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Weighted estimates for the averaging integral operator

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Abstract

Let $1 < p \le q < +\infty$ and let v, w be weights on $(0, +\infty)$ satisfying:

$(\star)$ $v(x)x^\rho$ is equivalent to a non-decreasing function on $(0, +\infty)$ for some $\rho \ge 0$

and

$$[w(x)x]^{1/q} \approx [v(x)x]^{1/p} \quad \text{for all } x \in (0, +\infty).$$

We prove that if the averaging operator $(Af)(x) := \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t$, $x \in (0, +\infty)$, is bounded from the weighted Lebesgue space $L^p((0, +\infty); v)$ into the weighted Lebesgue space $L^q((0, +\infty); w)$, then there exists $\varepsilon_0 \in (0, p-1)$ such that the operator A is also bounded from the space $L^{p-\varepsilon}((0, +\infty); v(x)^{1+\delta}x^\gamma)$ into the space $L^{q-\varepsilon q/p}((0, +\infty); w(x)^{1+\delta}x^{\delta(1-q/p)}x^{\gamma q/p})$ for all $\varepsilon, \delta, \gamma \in [0, \varepsilon_0)$. Conversely, assuming that the operator

$$A : L^{p-\varepsilon}((0, +\infty); v(x)^{1+\delta}x^\gamma) \to L^{q-\varepsilon q/p}((0, +\infty); w(x)^{1+\delta}x^{\delta(1-q/p)}x^{\gamma q/p})$$

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is bounded for some $\varepsilon \in [0, p-1)$, $\delta \geq 0$ and $\gamma \geq 0$, we prove that the operator A is also bounded from the space $L^p((0,+\infty); v)$ into the space $L^q((0,+\infty); w)$. In particular, our results imply that the class of weights v for which ($\star$) holds and the operator A is bounded on the space $L^p((0,+\infty); v)$ possesses properties similar to those of the A_p-class of B. Muckenhoupt.

1. Introduction

Let $1 < p < +\infty$ and let v be a weight on $(0,+\infty)$, i.e., a measurable function which is positive a.e. on $(0,+\infty)$. By $L^p(v) \equiv L^p((0,+\infty); v)$ we denote the weighted Lebesgue space of all measurable functions f on $(0,+\infty)$ for which the norm

$$\|f\|_{p,v} = \Big(\int_0^{+\infty} |f(x)|^p v(x)\,\mathrm{d}x\Big)^{1/p}$$

is finite.

We shall consider one of very important operators in the mathematical analysis, the averaging operator A defined by

$$(Af)(x) := \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t, \quad x \in (0,+\infty).$$

It is well known (see [2, 9]) that if $1 < p < +\infty$ and w, v are weights on $(0,+\infty)$, then the averaging operator $A : L^p(v) \to L^q(w)$ is bounded if and only if

$$B := \sup_{r>0}\Big(\int_r^{+\infty} w(t)t^{-q}\,\mathrm{d}t\Big)^{1/q}\Big(\int_0^r v(t)^{1-p'}\,\mathrm{d}t\Big)^{1/p'} < +\infty, \tag{1}$$

where $p' = p/(p-1)$.

Throughout the paper we use the following convention: For two non-negative expressions (i.e. functions or functionals) F and G the symbol $F \lesssim G$ (or $F \gtrsim G$) means that $F \leq cG$ (or $cF \geq G$), where c is a positive constant independent of appropriate quantities involved in F and G. We shall write $F \approx G$ (and say that F and G are equivalent) if both relations $F \lesssim G$ and $F \gtrsim G$ hold.

Our main results are the following two theorems.

Theorem 1.1

Let $1 < p \leq q < +\infty$ and let v, w be weights on $(0,+\infty)$ such that:

$$v(x)x^\rho \text{ is equivalent to a non-decreasing function on } (0,+\infty) \text{ for some } \rho \geq 0; \tag{2}$$

$$[w(x)x]^{1/q} \approx [v(x)x]^{1/p} \quad \text{for all } x \in (0,+\infty). \tag{3}$$

Assume that the averaging operator $A : L^p(v) \to L^q(w)$ is bounded. Then there exists $\varepsilon_0 \in (0, p-1)$ such that the operator

$$A : L^{p-\varepsilon}(v(x)^{1+\delta}x^\gamma) \to L^{q-\varepsilon q/p}(w(x)^{1+\delta}x^{\delta(1-q/p)}x^{\gamma q/p})$$

is also bounded for all $\varepsilon, \delta, \gamma \in [0, \varepsilon_0)$.

