cdot\|_Y)$-continuous and the family $\{m_k : k \in \mathbb{N}\}$ is uniformly variationally μ-continuous, i.e., $\sup_k \widetilde{m}_k(A_n) \to 0$ as $A_n \searrow_\mu \emptyset$, $(A_n) \subset \Sigma$.

Proof. By Theorem 2.5 and Corollary 3.4 $T \in \mathcal{L}_{\gamma_\infty}(L^\infty(\mu, X), Y)$. Then $\overline{T}_{m_k} \to T$ in $\mathcal{L}_{\gamma_\infty}(L^\infty(\mu, X), Y)$ for SOT, so $\{\overline{T}_{m_k} : k \in \mathbb{N}\} \cup \{T\}$ is a compact subset of $\mathcal{L}_{\gamma_\infty}(L^\infty(\mu, X), Y)$ for SOT. Hence by Theorem 3.6 the set $\{m_k : k \in \mathbb{N}\}$ is uniformly variationally μ-continuous. □

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