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Inequalities for Riemann–Liouville operator involving suprema

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Abstract

In the paper we obtain a characterization of an inequality for Riemann–Liouville operator involving suprema in case of nonincreasing weights.

1. Introduction

Let $b \in (0, \infty]$. Denote by $\mathfrak{M}^+$ the class of all nonnegative Lebesgue measurable functions on $(0, b)$. The weighted Riemann–Liouville operator

$$f \mapsto u(s) \int_0^s \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \tag{1}$$

was studied in papers [1, 5, 6, 7, 8], where criteria under some restrictions on the weight functions and relations on parameters p, q of the $L^p - L^q$ boundedness (and in some cases compactness) of the operator (1) was proved.

In various research projects some operators involving suprema have been recently encountered (see [3, 4]). In paper [3] a Hardy-type operator involving suprema was characterized. We study the inequality

$$\left(\int_0^b \left[(R_\alpha f)(x)\right]^q w(x)\,dx \right)^{1/q} \le C \left(\int_0^b f(x)^p\,dx \right)^{1/p}, \quad f \in \mathfrak{M}^+, \tag{2}$$

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where the Riemann–Liouville operator involving suprema R_α is defined by the formula

$$(R_\alpha f)(t) = \sup_{t\le s<b} u(s)\int_0^s \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}},$$

$\alpha \in (0,1)$, $w, v \in \mathfrak{M}^+$, u is a continuous nonnegative function and either u or v is nonincreasing on $(0,b)$.

Put $b_0 := \sup\{s \in (0,b) \,|\, u(s) \neq 0\}$. Since

$$(R_\alpha f)(t) = \sup_{t\le s<b_0} u(s)\int_0^s \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \quad \text{if } \ 0<t<b_0$$

and

$$(R_\alpha f)(t) = 0 \quad \text{if } \ t \in (b_0, b),$$

the inequality (2) is equivalent to the similar inequality with b_0 instead of b. So we assume that $b_0 = b$.

Throughout this paper $A \lesssim B$ and $B \gtrsim A$ means that $A \le cB$, where the constant c depends only on p, q, α and may be different in different places. If both $A \lesssim B$ and $A \gtrsim B$, then we write $A \approx B$.

2. Main results

Lemma 2.1

Let $\alpha \in (0,1)$, $\gamma \in (0,1]$, $[c,d) \subset (0,b)$, and let v be a measurable function such that the function $V(t) := \int_0^t |v(y)(t-y)^{\alpha-1}|^{p'}dy$ is bounded on $[c,d)$, and $f \in L^p(0,d)$. Then the integral $g(t) := \int_0^{\gamma t} f(y)v(y)(t-y)^{\alpha-1}dy$ is continuous from the right on $[c,d)$.

Proof. Fix an arbitrary point $t \in [c,d)$. The boundedness of the function V implies $K := \sup_{x\in[c,d)} |V(x)| < \infty$. Let $\delta > 0$ such that $[t, t+\delta) \subset [c,d)$ and $h \in (0,\delta)$. We have

$$|g(t+h) - g(t)| \le \int_{\gamma t}^{\gamma(t+h)} \frac{|f(y)v(y)|\,dy}{(t+h-y)^{1-\alpha}}$$
$$+ \int_0^{\gamma t} |f(y)v(y)| \Big| \frac{1}{(t-y)^{1-\alpha}} - \frac{1}{(t+h-y)^{1-\alpha}} \Big| dy =: I_1(h) + I_2(h).$$

Applying the Hölder inequality, we find

$$I_1(h) \le \|f\chi_{[\gamma t, \gamma(t+h))}\|_p \Big[\int_0^{t+h} \frac{|v(y)|^{p'}\,dy}{(t+h-y)^{(1-\alpha)p'}} \Big]^{1/p'}$$
$$\le K^{1/p'} \|f\chi_{[\gamma t,\gamma(t+h))}\|_p \to 0, \quad h \to 0.$$

Remark that if $s > x > 0$ and $\lambda \in (0,1)$, then

$$s^\lambda - x^\lambda = s^{\lambda-1}\Big(s - \big(\frac{x}{s}\big)^{\lambda-1} \cdot x\Big) < s^{\lambda-1}(s-x)$$

and by the mean value theorem there exists $\xi \in (x, s)$ such that

$$s^\lambda - x^\lambda = \lambda\,\xi^{\lambda-1}(s-x) > \lambda\,s^{\lambda-1}(s-x).$$

Hence for $y < t$ we have

$$\begin{aligned}\Big|\frac{1}{(t-y)^{1-\alpha}} - \frac{1}{(t+h-y)^{1-\alpha}}\Big| &= \frac{(t+h-y)^{1-\alpha} - (t-y)^{1-\alpha}}{(t-y)^{1-\alpha}(t+h-y)^{1-\alpha}} \\ &\approx \frac{(t+h-y)^{-\alpha}((t+h-y)-(t-y))}{(t-y)^{1-\alpha}(t+h-y)^{1-\alpha}} \\ &= \frac{h}{(t-y)^{1-\alpha}(t+h-y)},\end{aligned}$$

and we get the following estimate of $I_2(h)$

$$I_2(h) \approx \int_0^{\gamma t} \Big[\frac{h}{t+h-y}\Big] \frac{|f(y)v(y)|dy}{(t-y)^{1-\alpha}}.$$

Now for any $y \in (0, t)$ the inequality $|h(t+h-y)^{-1}| \le 1$ holds and $h(t+h-y)^{-1}$ monotonically tends to 0 as $h \to 0_+$. Besides that

$$\int_0^{\gamma t} \frac{|f(y)v(y)|dy}{(t-y)^{1-\alpha}} \le K^{1/p'} \|f\chi_{(0,d)}\|_p.$$