Theorem 1.2

Let $1 < p \leq q < +\infty$ and let v, w be weights on $(0,+\infty)$ such that (2) *and* (3) *hold. Assume that the averaging operator*

$$A : L^{p-\varepsilon}\big(v(x)^{1+\delta}x^{\gamma}\big) \to L^{q-\varepsilon q/p}\big(w(x)^{1+\delta}x^{\delta(1-q/p)}x^{\gamma q/p}\big)$$

is bounded for some $\varepsilon \in [0, p-1)$, $\delta \geq 0$ and $\gamma \geq 0$. Then the operator $A : L^p(v) \to L^q(w)$ is also bounded.

Remark 1.3 Assumptions of Theorem 1.1 (or Theorem 1.2) ensure that

$$\Big(\int_r^{+\infty} w(t)t^{-q}\,\mathrm{d}t\Big)^{1/q}\Big(\int_0^r v(t)^{1-p'}\,\mathrm{d}t\Big)^{1/p'} \approx 1 \quad \text{for all } r > 0,$$

which means that (w, v) is the optimal couple of weights for which (1) holds.

Note also that assumption (3) is satisfied when $w = v$ and $q = p$.

Theorem 1.1 is a particular case of the following assertion.

Theorem 1.4

Let $1 < p \leq q < +\infty$ and let v, w be weights on $(0,+\infty)$ such that (2) *and* (3) *hold. Assume that the averaging operator $A : L^p(v) \to L^q(w)$ is bounded. Then there exist $p_0 \in (1,p)$ and $\varepsilon_0 > 0$ such that the operator*

$$A : L^P\big(v(x)^{1+\delta}x^{\gamma}\big) \to L^Q\big(w(x)^{1+\delta}x^{\delta(1-Q/P)}x^{\gamma Q/P}\big)$$

is also bounded for all $P \in (p_0, +\infty)$ and for every $\delta, \gamma \in [0, \varepsilon_0)$, where $Q = Pq/p$.

Remark 1.5 If $1 < p < +\infty$ and v is a weight on $(0,+\infty)$, then we write $v \in M_p$ when the averaging operator A is bounded on the space $L^p(v)$, that is, when (1) holds with $q = p$ and $w = v$. Let A_p, $1 < p < +\infty$, be the A_p-class of B. Muckenhoupt of those weights v on $(0,+\infty)$ for which the Hardy-Littlewood maximal operator associated with the interval $(0,+\infty)$ is bounded on the space $L^p(v)$. Recall that $A_p \subset M_p$. Denote by C_p, $1 < p < +\infty$, the C_p-class of Calderón (introduced in [1]) of those weights v on $(0,+\infty)$ for which both the operator A and its adjoint operator A' are bounded on the space $L^p(v)$.

If (2) holds with $\rho = 0$, then v is equivalent to a non-decreasing function on $(0,+\infty)$. It is known (*cf.* [4, Theorem 6.1] or [3, Proposition 2.3]) that a non-decreasing weight v satisfies $v \in M_p$ if and only if it belongs to the A_p-class. Moreover, it can be shown that a non-decreasing weight v from the class M_p also belongs to the C_p-class. Since

$$\begin{aligned}
&v \in A_p \Longrightarrow v \in A_{p-\varepsilon} && \text{for some } \varepsilon \in (0, p-1),\\
&v \in A_p \Longrightarrow v^{1+\varepsilon} \in A_p && \text{for some } \varepsilon > 0,\\
&v \in A_p \Longrightarrow v \in A_q && \text{for all } q \in [p, +\infty],\\
&v \in C_p \Longrightarrow v(x)x^{\varepsilon} \in M_p && \text{for some } \varepsilon > 0
\end{aligned}$$

(*cf.* [6, 5] for the first three implications, and [1, Proposition 2.4] for the last one), Theorem 1.4 with $\rho = 0$ also follows from properties of weights $v \in A_p \cap C_p$. (This is clear if, in addition, $p = q$ in Theorem 1.4. If $p < q$, one can show that it is again true due to condition (3).)

