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Complex interpolation of spaces of integrable functions with respect to a vector measure*

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Abstract

Let (Ω, Σ) be a measurable space and $m : \Sigma \to X$ be a vector measure with values in the complex Banach space X. We apply the Calderón interpolation methods to the family of spaces of scalar $p-$integrable functions with respect to m with $1 \leq p \leq \infty$. Moreover we obtain a result about the relation between the complex interpolation spaces $[X_0, X_1]_{[\theta]}$ and $[X_0, X_1]^{[\theta]}$ for a Banach couple of interpolation (X_0, X_1) such that $X_1 \subset X_0$ with continuous inclusion.

1. Introduction

Let X be a Banach space and $m : \Sigma \longrightarrow X$ be a countably additive vector measure, where Σ is a σ–algebra of subsets of some nonempty set Ω. Associated with m are the Banach lattices $L^p(m)$ (and $L^p_w(m)$), with $1 \leq p < \infty$, of equivalence classes of functions $f : \Omega \longrightarrow \mathbb{K}$ (weakly) p–integrable with respect to m, equipped with the topology of convergence in p–mean. Here $\mathbb{K}$ denotes the scalar field $i.e.$, either the real

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numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ over which X is a vector space. In the case when $\mathbb{K} = \mathbb{R}$ the spaces $L^p(m)$ and $L^p_w(m)$ was intensively studied in [6].

We shall obtain the two Calderón's complex interpolation spaces, $[X_0, X_1]_{[\theta]}$ and $[X_0, X_1]^{[\theta]}$, with $0 < \theta < 1$ (see [3, 2] or [12]) of the complex Banach lattices couple (X_0, X_1), where X_0 and X_1 are the spaces $L^p(m)$ or $L^p_w(m)$, with $1 \leq p < \infty$. The results we will obtain in this paper are quite different from those in the classical setting of a positive scalar measure. As it is well–known, in the classical context we have $L^p(m) = L^p_w(m)$, and they are reflexive for $1 < p < \infty$. In general, for a vector measure m, the inclusion $L^p(m) \subset L^p_w(m)$ can be strict and these spaces can be non–reflexive spaces, even for $p > 1$, and moreover $L^1(m)$ or $L^1_w(m)$ can be reflexive.

An important tool in our work is the relationship of the complex interpolation spaces, namely, $[X_0, X_1]_{[\theta]}$ and $[X_0, X_1]^{[\theta]}$, with the so called Calderón–Lozanovskiĭ's product space $X_0^{1-\theta} X_1^{\theta}$, and also with the Gustavsson–Peetre's interpolation space $\langle X_0, X_1, \theta \rangle$. The basic properties of these interpolation spaces can be found in [3, 9].

Roughly speaking, we can say that $L^p(m)$ is, in a certain sense, a good prototype of $[X_0, X_1]_{[\theta]}$, and the same is true for $L^p_w(m)$ and $[X_0, X_1]^{[\theta]}$. Based in this idea together with the knowledge of the behavior of the relationships between the spaces $L^p(m)$ and $L^p_w(m)$, we obtain a general result about the relation between the complex interpolation spaces $[X_0, X_1]_{[\theta]}$ and $[X_0, X_1]^{[\theta]}$ for a Banach couple of interpolation (X_0, X_1) such that $X_1 \subset X_0$ with continuous inclusion.

2. The spaces of p–integrable functions

Let X be a Banach space over $\mathbb{K}$. The space of all continuous linear functionals $x^* : X \longrightarrow \mathbb{K}$ is denoted by X^*, and its unit ball by $B_1(X)$. Let $m : \Sigma \longrightarrow X$ be a vector measure defined on a σ–algebra of subsets Σ of a nonempty set Ω; this will always mean that m is countably additive on Σ. The semivariation of m over $A \in \Sigma$ is $\|m\|(A) := \sup \{ |\langle m, x^* \rangle|(A) : x^* \in B_1(X^*) \}$, where $|\langle m, x^* \rangle|$ is the total variation measure of the scalar measure $\langle m, x^* \rangle : \Sigma \longrightarrow \mathbb{K}$ defined by $\langle m, x^* \rangle(A) = \langle m(A), x^* \rangle$, for all $A \in \Sigma$. A measurable set A is said to be m–null when $\|m\|(A) = 0$. We assume that the σ–algebra Σ is complete with respect to m, that is, if $B \subset A \in \Sigma$ and $\|m\|(A) = 0$, then $B \in \Sigma$ (and so $\|m\|(B) = 0$).

A measurable function $f : \Omega \longrightarrow \mathbb{K}$ is called *weakly integrable*, with respect to m, if $f \in L^1(|\langle m, x^* \rangle|)$ for every $x^* \in X^*$. The space $L^1_w(m)$ of all (m–a.e. equivalence classes of) weakly integrable functions becomes a complex Banach space when is endowed with the norm defined by

$$\|f\|_1 := \sup \Big\{ \int_\Omega |f| \, d|\langle m, x^* \rangle| : x^* \in B_1(X^*) \Big\}, \quad f \in L^1_w(m).$$

A weakly integrable function f is said to be *integrable*, with respect to m, if for each $A \in \Sigma$ there exists an element (necessarily unique) $\int_A f \, dm \in X$ satisfying $\left\langle \int_A f \, dm, x^* \right\rangle = \int_A f \, d\langle m, x^* \rangle$, for all $x^* \in X^*$. The space $L^1(m)$ of all (m–a.e. equivalence classes of) integrable functions becomes a closed subspace of $L^1_w(m)$. Basic results about these spaces can be found in [6, 7, 11, 13, 14, 16]. If $\mathbb{K} = \mathbb{R}$ then, with the usual order, $L^1_w(m)$ is a Banach lattice with the Fatou property and $L^1(m)$ is an order continuous closed ideal of $L^1_w(m)$.