Consequently, by Lebesgue's Dominated Convergence Theorem, $I_2(h) \to 0$ as $h \to 0$. Thus the function g is continuous from the right on $[c, d)$. □

2.1. The case of nonincreasing function v

Let v be a nonnegative nonincreasing function. Since for any $f \in \mathfrak{M}^+$

$$(R_\alpha f)(t) \ge \Big[\sup_{t \le s < b} u(s)s^{\alpha-1}\Big] \int_0^t f(y)v(y)\,dy, \tag{3}$$

then in case of $p \in (0,1)$ by using the result [8, Theorem 2] for integral operator we get that the inequality (2) holds if and only if

$$\operatorname{mes}\Big(\Big\{t \in (0,b) \,\Big|\, w(t)^{1/q}\big[\sup_{t \le s < b} u(s)s^{\alpha-1}\big] \int_0^t v(y)\,dy \ne 0\Big\}\Big) = 0,$$

that is the left-hand side of (2) is equal 0 for any $f \in \mathfrak{M}^+$.

Lemma 2.2

Let

$$\alpha \in (0,1),\ 1 \le p \le \frac{1}{\alpha},\ 0 < q < \infty;\ w \in \mathfrak{M}^+,\ \int_0^t w(y)\,dy > 0$$

for all $t \in (0,b)$, u *be a continuous nonnegative function and* v *be a nonincreasing nonnegative function. Put*

$$b_1 := \sup\big\{s \in (0,b) \,\big|\, v(s) \neq 0\big\},\ b_2 := \inf\Big\{t \in (0,b_1] \,\Big|\, \int_t^{b_1} u(x)\,dx = 0\Big\}$$

and

$$b_3 := \sup\Big\{t \in [b_1,b) \,\Big|\, \int_{b_1}^{t} u(x)\,dx = 0\Big\}.$$

(a) *If* $b_2 > 0$, *then the inequality* (2) *is false.*

(b) *If* $b_2 = 0$ *and* $b_3 > b_1$, *then the inequality* (2) *holds if and only if* $A < \infty$, *where*

$$A := \sup_{x\in(0,b)} \bigg[\Big([\frac{\bar{u}(x)}{x}]^q \int_0^x w(y)\,dy + \int_x^b [\frac{\bar{u}(t)}{t}]^q w(t)\,dt\Big)^{1/q} \Big[\int_0^x v(t)^{p'}\,dt\Big]^{1/p'}\bigg] \tag{4}$$

and $\bar{u}(t) := t \sup_{t\le s<b} u(s)s^{\alpha-1}$.

(c) *If* $b_2 = 0$ *and* $b_1 = b_3$, *then the inequality* (2) *holds if and only if* $\max\{A, A'\} < \infty$, *where*

$$A' := \Big[\int_0^{b_1} w(x)\,dx\Big]^{1/q} \sup_{s\in[b_1,b)} u(s)\Big(\int_0^{b_1} \frac{v(y)^{p'}\,dy}{(s-y)^{(1-\alpha)p'}}\Big)^{1/p'}.$$

Proof. (a) Let the inequality (2) hold and $b_2 > 0$. Then there is the strictly increasing sequence $\{t_k\}_{k=1}^{\infty} \subset (0,b_2)$ such that $u(t_k) \neq 0$ and $\lim_{k\to\infty} t_k = b_2$. Since $(1-\alpha)p' \ge 1$, then $g_k(y) := (t_{k+1}-y)^{\alpha-1}\chi_{(t_k,t_{k+1})}(y)$ does not belong to the $L^{p'}(t_k,t_{k+1})$. Then there exists the function $f_k \in L^p(t_k,t_{k+1})$ such that $\int_{t_k}^{t_{k+1}} f_k(x)g_k(x)dx = \infty$. For instance, if $\alpha < \frac{1}{p}$ we can take $f_k(y) = (t_{k+1}-y)^{-\alpha}\chi_{(t_k,t_{k+1})}(y)$. Consequently, if we put $f := \sum_k 2^{-k} f_k \|f_k\|_p^{-1}$, we get $f \in L^p$ and $(R_\alpha f)(t) = \infty$ for any $t \in (0,b_2)$. Hence $\int_0^{b_2} w(y)\,dy = 0$ and we get contradiction.

(b) If $b_3 = b$, then $u = 0$ a.e. in $(0,b)$ and the statement is clear. Now let $b_3 < b$. Let the inequality (2) hold. The finiteness of the constant A follows directly from [3, Theorem 4.1] in accordance with

$$(R_\alpha f)(t) \ge \sup_{t\le s<b} \frac{u(s)s^\alpha}{s} \int_0^s f(y)v(y)\,dy, \quad f \in \mathfrak{M}^+. \tag{5}$$

Conversely, we have

$$\begin{aligned}(R_\alpha f)(t) &= \sup_{\max\{b_3,t\}\le s<b} u(s)\int_0^{b_1} \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \\ &\le \Big[1-\frac{b_1}{b_3}\Big]^{\alpha-1} \sup_{t\le s<b} \frac{u(s)}{s^{1-\alpha}} \int_0^s f(y)v(y)\,dy,\end{aligned}$$

since $s-y \ge s-b_1 = s(1-\frac{b_1}{s}) \ge s(1-\frac{b_1}{b_3})$. The statement follows from [3, Theorem 4.1].

(c) If $b_1 = b_3 = b$ or $b_1 = b_3 = 0$, then the statement is clear. Now let $0 < b_1 = b_3 < b$.

