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A fixed point theorem in locally convex spaces

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Abstract

For a locally convex space $\mathscr{X}$ with the topology given by a family $\{p(\cdot\,;\,\alpha)\}_{\alpha\in\Omega}$ of seminorms, we study the existence and uniqueness of fixed points for a mapping $\mathscr{K}:\mathscr{D}_{\mathscr{K}}\to\mathscr{D}_{\mathscr{K}}$ defined on some set $\mathscr{D}_{\mathscr{K}}\subset\mathscr{X}$. We require that there exists a linear and positive operator K, acting on functions defined on the index set Ω, such that for every $u,v\in\mathscr{D}_{\mathscr{K}}$

$$p(\mathscr{K}(u)-\mathscr{K}(v)\,;\,\alpha)\leq K(p(u-v\,;\,\cdot))(\alpha),\quad \alpha\in\Omega.$$

Under some additional assumptions, one of which is the existence of a fixed point for the operator $K+p(\mathscr{K}(0)\,;\,\cdot)$, we prove that there exists a fixed point of $\mathscr{K}$. For a class of elements satisfying $K^n(p(u\,;\,\cdot))(\alpha)\to 0$ as $n\to\infty$, we show that fixed points are unique. This class includes, in particular, the class for which we prove the existence of fixed points.

We consider several applications by proving existence and uniqueness of solutions to first and second order nonlinear differential equations in Banach spaces. We also consider pseudo-differential equations with nonlinear terms.

1. Introduction

In this article, we study existence and uniqueness of fixed points for a certain type of mappings on locally convex spaces. Spaces of this type arise in many applications, where there is no natural Banach space to work in and the topology is given by

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seminorms. Seminorms may also provide better understanding of local behaviour of solutions to, e.g., integral and differential equations.

We let $\mathscr{X}$ denote a locally convex topological space, where the topology is given by a family $\{p(\,\cdot\,;\alpha)\}_{\alpha\in\Omega}$ of seminorms that separates points. The index set Ω is not assumed to have any specific structure. We want to solve the equation

$$\mathscr{K}(u) = u, \quad u \in \mathscr{D}_{\mathscr{K}}, \tag{1}$$

where $\mathscr{K}\colon \mathscr{D}_{\mathscr{K}} \to \mathscr{D}_{\mathscr{K}}$ is a mapping defined on a subset $\mathscr{D}_{\mathscr{K}} \subset \mathscr{X}$. We assume that $0 \in \mathscr{D}_{\mathscr{K}}$. The novelty of our approach consists of the use of an auxiliary linear and positive operator K that acts on functions defined on the index set Ω. More specifically, we assume that $K\colon \mathscr{D}_K \to \mathbb{R}^\Omega$, where $\mathscr{D}_K \subset \mathbb{R}^\Omega$ is a linear subspace. By $\mathbb{R}^\Omega$ we denote the set of all real-valued functions on Ω, endowed with the topology of pointwise convergence.

We suppose that the operator $\mathscr{K}$ is subordinated to K; in particular, we require that

$$p(\mathscr{K}(u) - \mathscr{K}(v)\,;\alpha) \le K(p(u-v\,;\cdot))(\alpha), \quad \alpha \in \Omega.$$

Under some natural assumptions on the operator K (see $(K1)$–$(K4)$) which guarantee, in particular, the existence of a minimal non-negative solution σ in $\mathscr{D}_K$ to the equation

$$\sigma(\alpha) = K\sigma(\alpha) + p(\mathscr{K}(0)\,;\alpha), \quad \alpha \in \Omega, \tag{2}$$

we prove that equation (1) has a solution in $\mathscr{D}_{\mathscr{K}}$; see Theorem 2.4. In other words, the existence of a fixed point to a "simpler" operator $K + p(\mathscr{K}(0)\,;\,\cdot\,)$ implies the existence of a fixed point of the operator $\mathscr{K}$.

If the functions from $\mathscr{D}_{\mathscr{K}}$ satisfy $\lim_{n\to\infty} K^n(p(u\,;\cdot))(\alpha) = 0$, we also prove that the fixed point is unique; see Theorem 2.5.

In Section 2.6, we show how Banach's fixed point principle for contractive mappings on Banach spaces may be deduced from our result. For other generalisations of Banach's contraction principle, we refer to Dugundji and Granas [1] and references therein.

In Section 3, we then consider some applications, starting with two types of nonlinear differential equations in a Banach space. First, in Section 3.1, we treat a first order equation, where the right-hand side satisfies a Lipschitz-Carathéodory condition. In Section 3.2, we then consider a second order equation of Sturm-Liouville type, where the nonlinear term satisfies a similar condition. For both of these equations, we prove existence and uniqueness results. In the last section, we consider a class of nonlinear pseudo-differential equations on $\mathbb{R}^N$. For more examples of applications where the above fixed point theorems are useful, we refer to Kozlov and Maz'ya [5, 6].

2. Main results

2.1 The operator K

We suppose that $K\colon \mathscr{D}_K \to \mathbb{R}^\Omega$, where $\mathscr{D}_K \subset \mathbb{R}^\Omega$ and K is linear, and require that K is subject to the following conditions.

$(K1)$ *Positivity of K.* The operator K is positive, i.e., if $\eta \in \mathscr{D}_K$ is non-negative, then $K\eta \geq 0$.

$(K2)$ *Fixed point inequality.* The function $k_0(\,\cdot\,) = p(\mathscr{K}(0)\,;\,\cdot\,) \in \mathscr{D}_K$, and there exists a non-negative function $z \in \mathscr{D}_K$ such that

$$z(\alpha) \geq Kz(\alpha) + k_0(\alpha), \quad \alpha \in \Omega, \tag{3}$$

and $Kz \in \mathscr{D}_K$.

$(K3)$ *Monotone closedness of K.* The operator K is closed for non-negative, increasing sequences: if $\{\eta_n\}$ is a non-negative sequence in $\mathscr{D}_K$ such that $\eta_n \nearrow \eta$, $\eta \leq z$, and $K\eta_n \nearrow \zeta$, then $\eta \in \mathscr{D}_K$ and $K\eta = \zeta$.