On the other hand, there are weights in the M_p-class which satisfy (2) but which do not belong to $A_p \cap C_p$. A simple example is $v(t) = t^\beta$, $t > 0$, with $\beta \le -1$. (Note that the weight $v(t) = t^\beta$, $t > 0$, with $\beta \in \mathbb{R}$, belongs to the A_p-class or the C_p-class if and only if $-1 < \beta < p - 1$. However, v belongs to the M_p-class if and only if $\beta < p - 1$.) Another example is $v(t) = t^\beta(1 + |\ln t|)^\alpha$, $t > 0$, with $\beta \le -1$ and $\alpha \in \mathbb{R}$.

Remark 1.6 Denote by D_p, $1 < p < +\infty$, the subset of the M_p-class consisting of those weights v on $(0, +\infty)$ which satisfy condition (2). In particular, our results imply that the D_p-class possesses properties similar to those of the A_p-class. Namely,

$$\begin{aligned} &v \in D_p \Longrightarrow v \in D_{p-\varepsilon} && \text{for some } \varepsilon \in (0, p-1),\\ &v \in D_p \Longrightarrow v^{1+\varepsilon} \in D_p && \text{for some } \varepsilon > 0,\\ &v \in D_p \Longrightarrow v \in D_q && \text{for all } q \in [p, +\infty). \end{aligned} \tag{4}$$

Moreover,

$$v \in D_p \Longrightarrow v(x)x^\varepsilon \in D_p \qquad \text{for some } \varepsilon > 0.$$

It is well-known that a weight $v \in A_p$ possesses a better integrability than that mentioned in the A_p-condition and that such a weight v satisfies a reverse Hölder inequality. Implication (4) shows that also a weight $v \in D_p$ possesses better integrability properties than those mentioned in the definition of the D_p-class (*cf.* (1) with $w = v$ and $q = p$). It is even possible to prove that certain reverse Hölder inequalities hold for such a weight (*cf.* [8]).

The paper is organized as follows. In Section 2 we prove Theorem 1.1. Section 3 is devoted to the proof of Theorem 1.2. Finally, in Section 4 we prove Theorem 1.4.

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2. Proof of Theorem 1.1

To prove Theorem 1.1, we shall use the following two assertions.

Lemma A (see [7, Lemma 2])

Let $\varphi : (0, +\infty) \to (0, +\infty)$. If there is a constant $c_0 > 0$ such that

$$\int_r^{+\infty} \varphi(t) \frac{\mathrm{d}t}{t} \le c_0 \varphi(r) \quad \textit{for all } r > 0, \tag{5}$$

then there exist positive constants α_1 and c such that

$$\int_r^{+\infty} \varphi(t) t^\alpha \frac{\mathrm{d}t}{t} \le c \varphi(r) r^\alpha \quad \textit{for all } r > 0 \textit{ and } \alpha \in [0, \alpha_1).$$

Remark 2.1 In fact, it is proved in [7] that the last inequality holds for all $r > 0$ and some $\alpha > 0$. However, checking the [7, proof of Lemma 2], one can see that Lemma A holds, *e.g.*, with $\alpha_1 = (2c_0)^{-1}$ (and then one can put $c = 2c_0$), where c_0 is the constant in (5).

Lemma A*

Let $\varphi : (0, +\infty) \to (0, +\infty)$. If there is a constant $c_0 > 0$ such that

$$\int_0^r \varphi(t) \frac{\mathrm{d}t}{t} \le c_0 \varphi(r) \quad \text{for all } r > 0,$$

then there exist positive constants β_1 and c such that

$$\int_0^r \varphi(t) t^{-\beta} \frac{\mathrm{d}t}{t} \le c\varphi(r) r^{-\beta} \quad \text{for all } r > 0 \text{ and } \beta \in [0, \beta_1).$$

Proof. Lemma A* can be obtained from Lemma A by the change of variables $t \mapsto t^{-1}$. □

In addition, we shall also need the following lemma.