Necessity. The finiteness of the constant A is proved the same way as in part (b). Besides that, in this case

$$(R_\alpha f)(t) = \sup_{\max\{b_1,t\}\le s<b} u(s) \int_0^{b_1} \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}}, \tag{6}$$

and the inequality (2) implies

$$\Big(\int_0^{b_1} w(x)\,dx\Big)^{1/q} \sup_{b_1\le s<b} u(s) \int_0^{b_1} \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \le C\Big(\int_0^{b_1} f(x)^p\,dx\Big)^{1/p}.$$

Now, the sharpness of the Hölder inequality proves the finiteness of the constant A'.

Sufficiency. Since $A < \infty$, then $\int_0^x w(y)\,dy < \infty$ for any $x \in (0,b)$. There exists a point $b' \in (b_1, b)$ such that $\int_0^{b'} w(x)\,dx < 2\int_0^{b_1} w(x)\,dx$. In accordance with (6) and [3, Theorem 4.1], we have

$$\begin{aligned}\Big(\int_0^{b'} [(R_\alpha f)(x)]^q w(x)\,dx\Big)^{1/q} &\le \Big[\int_0^{b'} w(x)\,dx\Big]^{1/q} \sup_{b_1\le s<b} u(s) \int_0^{b_1} \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \\ &\le 2^{1/q} A' \Big(\int_0^b f(x)^p\,dx\Big)^{1/p},\end{aligned}$$

and

$$\begin{aligned}&\Big(\int_{b'}^b \big[(R_\alpha f)(x)\big]^q w(x)\,dx\Big)^{1/q} \\ &\le \Big[1-\frac{b_1}{b'}\Big]^{\alpha-1} \bigg(\int_{b'}^b \Big(\sup_{t\le s<b} \frac{u(s)}{s^{1-\alpha}} \int_0^s f(y)v(y)\,dy\Big)^q w(x)\,dx\bigg)^{1/q}, \\ &\lesssim \Big[1-\frac{b_1}{b'}\Big]^{\alpha-1} A\Big(\int_0^b f(x)^p\,dx\Big)^{1/p}.\end{aligned}$$

□

We also use the following Chebyshev inequality (see proof, for instance, in book [2, 2.18]).

Lemma 2.3

Let f be nonincreasing and g be nondecreasing nonnegative functions on (c,d), $-\infty < c < d < +\infty$. Then

$$\int_c^d f(x)g(x)\,dx \le \frac{1}{d-c}\int_c^d f(x)\,dx \cdot \int_c^d g(x)\,dx.$$

Corollary 2.4

Let $\alpha \in (0,1)$, $0 < p, q < \infty$; $w, u, \rho \in \mathfrak{M}^+$, $\mathfrak{M}^+_\downarrow$ *be the class of nonincreasing nonnegative functions on* $(0,b)$ *and* $v \in \mathfrak{M}^+_\downarrow$. *Then the inequality*

$$\left[\int_0^b \Big[\sup_{t \le s < b} u(s) \int_0^s \frac{f(y)v(y)dy}{(s-y)^{1-\alpha}}\Big]^q w(x)\,dx\right]^{1/q}$$
$$\le C\Big[\int_0^b f(x)^p \rho(x) dx\Big]^{1/p}, \ f \in \mathfrak{M}^+_\downarrow \tag{7}$$

is equivalent to the inequality

$$\left[\int_0^b \Big[\sup_{t \le s < b} \frac{u(s)}{s^{1-\alpha}} \int_0^s f(y)v(y)dy\Big]^q w(x)dx\right]^{1/q}$$
$$\le C\Big[\int_0^b f(x)^p \rho(x) dx\Big]^{1/p}, \ f \in \mathfrak{M}^+_\downarrow. \tag{8}$$

Proof. It is clear that

$$s^{\alpha-1}\int_0^s f(y)v(y)\,dy \le \int_0^s \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}}.$$

By the Chebyshev inequality we have

$$\int_0^s \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \le s^{-1}\int_0^s f(y)v(y)\,dy \int_0^s \frac{dy}{(s-y)^{1-\alpha}}$$
$$= \frac{s^{\alpha-1}}{\alpha}\int_0^s f(y)v(y)\,dy.$$

□

Thus the criterion of validity of the inequality (8), which was proved in paper [3, Theorem 3.5], is also a criterion of validity of the inequality (7).

Theorem 2.5

Let

$$\alpha \in (0,1), \ \frac{1}{\alpha} < p \le q < \infty; \ w \in \mathfrak{M}^+, \ \int_0^t w(y)\,dy > 0$$

for all $t \in (0,b)$, u *be a continuous nonnegative function and* v *be a nonincreasing nonnegative function. Then the inequality* (2) *holds if and only if* $A < \infty$, *where* A *is defined in* (4).

Proof. Necessity follows from [3, Theorem 4.1], since the estimate (5) is true.

Sufficiency. Since $A < \infty$, then $\int_0^x w(y)\,dy < \infty$ for any $x \in (0,b)$. If there exists $x \in (0,b)$ such that $\int_0^x v(y)^{p'}\,dy = \infty$, then $\int_0^t v(y)^{p'}\,dy = \infty$ for all $t \in (0,b)$ because of monotonicity of function v. Hence, the finiteness of A implies that $w(t)^{1/q}(R_\alpha f)(t) = 0$ for arbitrary $t \in (0,b)$ and $f \in \mathfrak{M}^+$. So in this case the inequality (2) holds.