$(K4)$ *Invariance property.* If $\eta, \zeta \in \mathscr{D}_K$ such that $0 \leq \eta \leq \zeta$ and $K\zeta \in \mathscr{D}_K$, then $K\eta \in \mathscr{D}_K$.

The existence of a non-negative solution z to (3) allows us to prove the existence of a non-negative solution to the equation

$$\sigma(\alpha) = K\sigma(\alpha) + k_0(\alpha), \quad \alpha \in \Omega, \tag{4}$$

which is minimal in the sense that if $\eta \in \mathscr{D}_K$ is another non-negative solution to (4), then $\sigma \leq \eta$.

Lemma 2.1

Suppose that $(K1)$ to $(K4)$ are satisfied. Then there exists a unique minimal solution $\sigma \in \mathscr{D}_K$ to (4) such that $\sigma \leq z$. This solution is the limit of the iterations

$$\sigma_0 = 0 \qquad \textit{and} \qquad \sigma_{k+1} = K\sigma_k + k_0, \quad k = 0, 1, 2, \ldots, \tag{5}$$

which are well-defined and converge for all $\alpha \in \Omega$. Moreover, for $n = 1, 2, \ldots$, $K^n\sigma$ belongs to $\mathscr{D}_K$ and $K^n\sigma \to 0$ as $n \to \infty$.

Proof. To see that the iterations are well-defined, we proceed by induction. Obviously σ_0 belongs to $\mathscr{D}_K$, $\sigma_0 \leq z$, and by $(K2)$, $\sigma_1 = k_0 \in \mathscr{D}_K$. Assume that $\sigma_k \in \mathscr{D}_K$ and $\sigma_k \leq z$. Then $K\sigma_k$ belongs to $\mathscr{D}_K$ by $(K4)$ and hence, σ_{k+1} belongs to $\mathscr{D}_K$ since $\mathscr{D}_K$ is a linear space. Now, $(K1)$ and $(K2)$ imply that $\sigma_{k+1} \leq Kz + k_0 \leq z$. Thus, the sequence $\{\sigma_k\}$ is well-defined and $\sigma_k \leq z$ for every $k \geq 0$. Furthermore, this sequence is increasing. Indeed, $\sigma_1 \geq \sigma_0$. Assume that $\sigma_k \geq \sigma_{k-1}$. Then, by $(K1)$,

$$\sigma_{k+1} - \sigma_k = K\sigma_k - K\sigma_{k-1} \geq 0.$$

This implies that $\sigma_k(\alpha)$ converges to a number $\sigma(\alpha)$ for every $\alpha \in \Omega$. Obviously $\sigma \leq z$. Moreover, we have $K\sigma_k = \sigma_{k+1} - k_0$, and therefore $K\sigma_k \nearrow \sigma - k_0$. Hence, by $(K3)$, we obtain that $\sigma \in \mathscr{D}_K$ and $K\sigma = \sigma - k_0$.

Let $0 \leq \eta \in \mathscr{D}_K$ be another solution to (4). The argument above with z replaced by η shows that $\sigma \leq \eta$. This proves that σ is minimal.

Let us finally show that $K^n\sigma$ belongs to $\mathscr{D}_K$ and that $K^n\sigma \to 0$ as $n \to \infty$. It is clear that $K\sigma$ belongs to $\mathscr{D}_K$ and that $K\sigma = \sigma - \sigma_1$. Assume that $K^n\sigma \in \mathscr{D}_K$ and that $K^n\sigma = \sigma - \sigma_n$. Then $K^n\sigma \leq z$ and $(K4)$ implies that $K^{n+1}\sigma$ belongs to $\mathscr{D}_K$. We also obtain that

$$K^{n+1}\sigma = K(\sigma - \sigma_n) = \sigma - \sigma_{n+1}. \tag{6}$$

Thus, $K^n\sigma$ is well-defined for all $n \in \mathbb{N}$ and tends to zero as $n \to \infty$. $\square$

2.2 The operator $\mathscr{K}$

Suppose that the operator $\mathscr{K}$ maps $\mathscr{D}_{\mathscr{K}}$ into $\mathscr{D}_{\mathscr{K}}$. We let σ be the minimal solution to (4), and put

$$\mathscr{D}_{\mathscr{K},\sigma} = \big\{u \in \mathscr{D}_{\mathscr{K}} : p(u\,;\,\alpha) \le \sigma(\alpha) \text{ for every } \alpha \in \Omega\big\}.$$

We shall require the following properties to hold.

($\mathscr{K}$1) *Subordination to K.* If u, v belong to $\mathscr{D}_{\mathscr{K},\sigma}$, then $p(u-v\,;\,\cdot\,)$ belongs to $\mathscr{D}_K$, and we have

$$p(\mathscr{K}(u) - \mathscr{K}(v)\,;\,\alpha) \le K(p(u-v\,;\,\cdot))(\alpha), \quad \alpha \in \Omega. \tag{7}$$

($\mathscr{K}$2) *Closedness of $\mathscr{D}_{\mathscr{K},\sigma}$.* If $\{v_k\}_{k=0}^{\infty}$ is a sequence in $\mathscr{D}_{\mathscr{K},\sigma}$ such that $v_0 = 0$ and

$$\sum_{k=0}^{\infty} p(v_{k+1} - v_k\,;\,\alpha) \le \sigma(\alpha), \quad \alpha \in \Omega, \tag{8}$$

then the limit of v_k exists and belongs to $\mathscr{D}_{\mathscr{K},\sigma}$.

As an example of the condition in ($\mathscr{K}$2), consider the following.

Remark 2.2 Let $\mathscr{X}$ be a sequentially complete space and suppose that $\mathscr{D}_{\mathscr{K}}$ is sequentially closed. If a sequence in $\mathscr{D}_{\mathscr{K},\sigma}$ satisfies (8), then it is clearly a Cauchy sequence. By completeness it converges to some element $v \in \mathscr{X}$ for which $p(v\,;\,\alpha) \le \sigma(\alpha)$, $\alpha \in \Omega$. Now, the fact that $\mathscr{D}_{\mathscr{K}}$ is sequentially closed immediately implies that $v \in \mathscr{D}_{\mathscr{K},\sigma}$.