Suppose now that X is a Banach space over $\mathbb{C}$. Let $X_\mathbb{R}$ denote X considered as a vector space over $\mathbb{R}$, in which case $X_\mathbb{R}$ is a real vector space. Note that the norm of X is also a norm on $X_\mathbb{R}$, and $X_\mathbb{R}$ becomes a Banach space with this norm. If $x^* \in (X_\mathbb{R})^*$, then it is easily checked that $x^*_\mathbb{C} : X \longrightarrow \mathbb{C}$ defined by $\langle x, x^*_\mathbb{C}\rangle := \langle x, x^*\rangle - i\langle ix, x^*\rangle$ is an element of X^*. Moreover, if $x^* \in X^*$ and

$$x^*_R : x \in X_\mathbb{R} \longrightarrow \langle x, x^*_R\rangle := \mathrm{Re}(\langle x, x^*\rangle) \in \mathbb{R},$$
$$x^*_I : x \in X_\mathbb{R} \longrightarrow \langle x, x^*_I\rangle := \mathrm{Im}(\langle x, x^*\rangle) \in \mathbb{R},$$

then both $x^*_R, x^*_I \in (X_\mathbb{R})^*$ and $\langle x, x^*_I\rangle = -\langle ix, x^*_R\rangle$ for all $x \in X$. Of course,

$$\langle x, x^*\rangle = \langle x, x^*_R\rangle + i\langle x, x^*_I\rangle, \quad x \in X.$$

Given a vector measure $m : \Sigma \longrightarrow X$, let $m_\mathbb{R} : \Sigma \longrightarrow X_\mathbb{R}$ denote m considered as taking its values in $X_\mathbb{R}$. It is clear that $m_\mathbb{R}$ is countably additive. Moreover, if $x^* \in X^*$, then we can write the complex measure $\langle m, x^*\rangle = \langle m_\mathbb{R}, x^*_R\rangle + i\langle m_\mathbb{R}, x^*_I\rangle$, where both $\langle m_\mathbb{R}, x^*_R\rangle$ and $\langle m_\mathbb{R}, x^*_I\rangle$ are $\mathbb{R}$–valued signed measures.

Proposition 2.1

Let X be a complex Banach space, $m : \Sigma \longrightarrow X$ a vector measure and $f : \Omega \longrightarrow \mathbb{C}$ a measurable function. Then

1) *$f \in L^1_w(m)$ if and only if both its real part and its imaginary part belong to $L^1_w(m_\mathbb{R})$. That is, $L^1_w(m) = L^1_w(m_\mathbb{R}) \oplus iL^1_w(m_\mathbb{R})$.*
2) *$f \in L^1(m)$ if and only if both its real part and its imaginary part belong to $L^1(m_\mathbb{R})$. That is, $L^1(m) = L^1(m_\mathbb{R}) \oplus iL^1(m_\mathbb{R})$.*

Proof. See [7, Lemma 2 and Lemma 3]. □

Remark 2.2 If X is a complex Banach space and $m : \Sigma \longrightarrow X$ a vector measure, the proposition above tells us that $L^1_w(m)$ and $L^1(m)$ are both *complex Banach lattices.* Moreover it is clear from the definition that $L^1(m) \subset L^1_w(m)$ and, in general, this inclusion is strict.

If $1 < p < \infty$, the space $L^p_w(m)$ of (m–a.e. equivalence classes of) *weakly p–integrable* functions with respect to m, is defined as the space of measurable functions $f : \Omega \longrightarrow \mathbb{K}$ such that $|f|^p \in L^1_w(m)$. When $L^p_w(m)$ is endowed with the norm

$$\|f\|_p := \sup\left\{\Big(\int_\Omega |f|^p \, d|\langle m, x^*\rangle|\Big)^{1/p} : x^* \in B_1(X^*)\right\}, \quad f \in L^p_w(m)$$

becomes a Banach space. Analogously, the space $L^p(m)$ of (m–a.e. equivalence classes of) *p–integrable* functions with respect to m, is defined as the space of measurable functions $f : \Omega \longrightarrow \mathbb{K}$ such that $|f|^p \in L^1(m)$. For $p = \infty$, the corresponding spaces $L^\infty_w(m)$ and $L^\infty(m)$ of m–a.e. equivalence classes of m–essentially bounded functions are equal and coincide, when endowed with the essential supremum norm, with the space $L^\infty(\mu)$ of essentially bounded functions with respect to every Rybakov's control measure μ of m.