Let $\int_0^t v(y)^{p'}\,dy<\infty$ for all $t\in(0,b)$. In particular, it implies that integral

$$g(t):=\int_0^{\gamma t} f(y)v(y)(t-y)^{\alpha-1}dy,\ \gamma\in(0,1]$$

of a function $f\in L^p(0,b)$ is continuous from the right on $(0,b)$, since for any $[c,d)\subset(0,b)$, by Lemma 2.3,

$$\int_0^t \frac{v(y)^{p'}\,dy}{(t-y)^{(1-\alpha)p'}}\lesssim t^{(\alpha-1)p'}\int_0^t v(y)^{p'}\,dy\le c^{(\alpha-1)p'}\int_0^d v(y)^{p'}\,dy<\infty.$$

Then for nonnegative $f\in L^p(0,b)$ we have

$$\int_0^b[(R_\alpha f)(x)]^q w(x)\,dx\lesssim\int_0^b w(x)\Big[\sup_{x\le s<b}u(s)\int_{s/2}^s\frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}}\Big]^q dx$$
$$+\int_0^b w(x)\Big[\sup_{x\le s<b}\frac{u(s)s^\alpha}{s}\int_0^s f(y)v(y)\,dy\Big]^q dx=:I_1+I_2.$$

The estimate $I_2\lesssim A^q\|f\|_p^q$ follows from [3, Theorem 4.1].

Put

$$N:=\begin{cases}\inf\big\{k\in\mathbb{Z}\,|\,2^k\ge b\big\}, & \text{if } b<\infty,\\ \infty, & \text{otherwise.}\end{cases}$$

Then $I_1\lesssim I_{11}+I_{12}$, where

$$I_{11}=\sum_{k<N}\int_{2^k}^{2^{k+1}} w(t)\sup_{t\le s<2^{k+1}}\Big(u(s)\int_{s/2}^s\frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}}\Big)^q dt,$$

$$I_{12}=\sum_{k<N}\int_{2^k}^{2^{k+1}} w(t)\,dt\sup_{2^{k+1}\le s<b}\Big(u(s)\int_{s/2}^s\frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}}\Big)^q.$$

Applying the Hölder inequality, Lemma 2.3 and monotonicity of function v, we obtain

$$I_{11}\le\sum_{k<N}\int_{2^k}^{2^{k+1}} w(t)\sup_{t\le s<2^{k+1}}\left[u(s)\Big[\int_{s/2}^s\frac{v(y)^{p'}\,dy}{(s-y)^{(1-\alpha)p'}}\Big]^{1/p'}\right]^q dt\Big[\int_{2^{k-1}}^{2^{k+1}} f(y)^p dy\Big]^{q/p}$$
$$\lesssim\sum_{k<N}\int_{2^k}^{2^{k+1}} w(t)\sup_{t\le s<2^{k+1}}\left[u(s)s^{\alpha-1}\Big[\int_{s/2}^s v(y)^{p'}dy\Big]^{1/p'}\right]^q dt\Big[\int_{2^{k-1}}^{2^{k+1}} f(y)^p dy\Big]^{q/p}$$
$$\le\sum_{k<N}\int_{2^k}^{2^{k+1}} w(t)\Big[\sup_{t\le s<2^{k+1}}u(s)s^{\alpha-1}\Big]^q dt\Big[\int_0^{2^k} v(y)^{p'}dy\Big]^{q/p'}\Big[\int_{2^{k-1}}^{2^{k+1}} f(y)^p dy\Big]^{q/p}.$$

Hence $I_{11}\lesssim A^q\|f\|_p^q$.

Moreover,

$$\begin{aligned}
I_{12} &= \sum_{k<N} \int_{2^k}^{2^{k+1}} w(t)\,dt \sup_{k+1\le i<N} \sup_{2^i\le s<2^{i+1}} \Big(u(s) \int_{s/2}^{s} \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \Big)^q \\
&\le \sum_{k<N} \int_{2^k}^{2^{k+1}} w(t)\,dt \sum_{k+1\le i<N} \sup_{2^i\le s<2^{i+1}} \Big(u(s) \int_{s/2}^{s} \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \Big)^q \\
&= \sum_{i<N} \int_{0}^{2^i} w(t)\,dt \sup_{2^i\le s<2^{i+1}} \Big(u(s) \int_{s/2}^{s} \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \Big)^q \\
&\lesssim \sum_{i<N} \int_{0}^{z_i} w(t)\,dt \Big(u(z_i) \int_{z_i/2}^{z_i} \frac{f(y)v(y)\,dy}{(z_i-y)^{1-\alpha}} \Big)^q,
\end{aligned}$$

where z_i is a point in $(2^i, 2^{i+1}]$ such that

$$\sup_{2^i\le s<2^{i+1}} \Big(u(s) \int_{s/2}^{s} \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \Big)^q \le 2\Big(u(z_i) \int_{z_i/2}^{z_i} \frac{f(y)v(y)\,dy}{(z_i-y)^{1-\alpha}} \Big)^q.$$

Applying the Hölder inequality, Lemma 2.3 and monotonicity of function v, we find

$$\begin{aligned}
I_{12} &\lesssim \sum_{i<N} \int_{0}^{z_i} w(t)\,dt \Big[\int_{2^{i-1}}^{2^{i+1}} f(y)^p\,dy \Big]^{q/p} \big(u(z_i) z_i^{\alpha-1}\big)^q \Big[\int_0^{z_i} v(y)^{p'}\,dy \Big]^{q/p'} \\
&\lesssim A^q \|f\|_p^q.
\end{aligned}$$

Thus the theorem is proved. □

2.2. The case of nonincreasing function u

Remark that, since u is a nonincreasing function, the assumption $b_0 = b$ (which we made in the Introduction) implies that $u(t) > 0$ for all $t \in (0, b)$.

By using the ideas from proof of the Theorem 4.1 of the paper [3] we get the following result.