Since $0 \in \mathscr{D}_{\mathscr{K},\sigma}$, ($\mathscr{K}$1) implies that $p(u\,;\,\cdot\,) \in \mathscr{D}_K$ for $u \in \mathscr{D}_{\mathscr{K}}$. We also obtain the following lemma concerning properties of elements in $\mathscr{D}_{\mathscr{K},\sigma}$.

Lemma 2.3

Suppose that ($\mathscr{K}$1) holds. Then

(i) *the operator $\mathscr{K}$ maps $\mathscr{D}_{\mathscr{K},\sigma}$ into itself;*

(ii) *if $u \in \mathscr{D}_{\mathscr{K},\sigma}$, then $K^n(p(u\,;\,\cdot))$ is well-defined for every non-negative integer n and $\lim_{n\to\infty} K^n(p(u\,;\,\cdot))(\alpha) = 0$ for every $\alpha \in \Omega$.*

Proof. Let $u \in \mathscr{D}_{\mathscr{K},\sigma}$. The assumption ($\mathscr{K}$1) implies that $p(u\,;\,\cdot\,) \in \mathscr{D}_K$ and

$$\begin{aligned} p(\mathscr{K}(u)\,;\,\alpha) &\le p(\mathscr{K}(u) - \mathscr{K}(0)\,;\,\alpha) + k_0(\alpha) \\ &\le K(p(u\,;\,\cdot)) + k_0(\alpha) \\ &\le K\sigma(\alpha) + k_0(\alpha) \\ &= \sigma(\alpha). \end{aligned}$$

This proves (i). To prove (ii), we notice that $K(p(u\,;\,\cdot)) \in \mathscr{D}_K$ by (K4). Assume that $K^n(p(u\,;\,\cdot))$ belongs to $\mathscr{D}_K$. Since $p(u\,;\,\cdot\,) \le \sigma$ and (6) holds for every $n \in \mathbb{N}$, it follows that

$$K^n(p(u\,;\,\cdot)) \le K^n\sigma = \sigma - \sigma_n \le \sigma \le z.$$

Then (K4) implies that $K^{n+1}(p(u\,;\,\cdot)) \in \mathscr{D}_K$. Moreover, by Lemma 2.1 we know that $\sigma_n \to \sigma$, which proves that $K^n(p(u\,;\,\cdot)) \to 0$ by the previous inequality. □

2.3 Existence of fixed points

We wish to find a fixed point of $\mathscr{K}$. This will be accomplished by the following iterations:

$$u_0 = 0 \in \mathscr{D}_{\mathscr{K},\sigma} \qquad \text{and} \qquad u_{k+1} = \mathscr{K}(u_k), \quad k = 0, 1, 2, \ldots. \tag{9}$$

By Lemma 2.3(i), this sequence is well-defined and every element of the sequence belongs to $\mathscr{D}_{\mathscr{K},\sigma}$. We will now prove that the iterations converge to a solution to (1).

Theorem 2.4

Suppose that K satisfies $(K1)$ to $(K4)$ and that $\mathscr{K}$ satisfies $(\mathscr{K}1)$ and $(\mathscr{K}2)$. Then there exists a fixed point of $\mathscr{K}$ in $\mathscr{D}_{\mathscr{K},\sigma}$. This fixed point is the limit of the iterations in (9)*.*

Proof. We have $u_1 = \mathscr{K}(0)$, so

$$p(u_1 - u_0\,;\, \alpha) = k_0(\alpha) = \sigma_1(\alpha) - \sigma_0(\alpha).$$

We proceed by induction to show that

$$p(u_{k+1} - u_k\,;\, \alpha) \le \sigma_{k+1}(\alpha) - \sigma_k(\alpha), \quad k = 0, 1, 2, \ldots. \tag{10}$$

Assume that $p(u_{k+1} - u_k\,;\, \alpha) \le \sigma_{k+1}(\alpha) - \sigma_k(\alpha)$. Then, by $(\mathscr{K}1)$,

$$\begin{aligned} p(u_{k+2} - u_{k+1}\,;\, \alpha) &= p(\mathscr{K}(u_{k+1}) - \mathscr{K}(u_k)\,;\, \alpha) \\ &\le K(p(u_{k+1} - u_k\,;\, \cdot))(\alpha) \\ &\le K\sigma_{k+1}(\alpha) - K\sigma_k(\alpha). \end{aligned}$$

Hence, (10) holds. Since σ_k converges to σ, it follows that

$$\sum_{k=0}^{\infty} p(u_{k+1} - u_k\,;\, \cdot\,) \le \sigma - \sigma_0 = \sigma.$$

Thus $(\mathscr{K}2)$ implies that u_k converges to some element u in $\mathscr{D}_{\mathscr{K},\sigma}$. We also see that

$$p(u - u_k\,;\, \alpha) \le \sigma(\alpha) - \sigma_k(\alpha). \tag{11}$$

By (11), (4) and the definition of σ_k, we now obtain that

$$\begin{aligned} p(\mathscr{K}(u) - \mathscr{K}(u_k)\,;\, \alpha) &\le K(p(u - u_k\,;\, \cdot))(\alpha) \\ &\le K\sigma(\alpha) - K\sigma_k(\alpha) \\ &= \sigma(\alpha) - \sigma_{k+1}(\alpha). \end{aligned}$$

Thus, $\mathscr{K}(u_k) \to \mathscr{K}(u)$. Since we also know that $\mathscr{K}(u_k) = u_{k+1} \to u$, it is clear that u is a fixed point of $\mathscr{K}$. $\square$

2.4 Uniqueness of fixed points

We now turn to prove a uniqueness result. Suppose that the operator $\mathscr{K}$ maps $\mathscr{D}_{\mathscr{K}}$ into itself. We shall assume that the following conditions hold.