Clearly $L^p(m) \subset L^p_w(m)$ and $L^p(m)$ is a closed subspace of $L^p_w(m)$. For $\mathbb{K} = \mathbb{R}$, both spaces, with the usual order, become Banach lattices and $L^p(m)$ is an order continuous ideal of $L^p_w(m)$ for $1 \leq p < \infty$. It is certainly true that many of the facts available for $L^p(m)$ or $L^p_w(m)$ over $\mathbb{R}$ as presented in [6], such as the usual limit theorems, various lattice properties of $L^p(m)$ or $L^p_w(m)$, the fact that all simple Σ–measurable functions are dense in $L^p(m)$, and so, carry over to the case $\mathbb{C}$; see [7] for example. In particular, if $\mathbb{K} = \mathbb{C}$, from the definition and Proposition 2.1 we can check that $L^p(m)$ and $L^p_w(m)$ are complex Banach lattices, that is, the equalities $L^p(m) = L^p(m_\mathbb{R}) \oplus iL^p(m_\mathbb{R})$ and $L^p_w(m) = L^p_w(m_\mathbb{R}) \oplus iL^p_w(m_\mathbb{R})$ hold for all $1 \leq p \leq \infty$.

For $\mathbb{K} = \mathbb{R}$ parts 1)–3) of the following result can be found in [6] and for $\mathbb{K} = \mathbb{C}$ the result follows from the last comment.

Proposition 2.3

Let X be a Banach space over $\mathbb{K}$, $m : \Sigma \longrightarrow X$ a vector measure and $1 \leq p < \infty$.

1) *The dominated convergence theorem holds, meaning if $(f_n)_n$ is a sequence in $L^p(m)$ converging m–a.e. to $f : \Omega \longrightarrow \mathbb{K}$ and $|f_n| \leq g$, for $n \geq 1$ and some $0 \leq g \in L^p(m)$, then $f \in L^p(m)$ and $f_n \to f$ in the norm of $L^p(m)$.*
2) *The Fatou property holds, meaning if $(f_n)_n$ is a sequence in $L^p_w(m)$ converging m–a.e. to $f : \Omega \longrightarrow \mathbb{K}$ and $\sup\limits_{n\geq 1} \|f_n\|_p < \infty$, then $f \in L^p_w(m)$ and $\|f\|_p \leq \sup_{n\geq 1} \|f_n\|_p$.*
3) *If $1 \leq p_0 < p_1 < \infty$, then $L^{p_1}_w(m) \subset L^{p_0}(m)$, and this inclusion is weakly compact.*

Note that the Fatou property holds in $L^\infty(m) \equiv L^\infty_w(m)$. However, unless trivial cases, we do not have a dominated convergence theorem in this space.

We end this section with the following important remark that we already mentioned in the introduction. In general the inclusion $L^p(m) \subset L^p_w(m)$ can be strict, and in contrast to the classical setting of a positive scalar measure, these spaces can be non–reflexive, even for $p > 1$, and moreover $L^1(m)$ or $L^1_w(m)$ can be reflexive. See [6] for details.

3. Complex interpolation of spaces of integrable functions

For a measure space (Ω, Σ, μ), let $L^0(\mu)$ the space of (μ–a.e. equivalence classes of) scalar measurable functions on Ω. A *Banach lattice ideal* on the measure space (Ω, Σ, μ) is a Banach space X which is a vector subspace of $L^0(\mu)$ and satisfies: if $f \in X$ and $g \in L^0(\mu)$ such that $|g(\omega)| \leq |f(\omega)|$, μ–a.e., then $g \in X$ and $\|g\| \leq \|f\|$. Now for a given vector measure m on a complex Banach space X, the spaces $L^p(m)$ and $L^p_w(m)$, with $1 \leq p < \infty$, are complex Banach lattice ideals on the measure space (Ω, Σ, μ) where μ is a *Rybakov control measure* for m, that is, $\mu = |\langle m, x^*\rangle|$ for some $x^* \in B_1(X^*)$, and m is absolutely continuous with respect to μ. Therefore each pair of spaces $L^p(m)$ or $L^p_w(m)$, forms a compatible couple of Banach spaces since they are imbedded continuously in the same topological vector space, namely, $L^0(\mu)$ endowed with the topology of convergence in measure.

Let X and Y be two Banach spaces such that $X \subset Y$. The *Gagliardo completion* of X in Y is defined as the set $\widetilde{X}^{\mathcal{G}Y}$ of all those elements $y \in Y$ such that there exists a bounded sequence $(x_n)_n$ in X which converges to y in Y. The space $\widetilde{X}^{\mathcal{G}Y}$ is a Banach space when it is endowed with the *Gagliardo norm* defined by

$$\|y\|_{\widetilde{X}^{\mathcal{G}Y}} := \inf \sup \{\|x_n\|_X : n \in \mathbb{N}\},$$

where the infimum is taken over all bounded sequences $(x_n)_n$ in X which converge to y in Y. See [1, §10] or [12, I.1.2].

For an arbitrary interpolation couple of Banach spaces (X_0, X_1) and $0 < \theta < 1$, from [2, Theorem 4.3.1] and the proof of [2, Lema 4.3.3, p. 94–95] it follows that

$$[X_0, X_1]_{[\theta]} \subset [X_0, X_1]^{[\theta]} \subset \widetilde{[X_0, X_1]_{[\theta]}}^{\mathcal{G}(X_0+X_1)}, \tag{3.1}$$

and both inclusions are continuous.