Lemma B

Let $1 < p \le q < +\infty$ and let v, w be weights on $(0, +\infty)$ such that (2) and (3) hold. Assume that the averaging operator $A : L^p(v) \to L^q(w)$ is bounded. Then there exists a positive constant α_0 such that

$$\int_0^r [v(t)t^{\alpha}]^{1-p'} \,\mathrm{d}t \approx [v(r) r^{\alpha+1-p}]^{1-p'} \tag{6}$$

and

$$\int_r^{+\infty} w(t) t^{\alpha - q} \,\mathrm{d}t \approx w(r) r^{\alpha+1-q} \tag{7}$$

for all $r > 0$ and $\alpha \in [0, \alpha_0)$.

Proof. Assume that all the assumptions of Lemma B are satisfied. Since the function $t \mapsto v(t)t^{\alpha+\rho}$, $\alpha \ge 0$, is equivalent to a non-decreasing function on $(0, +\infty)$,

$$\begin{aligned}
\int_0^r [v(t)t^{\alpha}]^{1-p'} \,\mathrm{d}t &= \int_0^r [v(t)t^{\alpha+\rho}]^{1-p'} t^{\rho(p'-1)} \,\mathrm{d}t \\
&\gtrsim [v(r) r^{\alpha+\rho}]^{1-p'} \int_0^r t^{\rho(p'-1)} \,\mathrm{d}t \\
&\approx [v(r) r^{\alpha+\rho}]^{1-p'} r^{\rho(p'-1)+1} \\
&= [v(r) r^{\alpha+1-p}]^{1-p'} \quad \text{for all } r > 0 \text{ and } \alpha \ge 0.
\end{aligned} \tag{8}$$

Consequently, we obtain from (1), (8) (with $\alpha = 0$) and (3) that

$$\int_r^{+\infty} w(t) t^{-q} \,\mathrm{d}t \le \frac{B^q}{\left(\int_0^r v(t)^{1-p'} \,\mathrm{d}t\right)^{q/p'}} \lesssim v(r)^{q/p} r^{-q/p'} \approx w(r) r^{1-q} \tag{9}$$

for all $r > 0$. Setting $\varphi(r) = w(r)r^{1-q}$, we can rewrite estimate (9) in the form

$$\int_r^{+\infty} \varphi(t)\frac{\mathrm{d}t}{t} \lesssim \varphi(r) \quad \text{for all } r > 0.$$

Thus, by Lemma A, there exist constants $\alpha_1 > 0$ and $c > 0$ such that

$$\int_r^{+\infty} w(t)t^{\alpha-q}\,\mathrm{d}t = \int_r^{+\infty} \varphi(t)t^{\alpha}\frac{\mathrm{d}t}{t} \le c\varphi(r)r^{\alpha} = cw(r)r^{\alpha+1-q} \tag{10}$$

for all $r > 0$ and $\alpha \in [0, \alpha_1)$.

On the other hand, using (3) and the fact that the function $t \mapsto [v(t)t^{\rho+1}]^{q/p}t^{\alpha}$, $\alpha \ge 0$, is equivalent to a non-decreasing function on $(0, +\infty)$, we arrive at

$$\begin{aligned}\int_r^{+\infty} w(t)t^{\alpha-q}\,\mathrm{d}t &\approx \int_r^{+\infty} [v(t)t^{\rho+1}]^{q/p}t^{\alpha}t^{-\rho q/p-q-1}\,\mathrm{d}t \\ &\gtrsim [v(r)r^{\rho+1}]^{q/p}r^{\alpha}\int_r^{+\infty} t^{-\rho q/p-q-1}\,\mathrm{d}t \\ &\approx [v(r)r]^{q/p}r^{\alpha}r^{-q} \\ &= w(r)r^{\alpha+1-q} \quad \text{for all } r > 0 \text{ and } \alpha \ge 0. \end{aligned} \tag{11}$$

Thus, (10) and (11) imply that (7) holds for all $r > 0$ and $\alpha \in [0, \alpha_1)$.