(I) If $u \in \mathscr{D}_{\mathscr{K}}$, then $K^n(p(u\,;\,\cdot))$ is defined and belongs to $\mathscr{D}_K$ for every non-negative integer n, and $\lim_{n\to\infty} K^n(p(u\,;\,\cdot)) = 0$.

(II) If $\eta, \zeta \in \mathscr{D}_K$ such that $0 \le \eta \le \zeta$ and $K\zeta \in \mathscr{D}_K$, then $K\eta \in \mathscr{D}_K$.

(III) If u, v belong to $\mathscr{D}_{\mathscr{K}}$, then the function $p(u - v\,;\,\cdot\,)$ belongs to $\mathscr{D}_K$, and (7) holds.

Theorem 2.5

Suppose that the operators $\mathscr{K}$ and K satisfy $(K1)$ and (I) *to* (III). *Then there exists at most one fixed point of $\mathscr{K}$.*

Proof. Let u and v be two fixed points of $\mathscr{K}$. Then (III) implies that

$$p(u - v\,;\,\alpha) = p(\mathscr{K}(u) - \mathscr{K}(v)\,;\,\alpha) \le K(p(u - v\,;\,\cdot))(\alpha), \quad \alpha \in \Omega. \tag{12}$$

Since $0 \in \mathscr{D}_{\mathscr{K}}$, it follows from (III) that $p(u\,;\,\cdot\,)$ and $p(v\,;\,\cdot\,)$ belong to $\mathscr{D}_K$. We also have $p(u - v\,;\,\alpha) \le p(u\,;\,\alpha) + p(v\,;\,\alpha) \in \mathscr{D}_K$ and, by (I), $K(p(u\,;\,\cdot\,) + p(v\,;\,\cdot\,))$ also belongs to $\mathscr{D}_K$, so it follows from (II) that $K(p(u - v\,;\,\cdot)) \in \mathscr{D}_K$. Assume that $K^n(p(u - v\,;\,\cdot))$ belongs to $\mathscr{D}_K$ for some $n \ge 1$. Then

$$K^n(p(u - v\,;\,\cdot)) \le K^n(p(u\,;\,\cdot)) + K^n(p(v\,;\,\cdot)) \in \mathscr{D}_K.$$

We also have $K(K^n p(u\,;\,\cdot\,) + K^n p(v\,;\,\cdot\,)) \in \mathscr{D}_K$, so $K^{n+1}(p(u - v\,;\,\cdot))$ belongs to $\mathscr{D}_K$ by (II). Thus, $K^n(p(u - v\,;\,\cdot))$ is well-defined for every $n \in \mathbb{N}$. Now, (12) implies that

$$p(u - v\,;\,\alpha) \le K^n(p(u\,;\,\cdot))(\alpha) + K^n(p(v\,;\,\cdot))(\alpha), \quad n \in \mathbb{N}, \alpha \in \Omega.$$

The assumption (I) now implies that $p(u - v\,;\,\alpha) = 0$ for every $\alpha \in \Omega$, which implies that $u = v$ since the seminorms separates points. $\square$

Let us restrict $\mathscr{K}$ to $\mathscr{D}_{\mathscr{K},\sigma}$. Suppose that K and $\mathscr{K}$ satisfy $(K1)$ to $(K4)$ and $(\mathscr{K}1)$ to $(\mathscr{K}2)$, respectively. It is immediate that $(\mathscr{K}1)$ implies (III). Since $p(u\,;\,\cdot\,) \le \sigma$ for every $u \in \mathscr{D}_{\mathscr{K}}$, $(K4)$ implies (II). Furthermore, (I) follows from Lemma 2.3(ii). Theorem 2.5 now shows that the fixed point in Theorem 2.4 is unique in the set $\mathscr{D}_{\mathscr{K},\sigma}$.

2.5 Error estimates

In the following theorem, σ_k are the iterations defined by (5) and σ is the limit, which exists by Lemma 2.1. Let $\{u_k\}$ be the iterations defined by (9) and let u be the limit.

Theorem 2.6

We assume that all assumptions in Theorem 2.4 are valid, that is, that K satisfy $(K1)$–$(K4)$ and that $\mathscr{K}$ satisfy $(\mathscr{K}1)$ and $(\mathscr{K}2)$. Then we have the following a priori estimate:

$$p(u - u_n\,;\,\alpha) \le \sigma(\alpha) - \sigma_n(\alpha), \quad n \in \mathbb{N}, \quad \alpha \in \Omega, \tag{13}$$

and the a posteriori estimate:

$$p(u - u_{n+1}\,;\,\alpha) \le \sum_{i=0}^{\infty} K^{i+1}(p(u_n - u_{n+1}\,;\,\cdot))(\alpha), \quad n \in \mathbb{N}, \quad \alpha \in \Omega, \tag{14}$$

where the series is finite for every $\alpha \in \Omega$.

Proof. Let $m, n \in \mathbb{N}$. From (10), it follows that

$$p(u_{n+m} - u_n\,;\,\alpha) \le \sigma_{n+m}(\alpha) - \sigma_n(\alpha) \tag{15}$$

If we let $m \to \infty$ in (15), (13) now follows.