Denote by S the class of all strictly increasing sequences $\{x_k\}_{k=n_1}^{k=n_2} \subset [0, b]$, where $n_1, n_2 \in \mathbb{Z} \cup \{\pm\infty\}$, $n_1 < n_2$, such that $[0, b] = \overline{\cup_{k=n_1}^{k=n_2-1} (x_k, x_{k+1})}$.

Theorem 2.6

Let $\alpha \in (0, 1)$, $1 < p < \infty$, $0 < q < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$; $w, v \in \mathfrak{M}^+$, u *be a continuous and nonincreasing nonnegative function. If* $p \le q$ *then the inequality* (2) *holds if and only if the inequality*

$$\left(\int_0^b w(x) u(x)^q \Big(\int_0^x \frac{f(y)v(y)\,dy}{(x-y)^{1-\alpha}} \Big)^q dx \right)^{1/q} \le C \Big(\int_0^b f(x)^p\,dx \Big)^{1/p}, \quad f \in \mathfrak{M}^+, \tag{9}$$

holds and $B < \infty$, *where*

$$B := \sup_{t\in(0,b)} B(t) := \sup_{t\in(0,b)} u(t) \Big[\int_0^t w(y)\,dy \Big]^{1/q} \Big[\int_0^t \frac{v(y)^{p'}\,dy}{(t-y)^{(1-\alpha)p'}} \Big]^{1/p'}.$$

If $p > q$ then the inequality (2) holds if and only if the inequality (9) holds and $D < \infty$, where

$$D := \sup_{\{x_k\}\in S} \left[\sum_k \Big[\int_{x_k}^{x_{k+1}} w(t)dt \Big]^{r/q} u(x_{k+1})^r \Big[\int_{x_k}^{x_{k+1}} \frac{v(y)^{p'}\,dy}{(x_{k+1}-y)^{(1-\alpha)p'}} \Big]^{r/p'} \right]^{1/r}.$$

Remark. The following simple estimate $\sup_{t\in(0,b)} B(t) \le D$ we will use in the proof of the theorem. For proof of this fact for arbitrary $t \in (0,b)$ we take the sequence $x_1 = 0$, $x_2 = t$ and $x_3 = b$.

Proof. Necessity. Fix an arbitrary $t \in (0, b)$. If $\int_0^t v(y)^{p'}(t-y)^{(\alpha-1)p'}\,dt = 0$ or $\int_0^t w(x)\,dx = 0$ or $u(t) = 0$ then $B(t) = 0 \le C$. Now let $\int_0^t w(x)\,dx > 0$, $u(t) > 0$ and $\int_0^t v(y)^{p'}(t-y)^{(\alpha-1)p'}\,dt > 0$. We take a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ such that $\gamma_n \downarrow 0$ as $n\to\infty$, $t+\gamma_n < b$, $n \in \mathbb{N}$ and $u(t+\gamma_1) > 0$ (see remark in the beginning of the Section 2.2). Substituting the function $f_t(y) = \min\{n, v(y)\}^{p'-1}(t+\gamma_n-y)^{(\alpha-1)(p'-1)}\chi_{(0,t)}(y)$ into (2), we obtain

$$\begin{aligned} C\|f_t\|_p &\ge \Big(\int_0^b [(R_\alpha f_t)(x)]^q w(x)\,dx \Big)^{1/q} \\ &\ge \Big(\int_0^t w(x)\,dx \Big)^{1/q} \sup_{t+\gamma_n\le s<b} u(s) \int_0^t \frac{\min\{n, v(y)\}^{p'}\,dy}{(s-y)^{1-\alpha}(t+\gamma_n-y)^{(1-\alpha)(p'-1)}} \\ &\ge u(t+\gamma_n)\Big(\int_0^t w(x)\,dx \Big)^{1/q} \int_0^t \frac{\min\{n, v(y)\}^{p'}\,dt}{(t+\gamma_n-y)^{(1-\alpha)p'}}. \end{aligned}$$

Since

$$\|f_t\|_p = \Big(\int_0^t \frac{\min\{n, v(y)\}^{p'}\,dy}{(t+\gamma_n-y)^{(1-\alpha)p'}} \Big)^{1/p} < \infty,$$

we have

$$C \ge u(t+\gamma_n)\Big(\int_0^t w(x)\,dx \Big)^{1/q} \Big(\int_0^t \frac{\min\{n, v(y)\}^{p'}\,dt}{(t+\gamma_n-y)^{(1-\alpha)p'}} \Big)^{1/p'}. \tag{10}$$

The Monotone Convergence Theorem implies that

$$\Big(\int_0^t \frac{\min\{n, v(y)\}^{p'}\,dt}{(t+\gamma_n-y)^{(1-\alpha)p'}} \Big)^{1/p'} \to \Big(\int_0^t \frac{v(y)^{p'}\,dt}{(t-y)^{(1-\alpha)p'}} \Big)^{1/p'} \quad \text{as } n\to\infty.$$

From this result, relation (10) and continuity of function u we get $C \ge B(t)$. Since also

$$(R_\alpha f)(t) \ge u(t) \int_0^t \frac{f(y)v(y)}{(s-y)^{1-\alpha}}\,dy, \quad f \in \mathfrak{M}^+,\ t \in (0,b), \tag{11}$$

the necessity is proved in the case $p \le q$.