Condition (1) and the first three estimates in (11) (with $\alpha = 0$) yield

$$\begin{aligned}\int_0^r v(t)^{1-p'}\,\mathrm{d}t &\le \frac{B^{p'}}{\left(\int_r^{+\infty} w(t)t^{-q}\,\mathrm{d}t\right)^{p'/q}} \\ &\lesssim \frac{1}{\left([v(r)r]^{q/p}r^{-q}\right)^{p'/q}} \\ &= v(r)^{1-p'}r \quad \text{for all } r > 0. \end{aligned} \tag{12}$$

Rewriting (12) in terms of the function $\psi(t) = v(t)^{1-p'}t$, $t > 0$, and applying Lemma A*, we obtain that there are constants $\beta_1 > 0$ and $c_1 > 0$ such that

$$\int_0^r v(t)^{1-p'}t^{-\beta}\,\mathrm{d}t \le c_1 v(r)^{1-p'}r^{1-\beta} \tag{13}$$

for all $r > 0$ and $\beta \in [0, \beta_1)$. Setting $\alpha = \beta/(p'-1)$ and $\alpha_2 = \beta_1/(p'-1)$, we can rewrite (13) in the form

$$\int_0^r [v(t)t^{\alpha}]^{1-p'}\,\mathrm{d}t \lesssim [v(r)r^{\alpha+1-p}]^{1-p'}$$

for all $r > 0$ and $\alpha \in [0, \alpha_2)$. Together with (8), this shows that (6) holds for all $r > 0$ and $\alpha \in [0, \alpha_2)$.

Now, it suffices to put $\alpha_0 = \min\{\alpha_1, \alpha_2\}$. □

Remark 2.2 On using (3), one can rewrite (7) as

$$\int_r^{+\infty} w(t)t^{\alpha-q}\,\mathrm{d}t \approx v(r)^{q/p}r^{\alpha-q+q/p} \tag{14}$$

for all $r > 0$ and $\alpha \in [0, \alpha_0)$.

Remark 2.3 Let all the assumptions of Lemma B be satisfied. Then the operator

$$A : L^p(v(x)x^\alpha) \to L^q(w(x)x^{\alpha q/p})$$

is also bounded for all $\alpha \in [0, \alpha_0\, p/q)$. Indeed, making use of estimates (6) and (14) (with α replaced by $\alpha q/p$), we see that (1) holds with $v(t)$ replaced by $v(t)t^\alpha$ and with $w(t)$ replaced by $w(t)t^{\alpha q/p}$ for all $\alpha \in [0, \alpha_0\, p/q)$.

Proof of Theorem 1.1. Let the assumptions of Theorem 1.1 be satisfied. By (6) and (7) (with $\alpha = 0$), for all $r > 0$,

$$\int_0^r v(t)^{1-p'}\,\mathrm{d}t \approx v(r)^{1-p'}\,r \tag{15}$$

and

$$\int_r^{+\infty} w(t)t^{-q}\,\mathrm{d}t \approx w(r)r^{1-q}. \tag{16}$$

Take $\delta,\ \gamma \geq 0$, $\varepsilon \in [0, p-1)$ and put $p(\varepsilon) := p - \varepsilon$, $q(\varepsilon) := q - \varepsilon q/p$. Clearly, $p(\varepsilon), p(\varepsilon)' \in (1, +\infty)$, $p' - p(\varepsilon)' \leq 0$ and $p(\varepsilon)/p = q(\varepsilon)/q = 1 - \varepsilon/p$. Thus,

$$\kappa := \frac{p' - p(\varepsilon)'}{1-p'} + \delta\frac{1-p(\varepsilon)'}{1-p'} \geq 0$$