To prove the a posteriori estimate, we observe that as in the proof of (ii) in Lemma 2.3, we have $K^i(p(u_n - u_m\,;\,\cdot)) \in \mathscr{D}_K$ for every $i \in \mathbb{N}$. This follows from $(K4)$ since $p(u_n - u_m\,;\,\cdot\,) \le \sigma_n - \sigma_m$ for $m \le n$. Thus,

$$\begin{aligned} p(u_{n+1} - u_{n+m+1}\,;\,\alpha) &\le \sum_{i=1}^{m} K^i(p(u_{n+1} - u_n\,;\,\cdot))(\alpha) \\ &= \sum_{i=0}^{m-1} K^i K(p(u_{n+1} - u_n\,;\,\cdot))(\alpha). \end{aligned} \tag{16}$$

Using the same argument as in the proof of (6), we obtain that $K^i\sigma_n = \sigma_{n+i} - \sigma_i$, and consequently

$$\begin{aligned} \sum_{i=0}^{m-1} K^i(\sigma_{n+2} - \sigma_{n+1}) &= \sum_{i=0}^{m-1} (\sigma_{n+i+2} - \sigma_{n+i+1}) \\ &= \sigma_{n+m+1} - \sigma_{n+1}. \end{aligned}$$

Since $K(p(u_{n+1} - u_n\,;\,\cdot)) \le \sigma_{n+2} - \sigma_{n+1}$, this shows that if we let $m \to \infty$ in (16), then the series in (14) is finite and the a posteriori estimate holds. $\square$

2.6 Comparison with Banach's fixed point theorem

We now demonstrate that our result is indeed a generalisation of Banach's fixed point theorem. Let $\mathscr{X}$ be a Banach space and $\mathscr{K} : \mathscr{D}_{\mathscr{K}} \to \mathscr{D}_{\mathscr{K}}$ a Lipschitz mapping, with constant γ, on a closed non-empty set $\mathscr{D}_{\mathscr{K}} \subset \mathscr{X}$. We have only one-seminorm, i.e., the norm $|\cdot|$ on $\mathscr{X}$. Thus, the index set Ω consists of one point and $\mathbb{R}^{\Omega} = \mathbb{R}$. Let K be the operator given by multiplication by γ. For a non-negative solution z to (3) to exist in general, it is necessary that $0 \le \gamma < 1$, i.e., that $\mathscr{K}$ is a contraction. The unique solution to (4) is given by $\sigma = (1-\gamma)^{-1}|\mathscr{K}(0)| \in \mathscr{D}_K$. Thus, if $\mathscr{K}$ is a contraction, then $(K2)$ holds. Obviously, K is linear and satisfies $(K1)$, $(K3)$, and $(K4)$. It is also obvious that $(\mathscr{K}1)$ and $(\mathscr{K}2)$ hold. Theorem 2.4 now shows that there exists a fixed point x of $\mathscr{K}$. Furthermore, $K^n|y| \to 0$ for every y in $\mathscr{X}$, so the fixed point is unique in $\mathscr{D}_{\mathscr{K}}$ by Theorem 2.5. From the expressions in (13) and (14), we also obtain the well-known prior and posterior estimates; see Zeidler [11, p. 19].

3.7 A first order differential equation

To show how to apply the abstract fixed point theorems, we give a new proof of a well-known solvability result for first order differential equations in a separable Banach space B, with the norm denoted by $|\cdot|$. We consider the following equation:

$$x'(t) = A(x(t), t), \quad t \geq 0, \tag{17}$$

where

$$x(0) = a \in B.$$

We suppose that $A\colon B \times [0, \infty) \to B$ satisfies the following Lipschitz-Carathéodory condition:

(A) For every fixed $x \in B$, $A(x, \cdot)$ is measurable[1], and there exists a function ω in $L^1_{\text{loc}}([0, \infty))$ such that for all $x, y \in B$ and every $t \geq 0$, we have

$$|A(x, t) - A(y, t)| \leq \omega(t)|x - y|. \tag{18}$$

This condition implies, in particular, that the composition $A(x(\cdot), \cdot)$ with a measurable function $x\colon [0, \infty) \to B$ is again measurable. We also require that $|A(0, \cdot)| \in L^1_{\text{loc}}([0, \infty))$.

Integrating (17), we obtain

$$x(t) = a + \int_0^t A(x(\tau), \tau)\, d\tau, \quad t \geq 0. \tag{19}$$

We let $L^1_{\text{loc}}([0, \infty)\,;\, B)$ denote the linear space of functions on $[0, \infty)$ into B that are locally Bochner integrable (see Hille [3, p. 78]). It is clear that the assumptions above imply that $A(x(\cdot), \cdot) \in L^1_{\text{loc}}([0, \infty)\,;\, B)$ if $x \in L^1_{\text{loc}}([0, \infty)\,;\, B)$, so equation (19) is well-defined. Similarly, by $L^\infty_{\text{loc}}([0, \infty)\,;\, B)$ we denote the linear space of measurable functions x mapping $[0, \infty)$ into B such that $|x(\cdot)|$ belongs to $L^\infty_{\text{loc}}([0, \infty))$. If there exists a solution $x \in L^\infty_{\text{loc}}([0, \infty)\,;\, B)$ to equation (19), x will be continuous and solves (17) in the sense of vector distributions; see Lions and Magenes [8, Section 1.3].

Theorem 3.1

There exists a unique solution x in $L^\infty_{\text{loc}}([0, \infty)\,;\, B)$ to (19), and this solution satisfies

$$|x(t)| \leq |a| \exp\biggl(\int_0^t \omega(\tau)\, d\tau\biggr), \quad t \geq 0.$$

Proof. Let $\mathscr{X}$ be the vector space of all mappings from $\Omega = [0, \infty)$ into B. We define the topology on $\mathscr{X}$ by the seminorms $p(x\,;\, t) = |x(t)|$, $x \in \mathscr{X}$, $t \geq 0$. Obviously these seminorms separate points, and it is easy to see that $\mathscr{X}$ is a complete, locally convex space with this topology.

[1] Since B is separable, strong measurability is equivalent with weak measurability (cf. Hille [3], Theorem 3.5.3).