Proposition 3.1

Let $1 \le p < \infty$ and p^* its conjugate exponent. Then

1) $L^p(m) \cdot L^{p^*}(m) = L^p(m) \cdot L^{p^*}_w(m) = L^1(m)$. *Moreover, for every $h \in L^1(m)$, we have*

$$\begin{aligned} \|h\|_1 &= \inf \left\{ \|f\|_p \, \|g\|_{p^*} : h = fg, f \in L^p(m), g \in L^{p^*}(m) \right\} \\ &= \inf \left\{ \|f\|_p \, \|g\|_{p^*} : h = fg, f \in L^p(m), g \in L^{p^*}_w(m) \right\}. \end{aligned}$$

2) $L^p_w(m) \cdot L^{p^*}_w(m) = L^1_w(m)$. *Moreover, for every $h \in L^1_w(m)$, we have*

$$\|h\|_1 = \inf \left\{ \|f\|_p \, \|g\|_{p^*} : h = fg, f \in L^p_w(m), g \in L^{p^*}_w(m) \right\}.$$

3) *For $p > 1$, the Gagliardo completion of $L^p(m)$ in $L^1_w(m)$ equals $L^p_w(m)$ isometrically. Therefore, for each $1 \le p_0 < p_1$, the Gagliardo completion of $L^{p_1}(m)$ in $L^{p_0}_w(m)$ equals $L^{p_1}_w(m)$ isometrically.*
4) *$L^p_w(m)$ is Gagliardo complete in $L^1_w(m)$, for $p > 1$. Therefore, $L^{p_1}_w(m)$ is Gagliardo complete in $L^{p_0}_w(m)$, if $1 \le p_0 < p_1$.*

Proof. 1) From Proposition 2.3–3) we know that $L^{p*}_w(m) \subset L^1(m)$, since $p^* > 1$. Note that this inclusion is trivial if $p^* = \infty$. Then, for each $g \in L^{p*}_w(m)$ and each simple function s, we have $sg \in L^1(m)$. Now, let us consider the functions $f \in L^p(m)$ and $g \in L^{p*}_w(m)$ and take a sequence $(s_n)_n$ of simple functions which converges to f in $L^p(m)$. Then, $(s_n g)_n$ converges to fg in norm in $L^1_w(m)$ since

$$\|fg - s_n g\|_1 = \|(f - s_n)g\|_1 \le \|f - s_n\|_p \, \|g\|_{p*} \longrightarrow 0.$$

Therefore, $fg \in L^1(m)$ because $L^1(m)$ is closed in $L^1_w(m)$. Moreover

$$\|fg\|_1 = \lim_n \|s_n g\|_1 \le \lim_n \|s_n\|_p \|g\|_{p^*} = \|f\|_p \|g\|_{p^*}.$$

To end the proof of 1), note that each function $h \in L^1(m)$ can be factorized as $|h| = |h|^{1/p}|h|^{1/p^*}$, where $|h|^{1/p} \in L^p(m)$ and $|h|^{1/p^*} \in L^{p^*}(m)$. With this factorization we have $\|h\|_1 = \|h\|_1^{1/p}\|h\|_1^{1/p^*} = \||h|^{1/p}\|_p\||h|^{1/p^*}\|_{p^*}$.
2) Obvious.
3) Let $(f_n)_n$ be a bounded sequence in $L^p(m)$ which converges to f in $L^1_w(m)$. For each $x^* \in X^*$, note that $(f_n)_n$ is a bounded sequence in $L^p(|\langle m, x^*\rangle|)$ which converges to f in $L^1(|\langle m, x^*\rangle|)$. Applying Fatou's Lemma to $(|f_n|^p)$ we obtain that $|f|^p$ is in $L^1(|\langle m, x^*\rangle|)$. Then f is in $L^p_w(m)$. On the other hand, for a given $f \in L^p_w(m)$ let us consider the sequence $(f_n)_n$ of bounded functions defined by $f_n := f\chi_{A_n}$, with

$$A_n := \{w \in \Omega : |f(\omega)| \leq n\}, \quad n = 1, 2, \dots$$

Then, $(f_n)_n$ is a bounded sequence in $L^p(m)$, since $\|f_n\|_p \leq \|f\|_p$ for every $n \geq 1$, and $(f_n)_n$ converges to f in $L^1_w(m)$ as a consequence of the Hölder inequality. Moreover $(\|f_n\|_p)_n$ converges to $\|f\|_p$, and therefore the Gagliardo norm of f coincides with $\|f\|_p$.
4) The same argument as in 3) works here. □

For a given couple (X_0, X_1) of (complex) Banach lattice ideals on the same measure space (Ω, Σ, μ), and the index $0 \leq \theta \leq 1$, the Calderón–Lozanovskiĭ's product space $X_0^{1-\theta}X_1^\theta$ is the space of all (μ–a.e. equivalence classes of) complex valued measurable functions f on (Ω, Σ, μ) such that there exist $f_0 \in B_1(X_0)$, $f_1 \in B_1(X_1)$ and $\lambda > 0$ for which

$$|f(\omega)| \leq \lambda |f_0(\omega)|^{1-\theta}|f_1(\omega)|^\theta, \quad w \in \Omega \ \ (\mu - \text{a.e.}) \tag{3.2}$$

The space $X_0^{1-\theta}X_1^\theta$ is a Banach space when endowed with the norm

$$\|x\|_{X_0^{1-\theta}X_1^\theta} := \inf \lambda,$$

where the infimum is taken over all λ satisfying (3.2), see [12, IV.§1.11].