and the function

$$t \mapsto \Big(\int_0^t v(\tau)^{1-p'}\,\mathrm{d}\tau\Big)^\kappa$$

is non-decreasing on $(0, +\infty)$. Consequently, applying (15), we obtain

$$\begin{aligned}
\int_0^r [v(t)^{1+\delta}t^\gamma]^{1-p(\varepsilon)'}\,\mathrm{d}t &= \int_0^r v(t)^{1-p'}v(t)^{\kappa(1-p')}t^{\gamma(1-p(\varepsilon)')}\,\mathrm{d}t \\
&\approx \int_0^r v(t)^{1-p'}\Big(t^{-1}\int_0^t v(\tau)^{1-p'}\,\mathrm{d}\tau\Big)^\kappa t^{\gamma(1-p(\varepsilon)')}\,\mathrm{d}t \\
&\leq \Big(\int_0^r v(\tau)^{1-p'}\,\mathrm{d}\tau\Big)^\kappa \int_0^r v(t)^{1-p'}t^{-\kappa+\gamma(1-p(\varepsilon)')}\,\mathrm{d}t \\
&\approx v(r)^{\kappa(1-p')}r^\kappa \int_0^r [v(t)t^\alpha]^{1-p'}\,\mathrm{d}t,
\end{aligned} \tag{17}$$

where

$$\begin{aligned}
\alpha &\equiv \alpha(\varepsilon, \delta, \gamma) \\
&:= \frac{-\kappa + \gamma(1-p(\varepsilon)')}{1-p'} \\
&= \frac{p'-p(\varepsilon)'}{(1-p')(p'-1)} + \delta\frac{1-p(\varepsilon)'}{(1-p')(p'-1)} + \gamma\frac{1-p(\varepsilon)'}{1-p'} \geq 0.
\end{aligned}$$

Since the function $(\varepsilon, \delta, \gamma) \mapsto \alpha(\varepsilon, \delta, \gamma)$ is non-negative and continuous on the set $[0, p-1) \times [0, +\infty) \times [0, +\infty)$ and $\alpha(0,0,0) = 0$, there is $\varepsilon_1 \in (0, p-1)$ such that $\alpha(\varepsilon, \delta, \gamma) \in [0, \alpha_0)$ provided that $\varepsilon, \delta, \gamma \in [0, \varepsilon_1)$, where the number α_0 is from Lemma B. Therefore, (17) and (6) imply that

$$\int_0^r \left[v(t)^{1+\delta} t^\gamma\right]^{1-p(\varepsilon)'} \mathrm{d}t \lesssim v(r)^{(1+\delta)(1-p(\varepsilon)')}\, r^{\gamma(1-p(\varepsilon)')+1}$$

for all $r > 0$ and $\varepsilon, \delta, \gamma \in [0, \varepsilon_1)$. Hence,

$$\Big(\int_0^r \left[v(t)^{1+\delta} t^\gamma\right]^{1-p(\varepsilon)'} \mathrm{d}t\Big)^{1/p(\varepsilon)'} \lesssim v(r)^{-(1+\delta)/p(\varepsilon)}\, r^{-\gamma/p(\varepsilon)}\, r^{1/p(\varepsilon)'} \tag{18}$$

for all $r > 0$ and $\varepsilon, \delta, \gamma \in [0, \varepsilon_1)$.

Applying (7) (with $\alpha = 0$), the fact that the function

$$t \mapsto \Big(\int_t^{+\infty} w(\tau)\tau^{-q}\,\mathrm{d}\tau\Big)^\delta t^{\delta(1-q/p)}, \qquad \delta \le 0,$$

is non-increasing on $(0, +\infty)$ and (14) (with $\alpha = 0$), we get

$$\begin{aligned}
&\int_r^{+\infty} w(t)^{1+\delta} t^{\delta(1-q/p)} t^{\gamma q/p} t^{-q(\varepsilon)}\,\mathrm{d}t \\
&\approx \int_r^{+\infty} w(t)\Big(t^{q-1}\int_t^{+\infty} w(\tau)\tau^{-q}\,\mathrm{d}\tau\Big)^\delta t^{(\gamma+\varepsilon)q/p-q} t^{\delta(1-q/p)}\,\mathrm{d}t \\
&\le \Big(\int_r^{+\infty} w(\tau)\tau^{-q}\,\mathrm{d}\tau\Big)^\delta r^{\delta(1-q/p)} \int_r^{+\infty} w(t) t^{(\gamma+\varepsilon)q/p+\delta(q-1)-q}\,\mathrm{d}t \\
&\approx [v(r)^{q/p} r^{-q+q/p}]^\delta r^{\delta(1-q/p)} \int_r^{+\infty} w(t) t^{(\gamma+\varepsilon)q/p+\delta(q-1)-q}\,\mathrm{d}t.
\end{aligned} \tag{19}$$