We first prove existence of a solution to (19). Let $\mathscr{K}: \mathscr{D}_{\mathscr{K}} \to \mathscr{X}$ be defined by the right-hand side in (19), with $\mathscr{D}_{\mathscr{K}}$ chosen as $L^\infty_{\mathrm{loc}}([0,\infty)\,;B)$. Define $K\colon \mathscr{D}_K \to \mathbb{R}^\Omega$ by

$$K\eta(t) = \int_0^t \omega(\tau)\eta(\tau)\,d\tau, \quad t \geq 0,$$

with $\mathscr{D}_K = L^\infty_{\mathrm{loc}}([0,\infty))$. Obviously, if $x \in \mathscr{D}_{\mathscr{K}}$, then $\mathscr{K}(x)$ belongs to $\mathscr{D}_{\mathscr{K}}$. The condition (A) implies that $(\mathscr{K}1)$ holds. Clearly, K is linear, positive, and closed for non-negative increasing sequences (by the monotone convergence theorem), so $(K1)$ and $(K3)$ hold. Moreover, the function $\sigma \in \mathscr{D}_K$, defined by

$$\sigma(t) = |a| \exp\biggl(\int_0^t \omega(\tau)\,d\tau\biggr), \quad t \geq 0, \tag{20}$$

is the unique solution to $\sigma = K\sigma + |a|$, so we may choose $z = \sigma$, which proves that $(K2)$ is valid. For $\eta \in \mathscr{D}_K$, $K\eta$ is continuous, so $(K4)$ is satisfied. Suppose that the sequence $\{x_n\}_{n=0}^\infty$ satisfy the condition (8) in $(\mathscr{K}2)$. Clearly, this is a Cauchy sequence in $\mathscr{D}_{\mathscr{K},\sigma}$. Then x_n converges to some measurable $x \in \mathscr{X}$ such that $|x| \leq \sigma$. From this it follows that x belongs to $L^\infty_{\mathrm{loc}}([0,\infty)\,;B)$, which proves that $(\mathscr{K}2)$ holds. Thus, all requirements for Theorem 2.4 are satisfied, so there exists a fixed point $x \in \mathscr{D}_{\mathscr{K}}$ of $\mathscr{K}$ such that $|x| \leq \sigma$.

Next, we prove that this fixed point is unique. It is obvious that (II) and (III) hold. To prove that (I) is valid as well, we will use the following identity:

$$\int_0^t \omega(\tau_n) \int_0^{\tau_n} \omega(t_{n-1}) \int_0^{\tau_{n-1}} \cdots \int_0^{\tau_2} \omega(\tau_1)\,d\tau_1\,d\tau_2 \cdots d\tau_n = \frac{1}{n!}\biggl(\int_0^t \omega(\tau)\,d\tau\biggr)^n,$$

which holds for every $t \geq 0$ and $n \in \mathbb{N}$. For continuous ω, this can be checked by differentiation and for general ω, it follows by density of smooth functions. From this, we obtain that, for $\eta \in \mathscr{D}_K$,

$$|K^n\eta(t)| \leq \frac{1}{n!}\|\omega\|^n_{L^1([0,t])}\|\eta\|_{L^\infty([0,t])} \tag{21}$$

for every $t \geq 0$ and $n \in \mathbb{N}$. This shows that $K^n(p(y\,;\cdot)) \to 0$ for every $y \in \mathscr{D}_{\mathscr{K}}$, so the fixed point is unique by Theorem 2.5. □

Remark 3.2 If the function ω in (A) instead belongs to $L^\infty_{\mathrm{loc}}([0,\infty))$, one can choose $\mathscr{D}_K = L^1_{\mathrm{loc}}([0,\infty))$ and $\mathscr{D}_{\mathscr{K}} = L^1_{\mathrm{loc}}([0,\infty)\,;B)$. With small changes to the proof of Theorem 3.1, we obtain existence and uniqueness of a solution to (19) in the space $L^1_{\mathrm{loc}}([0,\infty)\,;B)$.

3.8 A second order differential equation

Let us consider a second order differential equation in the Banach space B:

$$-x''(t) + k^2 x(t) = A(x(t), t), \quad t \in \mathbb{R}, \tag{22}$$

where k is a positive constant and $A\colon B \times \mathbb{R} \to B$ satisfies the following Lipschitz-Carathèodory condition:

(A') For every fixed $x \in B$, $A(x, \cdot)$ is measurable, and there exists a function ω in $L^\infty(\mathbb{R})$ such that for all $x, y \in B$ and every $t \in \mathbb{R}$, (18) holds.

It is easy to verify that the function $g(t) = (2k)^{-1} \exp(-k|t|)$, $t \in \mathbb{R}$, is a Green's function for the operator $-\partial_t^2 + k^2$. Using this, we can formally rewrite (22) as

$$x(t) = \int_{\mathbb{R}} g(t-\tau) A(x(\tau), \tau)\, d\tau, \quad t \in \mathbb{R}. \tag{23}$$

One can show that a solution in $L^1_{\text{loc}}(\mathbb{R}\,; B)$ to (23) is continuous and satisfies (22) in the sense of vector distributions.

In order to describe our results for (23), we introduce the auxiliary differential equation

$$-w''(t) + k^2 w(t) - \omega(t) w(t) = h(t), \quad t \in \mathbb{R}, \tag{24}$$

where $h(t) = |A(0,t)|$ for $t \in \mathbb{R}$. We will require that

$$\sup_{t \in \mathbb{R}} \omega(t) < k^2. \tag{25}$$

Under this condition, the operator $-\partial_t^2 + k^2 - \omega$ has a Green's function $g_\omega(t, \tau)$, represented by the Neumann series

$$\sum_{k=0}^{\infty} \int_{\mathbb{R}^k} g(t-\tau_1)\omega(\tau_1) g(\tau_1 - \tau_2) \cdots \omega(\tau_k) g(\tau_k - \tau)\, d\tau_1\, d\tau_2 \cdots d\tau_k; \tag{26}$$

see Kozlov and Maz'ya [4, Section 6]. This Green's function is uniquely defined if we require that g_ω is bounded. We also let $w_\pm$ be two positive solutions to the to (24) corresponding homogeneous equation, such that

(i) $w_\pm(t) > 0$ for $t \in \mathbb{R}$, $w_\pm(t) \to 0$ as $t \to \pm\infty$, $w_\pm(t) \to \infty$ as $t \to \mp\infty$,

One can show that $w_\pm$ satisfy

(ii) $|w_\mp(t)| + |\partial_t w_\mp(t)| \le Ce^{\pm kt}$ for $t \gtrless 0$.