In the following we consider for $1 \leq p_0 \neq p_1 \leq \infty$ and $0 < \theta < 1$ the index $p(\theta)$ defined by the equality

$$\frac{1}{p(\theta)} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \tag{3.3}$$

with the usual meaning if p_0 or p_1 is equal to ∞.

Proposition 3.2

The following identities hold isometrically.

1) $(L^{p_0}(m))^{1-\theta}(L^{p_1}(m))^\theta = L^{p(\theta)}(m) = (L^{p_0}(m))^{1-\theta}(L^{p_1}_w(m))^\theta$.
2) $(L^{p_0}_w(m))^{1-\theta}(L^{p_1}_w(m))^\theta = L^{p(\theta)}_w(m)$.

Proof. The proof follows directly from Proposition 3.1 and the definition of $p(\theta)$ since $\frac{p_0}{(1-\theta)p(\theta)}$ is the conjugate exponent of $\frac{p_1}{\theta p(\theta)}$. □

Now, we describe the *Gustavsson–Peetre's method* [9]. For a given couple of Banach spaces (X_0, X_1) and an index $0 < \theta < 1$, the Gustavsson–Peetre space $\langle X_0, X_1, \theta\rangle$ is the space of those elements $x \in X_0 + X_1$ for which there exists a double sequence $(x_k)_{k\in\mathbb{Z}}$ of elements $x_k \in X_0 \cap X_1$ such that:

(GP1) $x = \sum_{k\in\mathbb{Z}} x_k$, where the series converges in $X_0 + X_1$,

(GP2a) the series $\sum_{k\in\mathbb{Z}} \frac{1}{2^{k\theta}} x_k$ is weakly unconditionally Cauchy in X_0, and

(GP2b) the series $\sum_{k\in\mathbb{Z}} \frac{1}{2^{k(\theta-1)}} x_k$ is weakly unconditionally Cauchy in X_1.

Recall that a series $\sum_k z_k$ in a Banach space Z is weakly unconditionally Cauchy if for each $z^* \in Z^*$ the scalar series $\sum_k \langle z_k, z^*\rangle$ is absolutely convergent. It is well known that this is equivalent to the existence of a constant $C > 0$ such that for every finite subset $F \subset \mathbb{Z}$ and every finite scalar sequence $(\varepsilon_k)_{k\in F}$, with $|\varepsilon_k| \leq 1$, we obtain

$$\Big\| \sum_{k\in F} \varepsilon_k z_k \Big\| \leq C. \tag{3.4}$$

The Gustavsson–Peetre's interpolation space $\langle X_0, X_1, \theta\rangle$, with the norm

$$\|x\|_{\langle X_0,X_1,\theta\rangle} := \inf C,$$

where the infimum is taken over all sequences $(x_k)_{k\in\mathbb{Z}}$ verifying (GP1), (GP2a) and (GP2b), and $C > 0$ verifying (3.4), is a Banach space [9].

The relation of the Gustavsson–Peetre's interpolation space $\langle X_0, X_1, \theta\rangle$ and the Calderon's interpolation space $[X_0, X_1]^{[\theta]}$ for an arbitrary Banach couple (X_0, X_1) is given by the continuous inclusion $\langle X_0, X_1, \theta\rangle \subset [X_0, X_1]^{[\theta]}$. See [10, Theorem 5 and Section 7].

In the case of spaces of p–integrable functions we have the following result. For the proof we have adapted an argument used by Gagliardo in [8] (see also [15, Theorem 3.1]).

Proposition 3.3

If $1 \leq p_0 \neq p_1 \leq \infty$, then

$$L^{p(\theta)}_w(m) \subset \langle L^{p_0}(m), L^{p_1}(m), \theta\rangle,$$

and this inclusion is continuous.

Proof. Without loss of generality we can assume that $1 \leq p_0 < p_1 \leq \infty$. Then $p_0 < p(\theta) < p_1$, and so

$$L^{p_0}(m) \cap L^{p_1}(m) = L^{p_1}(m) \subset L^{p(\theta)}_w(m) \subset L^{p_0}(m) + L^{p_1}(m) = L^{p_0}(m),$$

where equalities mean equality as vector spaces with equivalent norms.

From (3.3) it is easy to verify that $\frac{-(1-\theta)p_1}{p_1 - p(\theta)} = \frac{-\theta p_0}{p(\theta) - p_0}$. Now define the constant $c := 2^{-\frac{\theta p_0}{p(\theta)-p_0}} = 2^{-\frac{(1-\theta)p_1}{p_1-p(\theta)}} < 1$.

For an arbitrary $f \in L_w^{p(\theta)}(m)$ let us consider the countable partition $\{A_k\}_{k\in\mathbb{Z}}$ of Ω given by the measurable sets

$$A_k := \left\{\omega \in \Omega : c^k \le |f(\omega)| < c^{k-1}\right\},$$

and also consider the sequence $(f_k)_{k\in\mathbb{Z}}$ of bounded functions defined by $f_k := f\chi_{A_k}$. We are going to check that f and the series $\sum_{k\in\mathbb{Z}} f_k$ verify the conditions on the definition of the Gustavsson–Peetre space.