Now, using (14) (with $(\gamma+\varepsilon)q/p + \delta(q-1)$ instead of α) to estimate the last integral, we arrive at

$$\int_r^{+\infty} w(t) t^{(\gamma+\varepsilon)q/p+\delta(q-1)-q}\,\mathrm{d}t \approx v(r)^{q/p} r^{(\gamma+\varepsilon+1)q/p+\delta(q-1)-q} \tag{20}$$

for all $r > 0$ provided that $(\gamma+\varepsilon)q/p + \delta(q-1) \in [0, \alpha_0)$. Therefore, (19) and (20) imply that

$$\Big(\int_r^{+\infty} w(t)^{1+\delta} t^{\delta(1-q/p)} t^{\gamma q/p} t^{-q(\varepsilon)}\,\mathrm{d}t\Big)^{1/q(\varepsilon)} \precsim v(r)^{(1+\delta)/p(\varepsilon)}\, r^{\gamma/p(\varepsilon)}\, r^{-1/p(\varepsilon)'} \tag{21}$$

for all $r > 0$ and $\varepsilon, \delta, \gamma \in [0, \varepsilon_2)$, where $\varepsilon_2 := \min\{\alpha_0\, p/(3q), \alpha_0/(3(q-1))\}$.

Putting $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ and using estimates (18) and (21) in (1) (with $w(t)$, $v(t)$, q and p replaced by $w(t)^{1+\delta} t^{\delta(1-q/p)} t^{\gamma q/p}$, $v(t)^{1+\delta} t^\gamma$, $q(\varepsilon)$ and $p(\varepsilon)$, respectively), we obtain the desired result. $\square$

3. Proof of Theorem 1.2

Assume that the assumptions of Theorem 1.2 are satisfied. Put $p(\varepsilon) := p - \varepsilon$ and $q(\varepsilon) := q - \varepsilon q/p$. The Hölder inequality with the exponents

$$\frac{(p(\varepsilon)' - 1)(1+\delta)}{(p' - 1)} \quad \text{and} \quad \frac{(p(\varepsilon)' - 1)(1+\delta)}{(p(\varepsilon)' - 1)(1+\delta) - (p' - 1)}$$

implies that, for all $r > 0$,

$$\int_0^r v(t)^{1-p'}\,\mathrm{d}t \le \Big(\int_0^r [v(t)^{1+\delta}]^{1-p(\varepsilon)'}\,\mathrm{d}t\Big)^{\frac{p'-1}{(p(\varepsilon)'-1)(1+\delta)}} r^{\frac{(p(\varepsilon)'-1)(1+\delta)-p'+1}{(p(\varepsilon)'-1)(1+\delta)}}. \tag{22}$$

Using the fact that the function $t \mapsto t^{\gamma(p(\varepsilon)'-1)}$ is non-decreasing on the interval $(0, +\infty)$, we obtain

$$\int_0^r [v(t)^{1+\delta}]^{1-p(\varepsilon)'}\,\mathrm{d}t \le r^{\gamma(p(\varepsilon)'-1)} \int_0^r [v(t)^{1+\delta} t^{\gamma}]^{1-p(\varepsilon)'}\,\mathrm{d}t \quad \text{for all } r > 0. \tag{23}$$

Fix $\overline{\rho} \ge \max\{\rho(1+\delta) - \gamma, 0\}$. One can easily verify that (2) and (3) holds with $v(x)x^{\rho}$, $w(x)$, $v(x)$, q and p replaced by $(v(x)^{1+\delta}x^{\gamma})x^{\overline{\rho}}$, $w(x)^{1+\delta}x^{\delta(1-q/p)}x^{\gamma q/p}$, $v(x)^{1+\delta}x^{\gamma}$, $q(\varepsilon)$ and $p(\varepsilon)$, respectively. Thus, we can apply Lemma B (with $v(x)x^{\rho}$, $w(x)$, $v(x)$, q and p replaced by $(v(x)^{1+\delta}x^{\gamma})x^{\overline{\rho}}$, $w(x)^{1+\delta}x^{\delta(1-q/p)}x^{\gamma q/p}$, $v(x)^{1+\delta}x^{\gamma}$, $q(\varepsilon)$ and $p(\varepsilon)$, respectively). Hence, taking $\alpha = 0$ in (6) and (7), we obtain, for all $r > 0$,