These solutions exist and are unique up to a positive constant factor; see the proof of Theorem 6.4.1 in Kozlov and Maz'ya [4]. Using these, one can give another representation for g_ω:

$$g_\omega(t, \tau) = \begin{cases} D\, w_+(t) w_-(\tau), & t \ge \tau, \\ D\, w_-(t) w_+(\tau), & t \le \tau, \end{cases} \tag{27}$$

where D is some positive constant.

Theorem 3.3

Suppose that

$$\int_{\mathbb{R}} g_\omega(0, \tau) |A(0, \tau)|\, d\tau < \infty. \tag{28}$$

Then there exists a solution $x \in L^1_{\text{loc}}(\mathbb{R}\,; B)$ to (23) that satisfies

$$|x(t)| \le \int_{\mathbb{R}} g_\omega(t, \tau) |A(0, \tau)|\, d\tau, \quad t \in \mathbb{R}, \tag{29}$$

and

$$|x(t)| = o(w_\pm(t)) \quad \text{as } t \to \mp\infty. \tag{30}$$

Moreover, a solution in $L^1_{\mathrm{loc}}(\mathbb{R}\,;\,B)$ *to* (23)*, that satisfies* (30)*, is unique.*

To verify (28) in specific cases, one can employ well-known asymptotic properties of solutions to ordinary differential equations; see Eastham [2], Kozlov and Maz'ya [4], and Wasov [10].

Proof. As in the proof of Theorem 3.1, we let $\mathscr{X}$ be the vector space of all mappings from $\Omega = \mathbb{R}$ into B. The topology on $\mathscr{X}$ is given by the seminorms $p(x\,;\,t) = |x(t)|$, $x \in \mathscr{X}$, $t \in \mathbb{R}$. With this topology, $\mathscr{X}$ is locally convex and complete. We define the operator $K\colon \mathscr{D}_K \to \mathbb{R}^\Omega$ by

$$K\eta(t) = \int_{\mathbb{R}} g(t-\tau)\omega(\tau)\eta(\tau)\,d\tau, \quad t \in \mathbb{R},$$

where the domain $\mathscr{D}_K$ of K is the set of measurable functions η on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} g(\tau)\omega(\tau)|\eta(\tau)|\,d\tau < \infty. \tag{31}$$

Furthermore, let $\mathscr{K}$ be the right-hand side in (23). It follows from (A') that, for example, if $x \in L^1_{\mathrm{loc}}(\Omega\,;\,B)$ such that $|x| \in \mathscr{D}_K$, then $\mathscr{K}(x)$ is defined:

$$\begin{aligned} |\mathscr{K}(x)(t)| &\le \int_{\mathbb{R}} g(t-\tau)|A(x(\tau),\tau) - A(0,\tau)|\,d\tau + |\mathscr{K}(0)(t)| \\ &\le K|x|(t) + |\mathscr{K}(0)(t)|. \end{aligned} \tag{32}$$

The last term is finite since (28) holds.

We will next show that the functions $w_\pm$ both belong to $\mathscr{D}_K$. First of all,

$$e^{-k\tau}\omega(\tau)w_-(\tau) = -\partial_\tau(e^{-k\tau}(\partial_\tau + k)w_-(\tau))$$

and $|(\partial_\tau + k)w_-(\tau)| \le Ce^{k\tau}$ for $\tau > 0$. Thus,

$$\int_0^M g(\tau)\omega(\tau)w_-(\tau)\,d\tau = -\int_0^M \partial_\tau\big(e^{-k\tau}(\partial_\tau + k)w_-(\tau)\big)\,d\tau$$

is bounded with respect to $M > 0$. Moreover, $w_-(\tau)$ is bounded for $\tau < 0$, so

$$\int_{-\infty}^0 g(\tau)\omega(\tau)w_-(\tau)\,d\tau < \infty.$$

This shows that $w_- \in \mathscr{D}_K$. In a similar manner, one can show that $w_+ \in \mathscr{D}_K$.

We now prove that $(K2)$ is satisfied. Let $h(t) = |A(0,t)|$, $t \in \mathbb{R}$. By Theorem 6.5.2 in Kozlov and Maz'ya [4], there exists a solution z to (24), which is given by

$$z(t) = \int_{\mathbb{R}} g_\omega(t,\tau)h(t)\,d\tau, \quad t \in \mathbb{R}, \tag{33}$$

and this solution satisfies $z(t) = o(w_\pm(t))$ as $t \to \mp\infty$. It is also clear that z belongs to $\mathscr{D}_K$ since $w_\pm$ belong to $\mathscr{D}_K$. Moreover, the fact that g is a Green's function for $-\partial_t^2 + k^2$ implies that z also solves the equation

$$Kz(t) + h * g(t) = z(t), \quad t \in \mathbb{R},$$

and hence, $(K2)$ is satisfied. Let $\eta, \zeta \in \mathscr{D}_K$ such that $0 \le \eta \le \zeta$ and $K\zeta$ belongs to $\mathscr{D}_K$. Clearly $K\eta$ is measurable, so it follows that $K\eta \in \mathscr{D}_K$ since $K\eta \le K\zeta$ and $K\zeta \in \mathscr{D}_K$. Therefore it is clear that $(K4)$ holds. By monotone convergence, it follows that K is closed for non-negative, increasing sequences. Hence, $(K3)$ is valid. Thus, $(K1)$ to $(K4)$ are satisfied, so Lemma 2.1 shows that a unique minimal $\sigma \in \mathscr{D}_K$ exists.

Existence. We let $\mathscr{D}_{\mathscr{K}}$ be defined as the set of those measurable functions x that maps $\mathbb{R}^N$ into B for which $|x| \le \sigma$. It follows from (18) that $\mathscr{K}$ is defined on $\mathscr{D}_{\mathscr{K}}$ since (A') holds and $\sigma \in \mathscr{D}_K$; compare with (32). It is clear that $\mathscr{K}(x)$ is measurable for every x in $\mathscr{D}_{\mathscr{K}}$, and using the same argument as in the proof of (i) in Lemma 2.3, we obtain that $|\mathscr{K}(x)| \le \sigma$ if $x \in \mathscr{D}_{\mathscr{K}}$. Thus, $\mathscr{K}$ maps $\mathscr{D}_{\mathscr{K}}$ into itself. As above, we also have $|\mathscr{K}(x) - \mathscr{K}(y)| \le K|x-y|$ for all $x, y \in \mathscr{D}_{\mathscr{K}}$. Hence, $(\mathscr{K}1)$ holds.