Now let $q < p$. Fix any sequence $\{x_k\} \in S$ and for $n \in \mathbb{N}$ put

$$V_k := \int_{x_k}^{x_{k+1}} \frac{v(y)^{p'}\,dy}{(x_{k+1}-y)^{(1-\alpha)p'}},$$

$$g_n(y) = \sum_{|k|<n} u(x_{k+1})^{r/p} V_k^{r/(q'p)} \Big(\int_{x_k}^{x_{k+1}} w(t)\,dt \Big)^{r/(qp)} \frac{v(y)^{p'-1}\chi_{[x_k,x_{k+1})}(y)}{(x_{k+1}-y)^{(1-\alpha)(p'-1)}}.$$

Then

$$\|g_n\|_p^p = \sum_{|k|<n} \Big(\int_{x_k}^{x_{k+1}} w(t)\,dt \Big)^{r/q} u(x_{k+1})^r V_k^{r/p'} < \infty$$

and

$$\begin{aligned} \|(R_\alpha g_n) w^{1/q}\|_q^q &\geq \sum_{|k|<n} \int_{x_k}^{x_{k+1}} w(t)\,dt \Big[\sup_{x_{k+1} \leq s < b} u(s) \int_{x_k}^{x_{k+1}} \frac{g_n(y)v(y)\,dy}{(s-y)^{1-\alpha}} \Big]^q \\ &\geq \sum_{|k|<n} u(x_{k+1})^q \int_{x_k}^{x_{k+1}} w(t)\,dt \Big[\int_{x_k}^{x_{k+1}} \frac{g_n(y)v(y)\,dy}{(x_{k+1}-y)^{1-\alpha}} \Big]^q \\ &= \sum_{|k|<n} \Big(\int_{x_k}^{x_{k+1}} w(t)\,dt \Big)^{r/q} u(x_{k+1})^r V_k^{r/p'}. \end{aligned}$$

Hence, $C \geq D$.

Inequality (9) follows from (2) and (11).

Sufficiency. If $\int_0^b w(y)\,dy = 0$ then the inequality (2) holds. Now let $\int_0^b w(y)\,dy > 0$. Put

$$N := \begin{cases} \inf \Big\{ k \in \mathbb{Z} \,|\, 2^k \geq \int_0^b w(x)\,dx \Big\}, & \text{if } \int_0^b w(x)\,dx < \infty, \\ \infty, & \text{otherwise,} \end{cases}$$

and construct the sequence $\{a_k\}_{k \leq N}$ satisfying $\int_0^{a_k} w(x)\,dx = 2^k$, $k < N$; $a_N = b$. Remark that for arbitrary $k < N$

$$\sup_{t \in [a_k, a_{k+1}]} \int_0^t |v(y)(t-y)^{\alpha-1}|^{p'} dy \leq B^{p'} \Big(\int_0^{a_k} w(y)\,dy \Big)^{-p'/q} u(a_{k+1})^{-p'} < \infty.$$

Hence, by Lemma 2.1, for any $f \in L^p(0,b)$, the Riemann–Liouville integral $\int_0^t f(y)v(y)(t-y)^{\alpha-1} dy$ is bounded on $[a_k, a_{k+1}]$ and it is continuous from the right on $[a_k, a_{k+1})$.

Fix a nonnegative function $f \in L^p(0,b)$. We have

$$\begin{aligned} \int_0^b [(R_\alpha f)(x)]^q w(x)\,dx &\leq \sum_{k<N} \int_{a_k}^{a_{k+1}} w(t)\,dt \Big(\sup_{a_k \leq s < b} u(s) \int_0^s \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \Big)^q \\ &= \sum_{k<N} 2^k \Big(\sup_{k \leq i < N} \sup_{a_i \leq s < a_{i+1}} u(s) \int_0^s \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \Big)^q \\ &\leq \sum_{k<N} 2^k \sum_{k \leq i < N} \sup_{a_i \leq s < a_{i+1}} \Big(u(s) \int_0^s \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \Big)^q \\ &\approx \sum_{i<N} 2^{i-1} \sup_{a_i \leq s < a_{i+1}} \Big(u(s) \int_0^s \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \Big)^q \\ &\lesssim \sum_{i<N} \int_{a_{i-1}}^{a_i} w(t)\,dt \Big(u(z_i) \int_0^{z_i} \frac{f(y)v(y)\,dy}{(z_i-y)^{1-\alpha}} \Big)^q, \end{aligned}$$

where z_i is a point in $(a_i, a_{i+1}]$ such that

$$\sup_{a_i \le s < a_{i+1}} \Big(u(s) \int_0^s \frac{f(y)v(y)\,dy}{(s-y)^{1-\alpha}} \Big)^q \le 2 \Big(u(z_i) \int_0^{z_i} \frac{f(y)v(y)\,dy}{(z_i-y)^{1-\alpha}} \Big)^q.$$

The existence of such a point $z_i \in (a_i, a_{i+1}]$ follows from the continuity from the right of the Riemann–Liouville integral. Consequently,

$$\int_0^b [(R_\alpha f)(x)]^q w(x)\,dx \lesssim \sum_{i<N} \int_{z_{i-2}}^{z_i} w(t)\,dt \Big(u(z_i) \int_0^{z_i} \frac{f(y)v(y)\,dy}{(z_i-y)^{1-\alpha}} \Big)^q \lesssim I_1 + I_2, \quad (12)$$

where

$$I_1 := \sum_{i<N} \int_{z_{i-2}}^{z_i} w(t)\,dt \Big(u(z_i) \int_{z_{i-2}}^{z_i} \frac{f(y)v(y)\,dy}{(z_i-y)^{1-\alpha}} \Big)^q,$$

$$\begin{aligned} I_2 &:= \sum_{i<N} \int_{z_{i-2}}^{z_i} w(t)\,dt \Big(u(z_i) \int_0^{z_{i-2}} \frac{f(y)v(y)\,dy}{(z_i-y)^{1-\alpha}} \Big)^q \\ &\le \int_0^b w(x)u(x)^q \Big(\int_0^x \frac{f(y)v(y)\,dy}{(x-y)^{1-\alpha}} \Big)^q dx. \end{aligned}$$

Applying the Hölder inequality, we obtain

$$I_1 \le \sum_{i<N} \int_{z_{i-2}}^{z_i} w(t)\,dt \Big[\int_{z_{i-2}}^{z_i} f(y)^p\,dy \Big]^{q/p} u(z_i)^q \Big[\int_{z_{i-2}}^{z_i} \frac{v(y)^{p'}\,dy}{(z_i-y)^{(1-\alpha)p'}} \Big]^{q/p'}. \quad (13)$$

Now, using the Jensen inequality in case of $p \le q$, we get the estimate $I_1 \lesssim B^q \|f\|_p^q$.