$$\int_0^r [v(t)^{1+\delta} t^{\gamma}]^{1-p(\varepsilon)'}\,\mathrm{d}t \approx [v(r)^{1+\delta} r^{\gamma}]^{1-p(\varepsilon)'} r \tag{24}$$

and

$$\int_r^{+\infty} w(t)^{1+\delta} t^{\delta(1-q/p)} t^{\gamma q/p} t^{-q(\varepsilon)}\,\mathrm{d}t \approx w(r)^{1+\delta} r^{\delta(1-q/p)} r^{\gamma q/p} r^{1-q(\varepsilon)}. \tag{25}$$

Combining estimates (22)–(24), we arrive at

$$\Big(\int_0^r v(t)^{1-p'}\,\mathrm{d}t\Big)^{1/p'} \lesssim v(r)^{-1/p} r^{1/p'} \quad \text{for all } r > 0. \tag{26}$$

On the other hand, Hölder's inequality with the exponents $1 + \delta$ and $(1 + \delta)/\delta$ gives

$$\int_r^{+\infty} w(t) t^{-q}\,\mathrm{d}t \le \Big(\int_r^{+\infty} w(t)^{1+\delta} t^{\delta(1-q/p)} t^{\gamma q/p} t^{-q(\varepsilon)}\,\mathrm{d}t\Big)^{1/(1+\delta)} \Big(r^{\frac{q}{p} - \frac{\gamma q}{\delta p} - \frac{\varepsilon q}{\delta p} - q}\Big)^{\delta/(1+\delta)},$$

which, together with (25) and (3), implies that

$$\Big(\int_r^{+\infty} w(t) t^{-q}\,\mathrm{d}t\Big)^{1/q} \lesssim w(r)^{1/q} r^{-1/q'} \approx v(r)^{1/p} r^{-1/p'} \quad \text{for all } r > 0. \tag{27}$$

Estimates (26) and (27) used in (1) yield the desired result. $\square$

4. Proof of Theorem 1.4

With respect to Theorem 1.1, it is sufficient to prove that the operator $A : L^P(v(x)) \to L^Q(w(x))$ is bounded if $p < P < +\infty$ and $Q/P = q/p$.

Using the monotonicity of the function $t \mapsto t^{q-Q}$, $t > 0$, and (14) (with $\alpha = 0$), we obtain

$$\begin{aligned}\Big(\int_r^{+\infty} w(t)t^{-Q}\,\mathrm{d}t\Big)^{1/Q} &\le \Big(r^{q-Q}\int_r^{+\infty} w(t)t^{-q}\,\mathrm{d}t\Big)^{1/Q}\\ &\approx \Big(r^{q-Q}v(r)^{q/p}r^{-q+q/p}\Big)^{1/Q}\\ &= v(r)^{1/P}r^{-1/P'} \quad \text{for all } \ r > 0.\end{aligned}$$

Moreover, the Hölder inequality (with the exponents $\frac{1-p'}{1-P'}$ and $\frac{1-p'}{P'-p'}$) and (6) (with $\alpha = 0$) imply that

$$\begin{aligned}\Big(\int_0^r v(t)^{1-P'}\,\mathrm{d}t\Big)^{1/P'} &\le \Big(\int_0^r v(t)^{1-p'}\,\mathrm{d}t\Big)^{(1-P')/((1-p')P')} r^{(P'-p')/((1-p')P')}\\ &\approx [v(r)^{1-p'}r]^{(1-P')/((1-p')P')}\, r^{(P'-p')/((1-p')P')}\\ &= v(r)^{-1/P}r^{1/P'} \quad \text{for all } \ r > 0.\end{aligned}$$

Consequently, the result follows from (1) (with p and q replaced by P and Q, respectively). $\square$

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