(GP1) First, let us note that $f = \sum_{k\in\mathbb{Z}} f_k$ pointwise and each of its partial sums is pointwise bounded by $|f| \in L_w^{p(\theta)}(m) \subset L^{p_0}(m)$. The dominated convergence theorem in $L^{p_0}(m)$ implies the convergence of $\sum_{k\in\mathbb{Z}} f_k$ to f in the norm of $L^{p_0}(m)$.

(GP2a) Let us verify that $\sum_{k\in\mathbb{Z}} \frac{1}{2^{k\theta}} f_k$ is weakly unconditionally Cauchy in $L^{p_0}(m)$. More precisely, let us verify that for every finite set $F \subset \mathbb{Z}$ and every finite scalar sequence $(\varepsilon_k)_{k\in F}$, with $|\varepsilon_k| \le 1$, we have

$$\Big\| \sum_{k\in F} \frac{\varepsilon_k}{2^{k\theta}} f_k \Big\|_{p_0}^{p_0} \le \|f\|_{p(\theta)}^{p(\theta)}. \tag{3.5}$$

For every $x^* \in B_1(X^*)$ we have

$$\begin{aligned}
\int_\Omega \Big| \sum_{k\in F} \frac{\varepsilon_k}{2^{k\theta}} f_k \Big|^{p_0} d|\langle m, x^*\rangle| &\le \sum_{k\in F} \frac{1}{2^{k\theta p_0}} \int_{A_k} |f|^{p_0} d|\langle m, x^*\rangle| \\
&\le \sum_{k\in F} \frac{1}{2^{k\theta p_0}} \frac{1}{c^{k(p(\theta)-p_0)}} \int_{A_k} |f|^{p(\theta)} d|\langle m, x^*\rangle| \\
&= \int_{\bigcup_{k\in F} A_k} |f|^{p(\theta)} d|\langle m, x^*\rangle| \le \|f\|_{p(\theta)}^{p(\theta)}.
\end{aligned}$$

Taking supremum when x^* runs through $B_1(X^*)$ we obtain (3.5).

(GP2b) In a similar way we can prove that for every finite subset $F \subset \mathbb{Z}$ and every finite sequence $(\varepsilon_k)_{k\in F}$, with $|\varepsilon_k| \le 1$, we have that, when $p_1 < \infty$,

$$\Big\| \sum_{k\in F} \frac{\varepsilon_k}{2^{k(\theta-1)}} f_k \Big\|_{p_1}^{p_1} \le 2^{(1-\theta)p_1} \|f\|_{p(\theta)}^{p(\theta)}. \tag{3.6}$$

That is, the series $\sum_{k\in\mathbb{Z}} \frac{1}{2^{k(\theta-1)}} f_k$ is weakly unconditionally Cauchy in $L^{p_1}(m)$. For $p_1 = \infty$ we have that $c = 2^{-(1-\theta)}$ and

$$\Big\| \sum_{k\in F} \frac{\varepsilon_k}{2^{k(\theta-1)}} f_k \Big\|_\infty = \max_{k\in F} \frac{|\varepsilon_k|}{2^{k(\theta-1)}} \|f_k\|_\infty \le \max_{k\in F} \frac{c^{k-1}}{2^{k(\theta-1)}} = 2^{1-\theta}. \tag{3.7}$$

Finally, from (3.5) and (3.6) (or (3.7)) it follows that $\|f\|_{\langle\theta\rangle} \le 2^{(1-\theta)}$, if $\|f\|_{p(\theta)} = 1$, and $f \in L_w^{p(\theta)}(m)$. Therefore the inclusion

$$L_w^{p(\theta)}(m) \subset \langle L^{p_0}(m), L^{p_1}(m), \theta\rangle$$

is continuous. □

For the Calderón interpolation spaces we obtain the following result.

Theorem 3.4

Let $1 \le p_0 \ne p_1 \le \infty$ and $0 < \theta < 1$. The following equalities hold isometrically.

1) $[L^{p_0}(m), L^{p_1}(m)]_{[\theta]} = [L_w^{p_0}(m), L^{p_1}(m)]_{[\theta]} = [L_w^{p_0}(m), L_w^{p_1}(m)]_{[\theta]} = L^{p(\theta)}(m)$.
2) $[L^{p_0}(m), L^{p_1}(m)]^{[\theta]} = [L_w^{p_0}(m), L^{p_1}(m)]^{[\theta]} = [L_w^{p_0}(m), L_w^{p_1}(m)]^{[\theta]} = L_w^{p(\theta)}(m)$.

Proof. Without loss of generality we can assume that $1 \le p_0 < p_1 \le \infty$. Then, $p_0 < p(\theta) < p_1$ and having in mind Proposition 2.3–3), the intersection $L_w^{p_0}(m) \cap L_w^{p_1}(m)$ is, in any case, included in $L^{p(\theta)}(m)$, that is,

$$L_w^{p_0}(m) \cap L_w^{p_1}(m) = L_w^{p_1}(m) \subset L^{p(\theta)}(m).$$

Moreover, clearly, we also have $L_w^{p_0}(m) + L_w^{p_1}(m) = L_w^{p_0}(m)$.