If $q < p$ then applying the Hölder inequality with exponents $\frac{r}{q}$, $\frac{p}{q}$ in formula (13) we obtain

$$\begin{aligned} I_1 &\le \Bigg[\sum_{i<N} \Big[\int_{z_{i-2}}^{z_i} w(t)\,dt \Big]^{r/q} u(z_i)^r \Big[\int_{z_{i-2}}^{z_i} \frac{v(y)^{p'}\,dy}{(z_i-y)^{(1-\alpha)p'}} \Big]^{r/p'} \Bigg]^{q/r} \\ &\quad \times \Big[\sum_{i<N} \int_{z_{i-2}}^{z_i} f(y)^p\,dy \Big]^{q/p} \lesssim \Big(\sum_{\substack{i<N \\ |i|\,\text{odd}}} + \sum_{\substack{i<N \\ |i|\,\text{even}}} \Big)^{q/r} \|f\|_p^q \lesssim D^q \|f\|_p^q. \end{aligned}$$

□

Theorem 2.7

Let $\alpha \in (0,1)$, $1 < p < \infty$, $0 < q < p < \infty$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$; $w, v \in \mathfrak{M}^+$, u be a continuous and nonincreasing nonnegative function. Then the inequality (2) *holds if and only if the inequality* (9) *holds,* $\sup_{t \in (0,b)} B(t) < \infty$ *and* $\mathcal{D} < \infty$, *where*

$$\mathcal{D} := \Bigg(\int_0^b w(t) \sup_{t \le s < b} \Big\{ u(s)^r \Big(\int_t^s w(y)\,dy \Big)^{r/p} \Big(\int_t^s \frac{v(y)^{p'}\,dy}{(s-y)^{(1-\alpha)p'}} \Big)^{r/p'} \Big\} dt \Bigg)^{1/r}.$$

Proof. Sufficiency. Fix a nonnegative function $f \in L^p(0,b)$. Using the same arguments as in proof of Theorem 2.6, we get the estimate (12). Applying the Hölder inequality for integral with exponents p, p' and the Hölder inequality for sum with exponents $\frac{r}{q}$, $\frac{p}{q}$, we obtain

$$I_1 \le \sum_{i<N} \Big[\int_{z_{i-2}}^{z_i} f(y)^p\,dy\Big]^{q/p} u(z_i)^q \Big[\int_{z_{i-2}}^{z_i} w(t)\,dt\Big]\Big[\int_{z_{i-2}}^{z_i} \frac{v(y)^{p'}\,dy}{(z_i-y)^{(1-\alpha)p'}}\Big]^{q/p'}$$

$$\lesssim \left[\sum_{i<N} u(z_i)^r \Big[\int_{z_{i-2}}^{z_i} w(t)\,dt\Big]^{r/q}\Big[\int_{z_{i-2}}^{z_i} \frac{v(y)^{p'}\,dy}{(z_i-y)^{(1-\alpha)p'}}\Big]^{r/p'}\right]^{q/r} \|f\|_p^q$$

$$\lesssim \left[\sum_{i<N} \int_{z_{i-4}}^{z_{i-2}} w(t)dt \Big\{u(z_i)^r \Big[\int_{z_{i-2}}^{z_i} w(y)dy\Big]^{r/p}\Big[\int_{z_{i-2}}^{z_i} \frac{v(y)^{p'}\,dy}{(z_i-y)^{(1-\alpha)p'}}\Big]^{r/p'}\Big\}\right]^{q/r} \|f\|_p^q$$

$$\lesssim \left[\sum_{i<N} \int_{z_{i-4}}^{z_{i-2}} w(t) \Big\{u(z_i)^r \Big[\int_{t}^{z_i} w(y)dy\Big]^{r/p}\Big[\int_{t}^{z_i} \frac{v(y)^{p'}\,dy}{(z_i-y)^{(1-\alpha)p'}}\Big]^{r/p'}\Big\}dt\right]^{q/r} \|f\|_p^q.$$

Hence $I_1 \lesssim \mathcal{D}^q \|f\|_p^q$.

Necessity. By Theorem 2.6 the constant D is finite. Since $\sup\limits_{t\in(0,b)} B(t) \le D$ (see Remark after Theorem 2.6), then the equality $\int_0^t w(y)\,dy = \infty$ for some $t \in (0,b)$ implies $v = 0$ a.e. on $(0,b)$ and $\mathcal{D} = 0$. Also if $\int_0^b w(y)\,dy = 0$ then $\mathcal{D} = 0$. We show that finiteness of the constant D implies finiteness $\mathcal{D}$. Let

$$a_0 := \sup\Big\{t \in (0,b) \mid \int_0^{a_0} w(y)\,dy = 0\Big\},$$

$$h(s,t) := u(s)^r \Big(\int_t^s w(y)\,dy\Big)^{r/p}\Big(\int_t^s \frac{v(y)^{p'}\,dy}{(s-y)^{(1-\alpha)p'}}\Big)^{r/p'},$$

$$\tilde{b} := \sup\Big\{t \in (0,b) \mid \sup_{t\le s<b} h(s,t) > 0\Big\}$$

and $a_0 < \tilde{b}$. Fix an arbitrary $a \in (a_0, \tilde{b})$ and $\varepsilon > 0$. Then, for any $t \in [a, \tilde{b})$,

$$0 < \sup_{t\le s<b} h(s,t) \le \sup_{t\le s<b} B(s)^r \Big(\int_0^s w(y)\,dy\Big)^{-1} \le D^r \Big[\int_0^a w(y)\,dy\Big]^{-1} < \infty.$$