1) Since $L^{p_0}(m) \cap L^{p_1}(m) \subset [L^{p_0}(m), L^{p_1}(m)]_{[\theta]} \subset [L_w^{p_0}(m), L_w^{p_1}(m)]_{[\theta]}$ and simple functions are dense in $L^{p(\theta)}(m)$ it is enough to prove that

$$[L_w^{p_0}(m), L_w^{p_1}(m)]_{[\theta]} \subset L^{p(\theta)}(m). \tag{3.8}$$

Since the closure of $L_w^{p_1}(m)$ in $L_w^{p_0}(m)$ is $L^{p_0}(m)$, by applying [2, Theorem 4.2.2(b)] we have that

$$[L_w^{p_0}(m), L_w^{p_1}(m)]_{[\theta]} = [L^{p_0}(m), L_w^{p_1}(m)]_{[\theta]}\,.$$

But we know, from [3], that $[L^{p_0}(m), L_w^{p_1}(m)]_{[\theta]} \subset (L^{p_0}(m))^{1-\theta}\,(L_w^{p_1}(m))^{\theta}$. Then inclusion (3.8) follows from Proposition 3.2–1).

To end the proof of 1) note that the norms in the spaces

$$[L^{p_0}(m), L^{p_1}(m)]_{[\theta]}\,, [L_w^{p_0}(m), L^{p_1}(m)]_{[\theta]} \text{ and } [L_w^{p_0}(m), L_w^{p_1}(m)]_{[\theta]}$$

coincide since $[\,\cdot\,,\,\cdot\,]_{[\theta]}$ is an exact interpolation functor (see [2, Theorem 4.1.2]). On the other hand, the norms in the spaces $[L^{p_0}(m), L^{p_1}(m)]_{[\theta]}$ and $L^{p(\theta)}(m)$ coincide because $L^{p(\theta)}(m)$ has order continuous norm (see [3, 13.6.(ii)] or [12, Theorem IV.1.14]).

2) Note that it is enough to prove that

$$[L_w^{p_0}(m), L_w^{p_1}(m)]^{[\theta]} \subset L_w^{p(\theta)}(m) \subset [L^{p_0}(m), L^{p_1}(m)]^{[\theta]}\,.$$

The first inclusion is obtained as follows. We always have

$$[L_w^{p_0}(m), L_w^{p_1}(m)]^{[\theta]} \subset \widetilde{[L_w^{p_0}(m), L_w^{p_1}(m)]_{[\theta]}}^{\mathcal{G}(L_w^{p_0}(m)+L_w^{p_1}(m))},$$

(see (3.1)), but we know, from part 1), that $[L_w^{p_0}(m), L_w^{p_1}(m)]_{[\theta]} = L^{p(\theta)}(m)$, and so

$$[L_w^{p_0}(m), L_w^{p_1}(m)]^{[\theta]} \subset \widetilde{L^{p(\theta)}(m)}^{\mathcal{G}L_w^{p_0}(m)}.$$

Finally recall that we obtained in Proposition 2.3–3) that

$$\widetilde{L^{p(\theta)}(m)}^{\mathcal{G}L_w^{p_0}(m)} = L_w^{p(\theta)}(m).$$

The second inclusion follows directly from Proposition 3.3 and [10, Theorem 5 and Section 7].

The equality of the norms of the spaces

$$[L^{p_0}(m), L^{p_1}(m)]^{[\theta]}, [L_w^{p_0}(m), L^{p_1}(m)]^{[\theta]} \text{ and } [L_w^{p_0}(m), L_w^{p_1}(m)]^{[\theta]}$$

follows from the fact that $[\,\cdot\,,\,\cdot\,]^{[\theta]}$ is also an exact interpolation functor (see [2, Theorem 4.1.4]). The equality of the norms of $L_w^{p(\theta)}(m)$ and $[L_w^{p_0}(m), L_w^{p_1}(m)]^{[\theta]}$ follows from [3, 13.6.(ii)] having in mind that $L_w^{p(\theta)}(m)$ has the Fatou property. □

A basic tool in the proof of the theorem above is the fact that $L_w^{p_1}(m)$ is included in $L^{p_0}(m)$, when $1 \leq p_0 < p_1$. This inclusion can be interpreted as a particular case of a more general result on the relationship between the two Calderón complex interpolation methods that it is interesting by itself. We formulate this comment precisely in the following theorem.

Theorem 3.5

Let (X_0, X_1) be a Banach couple such that X_1 is continuously included in X_0, and let $0 \leq \alpha < \beta < 1$. Then

$$[X_0, X_1]^{[\beta]} \subset \widetilde{[X_0, X_1]_{[\beta]}}^{\mathcal{G}X_0} \subset [X_0, X_1]_{[\alpha]}.$$

These inclusions are both continuous.

Proof. Since the inclusion of X_1 into X_0 is continuous, $X_0 \cap X_1$ coincides isomorphically with X_1 and $X_0 + X_1$ with X_0. The first inclusion is always true, see (3.1). To prove the second inclusion recall that the following inclusions

$$[X_0, X_1]_{[\beta]} \subset [X_0, X_1]_{[\alpha]} \subset [X_0, X_1]_{[0]} \subset X_0,$$

are all continuous, and $[X_0, X_1]_{[0]}$ is a closed subspace of X_0. Moreover, $\|y\|_{[X_0,X_1]_{[0]}} = \|y\|_{X_0}$ for all $y \in [X_0, X_1]_{[0]}$ (see [2, Theorem 4.2.2(c)]). Now let us consider $x \in \widetilde{[X_0, X_1]_{[\beta]}}^{\mathcal{G}X_0}$ and a bounded sequence $(x_n)_n$ in $[X_0, X_1]_{[\beta]}$ which converges to x in the norm of X_0 (and therefore in $[X_0, X_1]_{[0]}$). Put $M := \sup\left\{\|x_n\|_{[X_0,X_1]_{[\beta]}} : n \geq 1\right\}$. We are going to prove that $(x_n)_n$ is a Cauchy sequence in $[X_0, X_1]_{[\alpha]}$. By applying the Reiteration Theorem of [4] we can obtain $[X_0, X_1]_{[\alpha]}$ as

$$[X_0, X_1]_{[\alpha]} = \Big[[X_0, X_1]_{[0]}, [X_0, X_1]_{[\beta]}\Big]_{[\eta]},$$

with equality of norms, taking $\eta := \dfrac{\alpha}{\beta}$. From [12, IV.§1.9] we get

$$\begin{aligned} \|x_n - x_k\|_{[X_0,X_1]_{[\alpha]}} &\leq \|x_n - x_k\|_{[X_0,X_1]_{[0]}}^{1-\eta} \|x_n - x_k\|_{[X_0,X_1]_{[\beta]}}^{\eta} \\ &\leq (2M)^{\eta} \|x_n - x_k\|_{[X_0,X_1]_{[0]}}^{1-\eta}, \end{aligned}$$

for every $n, k \in \mathbb{N}$, and therefore $(x_n)_n$ is a Cauchy sequence in $[X_0, X_1]_{[\alpha]}$. Clearly $(x_n)_n$ converges to x in $[X_0, X_1]_{[\alpha]}$ and the inclusion

$$\widetilde{[X_0, X_1]_{[\beta]}}^{\mathcal{G}X_0} \subset [X_0, X_1]_{[\alpha]}$$

is proved. In order to show that this inclusion is continuous we can use similar arguments. Indeed, if the Gagliardo norm of $x \in \widetilde{[X_0, X_1]_{[\beta]}}^{\mathcal{G}X_0}$ equals 1, and $\varepsilon > 0$ is given, we can select a bounded sequence $(x_n)_n$ in $[X_0, X_1]_{[\beta]}$ such that $\|x_n\|_{[X_0,X_1]_{[\beta]}} \leq 1 + \varepsilon$, for all $n \geq 1$. If we denote by $C > 0$ the norm of the continuous inclusion $[X_0, X_1]_{[\beta]} \subset [X_0, X_1]_{[0]}$, then

$$\begin{aligned} \|x\|_{[X_0,X_1]_{[\alpha]}} &= \lim_n \|x_n\|_{[X_0,X_1]_{[\alpha]}} \leq \limsup_n \|x_n\|^{1-\eta}_{[X_0,X_1]_{[0]}} \|x_n\|^{\eta}_{[X_0,X_1]_{[\beta]}} \\ &\leq C^{1-\eta} \limsup_n \|x_n\|^{1-\eta}_{[X_0,X_1]_{[\beta]}} \|x_n\|^{\eta}_{[X_0,X_1]_{[\beta]}} \leq C^{1-\eta}(1+\varepsilon), \end{aligned}$$

and therefore $\|x\|_{[X_0,X_1]_{[\alpha]}} \leq C^{1-\eta}$. □

Remark 3.6 1) Related to the theorem above, we would like to mention that, as far as we know, it is not known in general, even for Banach lattices, whether the second complex interpolation space $[X_0, X_1]^{[\theta]}$ coincides with the Gagliardo completion of the first one $[X_0, X_1]_{[\theta]}$.

2) We would like to thank F. Cobos (Universidad Complutense de Madrid) for pointing out another proof of the inclusion $[X_0, X_1]^{[\beta]} \subset [X_0, X_1]_{[\alpha]}$, for $0 < \alpha < \beta < 1$, when X_1 is included continuously in X_0, by using certain relationships between the complex interpolation methods with the real interpolation methods. Since $[\,\cdot\,,\,\cdot\,]_{[\theta]}$ and $[\,\cdot\,,\,\cdot\,]^{[\theta]}$ are interpolation functors of exponent $0 < \theta < 1$, by applying [2, Theorem 4.7.1] and [5, Lemma 1.1] we have, for an arbitrary Banach couple (X_0, X_1), the continuous inclusions

$$\begin{aligned} (X_0, X_1)_{\theta,1} \subset [X_0, X_1]_{[\theta]} \subset (X_0, X_1)_{\theta,\infty}\,, \\ (X_0, X_1)_{\theta,1} \subset [X_0, X_1]^{[\theta]} \subset (X_0, X_1)_{\theta,\infty}\,. \end{aligned}$$

Now, if the inclusion $X_1 \subset X_0$ is continuous, by applying [2, Theorem 3.4.1] we have the continuous inclusion $(X_0, X_1)_{\theta_1,q_1} \subset (X_0, X_1)_{\theta_0,q_0}$, for every $\theta_0 < \theta_1$ and every $1 \leq q_0, q_1 \leq \infty$. Therefore, we obtain the continuous inclusions

$$[X_0, X_1]^{[\beta]} \subset (X_0, X_1)_{\beta,\infty} \subset (X_0, X_1)_{\alpha,1} \subset [X_0, X_1]_{[\alpha]},$$

for every $0 < \alpha < \beta < 1$.

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