Put $a_1 := a$, $c_1 := a$ for $k \in \mathbb{N}$ we take

$$a_{k+1} := \sup\Big\{t \in (a_k, \tilde{b}] \mid \sup_{a_k\le s<b} h(s,a_k) \le 2h(t,a_k)\Big\},$$

and $c_{k+1} \in (a_k, a_{k+1}]$ such that

$$\sup_{a_k\le s<b} h(s,a_k) \le 2h(c_{k+1},a_k) \quad\text{and}\quad \int_{c_{k+1}}^{a_{k+1}} w(y)\,dy < \varepsilon\Big(2^k \sup_{a_k\le s<b} h(s,a_k)\Big)^{-1}.$$

There are two opportunities: 1) there exists a number $N \in \mathbb{N}$ such that $a_N = \tilde{b}$ or 2) $a_k < \tilde{b}$ for any $k \in \mathbb{N}$ and in this case we put $N = \infty$ and $\tilde{a} := \lim\limits_{k\to\infty} a_k$.

Let $N = \infty$. By the definition of sequence $\{a_k\}_{k\in\mathbb{N}}$ the inequality

$$h(t, a_k) \le 2^{-1} \sup_{a_k \le s < b} h(s, a_k)$$

holds for any $t > a_{k+1}$. Together with the facts that the function $s \mapsto h(s, a_{k+1})$ is right continuous and that the function $t \mapsto h(s, t)$ is non-increasing, the last inequality implies that

$$\begin{aligned}\sup_{a_{k+1} \le s < b} h(s, a_{k+1}) &= \sup_{a_{k+1} < s < b} h(s, a_{k+1}) \\ &\le \sup_{a_{k+1} < s < b} h(s, a_k) \le 2^{-1} \sup_{a_k \le s < b} h(s, a_k).\end{aligned}$$

Hence,

$$\sup_{\tilde{a} \le s < b} h(s, \tilde{a}) \le \sup_{a_k \le s < b} h(s, a_k) \le 2^{-(k-1)} \sup_{a_1 \le s < b} h(s, a_1) \to 0 \quad \text{as } k \to \infty$$

and $\tilde{a} = \tilde{b}$.

In both cases we have

$$\begin{aligned}I(a) &:= \int_a^b w(t) \sup_{t \le s < b} h(s, t)\, dt = \int_a^{\tilde{b}} w(t) \sup_{t \le s < b} h(s, t)\, dt \\ &\le \sum_{1 \le k < N} \int_{a_k}^{a_{k+1}} w(y)\, dy \sup_{a_k \le s < b} h(s, a_k) = I_1 + I_2,\end{aligned}$$

where

$$\begin{aligned}I_1 &:= \sum_{1 \le k < N} \int_{a_k}^{c_{k+1}} w(y)\, dy \sup_{a_k \le s < b} h(s, a_k), \\ I_2 &:= \sum_{1 \le k < N} \int_{c_{k+1}}^{a_{k+1}} w(y)\, dy \sup_{a_k \le s < b} h(s, a_k) \le \sum_{1 \le k < N} \varepsilon\, 2^{-k} \le \varepsilon.\end{aligned}$$

Besides that,

$$\begin{aligned}I_1 &\le 2 \sum_{1 \le k < N} \int_{a_k}^{c_{k+1}} w(y)\, dy\, h(c_{k+1}, a_k) \\ &\le 2 \sum_{1 \le k < N} \int_{c_k}^{c_{k+1}} w(y)\, dy\, h(c_{k+1}, c_k) \le 2D^r.\end{aligned}$$

Thus, for arbitrary $\varepsilon > 0$ we obtain $I(a) \le 2D^r + \varepsilon$ that is $I(a) \le 2D^r$. Since $\mathcal{D} = \lim\limits_{a \to a_0 + 0} I(a)^{1/r}$, we get the estimate $\mathcal{D} \le 2^{1/r} D$. □

Using results of the papers [1, 5, 6, 7, 8], where the inequality (9) was characterized, and Theorems 2.6, 2.7 we obtain criteria of validity of the inequality (2). For example, if w is a nonincreasing nonnegative function, by using [8, Theorem 5], we get the following ctiterion.

Theorem 2.8

Let $1 < p \le q < \infty$, $1 - \frac{p}{q} < \alpha < 1$, $1 - \gamma := \frac{(1-\alpha)q}{p}$; $v \in \mathfrak{M}^+$, *let* u, w *be nonincreasing nonnegative functions and* u *be a continuous function, also. Then the inequality* (2) *holds if and only if*

$$\int_0^t \frac{v(y)^{p'}dy}{(t-y)^{1-\alpha}} < +\infty \quad \text{for almost all} \quad t \in (0,\, b);$$

$$\operatorname*{ess\,sup}_{0<t<b} \left[\int_0^t \frac{w(x)u(x)^q\,dx}{(t-x)^{1-\gamma}} \Big[\int_0^x \frac{v(y)^{p'}dy}{(x-y)^{1-\alpha}} \Big]^q \right]^{1/q} \Big[\int_0^t \frac{v(y)^{p'}dy}{(t-y)^{1-\alpha}} \Big]^{-1/p} < \infty;$$

and $B < \infty$.

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