We next show that $(\mathscr{K}2)$ is satisfied. Suppose that $\{x_n\}_{n=0}^\infty$ satisfies condition (8) in $(\mathscr{K}2)$. Then the sequence is a Cauchy sequence in $\mathscr{D}_{\mathscr{K},\sigma}$. Since $\mathscr{X}$ is (sequentially) complete, we have $x_n \to x$ for some measurable $x \in \mathscr{X}$. Clearly $|x| \le \sigma$, so $x \in \mathscr{D}_{\mathscr{K},\sigma}$. Thus, all requirements for Theorem 2.4 are satisfied and there exists a fixed point x in $\mathscr{D}_{\mathscr{K},\sigma}$ of $\mathscr{K}$. Moreover, it is straightforward to verify, using (27) and the properties of $w_\pm$, that the estimate in (29) implies (30).

Uniqueness. Choose $\mathscr{D}_{\mathscr{K}}$ as the linear space of functions $x \in L^1_{\mathrm{loc}}(\mathbb{R}; B)$ such that (30) holds. If $x \in \mathscr{D}_{\mathscr{K}}$, then $|x| \in \mathscr{D}_K$ since $w_\pm \in \mathscr{D}_K$, so $\mathscr{K}(x)$ is defined; compare with (32). We next show that the operator $\mathscr{K}$ maps $\mathscr{D}_{\mathscr{K}}$ into itself. To see that this is true, let x belong to $\mathscr{D}_{\mathscr{K}}$ and let $\varepsilon > 0$ be arbitrary. Choose $M > 0$ such that $|x(t)| \le \varepsilon w_\mp(t)$ for $t \gtrless \pm M$. Since (28) holds, it is sufficient to prove that

$$\int_{\mathbb{R}} g(t-\tau)\omega(\tau)|x(\tau)|\,d\tau = o(w_\pm(t)) \quad \text{as } t \to \mp\infty. \tag{34}$$

It is straightforward to show that

$$\int_{-M}^{M} g(t-\tau)\omega(\tau)|x(\tau)|\,d\tau \to 0 \quad \text{as } t \to \mp\infty.$$

Since $\omega(\tau)w_\pm(\tau) = (-\partial_t^2 + k^2)w_\pm(\tau)$, we also have

$$\begin{aligned}\int_{-\infty}^{-M} g(t-\tau)\omega(\tau)|x(\tau)|\,d\tau &\le \varepsilon \int_{-\infty}^{\infty} g(t-\tau)\omega(\tau)w_+(\tau)\,d\tau \\ &= \varepsilon w_+(t),\end{aligned}$$

and similarly

$$\int_{M}^{\infty} g(t-\tau)\omega(\tau)|x(\tau)|\,d\tau \le \varepsilon w_-(t).$$

This implies that (34) is valid. The relation in (34) also shows that (II) holds. The fact that (III) holds follows from (A') and the definition of $\mathscr{D}_{\mathscr{K}}$ and $\mathscr{D}_K$.

We now turn to show that (I) holds. For an arbitrary but fixed function u in $\mathscr{D}_{\mathscr{K}}$, let $\eta(t) = |u(t)|$, $t \in \mathbb{R}$. Then $\eta \in \mathscr{D}_K$ and $\eta(t) = o(w_\pm(t))$ as $t \to \mp\infty$. We prove, for a fixed $t = t_0$, that $K^n\eta(t_0) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Choose a positive number M so that $|\eta(t)| \le \varepsilon w_\mp(t)$ when $t \gtrless \pm M$. Put $\eta_M(t) = |\eta(t)|$ when $|t| \le M$, $\eta_-(t) = |\eta(t)|$ when $t < -M$, and $\eta_+(t) = |\eta(t)|$ when $t > M$. All three functions are defined as zero elsewhere.

For η_M, we multiply (26) by η_M and integrate. The monotone convergence theorem implies that

$$\sum_{n=0}^{\infty} g_n(t) = \int_{\mathbb{R}} g_\omega(t,\tau)\eta_M(\tau)\,d\tau < \infty, \quad t \in \mathbb{R}, \tag{35}$$

where

$$g_n(t) = \int_{\mathbb{R}^n} g(t-\tau_1)\omega(\tau_1)g(\tau_1-\tau_2)\cdots\omega(\tau_{n-1})g(\tau_{n-1}-\tau_n)\eta_M(\tau_n)\,d\tau_1\,d\tau_2\cdots d\tau_n.$$

Since $K^n\eta_M(t) \le k^2 g_n(t)$ for every t, the convergence of the series in (35) implies that $|K^n\eta_M(t_0)| \le \varepsilon$ if n is large enough.

For $\eta_\pm$, the fact that $Kw_\pm = w_\pm$ shows that

$$K^n\eta_\pm(t_0) \le \varepsilon K^n w_\pm(t_0) = \varepsilon w_\pm(t_0).$$

Hence, we obtain that for all large n, we have

$$|K^n\eta(t_0)| < (1 + w_-(t_0) + w_+(t_0))\varepsilon.$$

This proves that $K^n\eta(t_0) \to 0$ as $n \to \infty$. Hence, we may apply Theorem 2.5, and find that the fixed point is indeed unique. □

3.9 A pseudo-differential equation

Let $1 < p < \infty$ and let S be a pseudo-differential operator on $\mathbb{R}^N$ with an invertible symbol $a(\xi)$ which is smooth outside the origin and positively homogeneous of order zero. We consider the equation

$$Su(x) = Q(u)(x), \quad x \in \mathbb{R}^N, \tag{36}$$

where the mapping $Q\colon L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\}) \to L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ is assumed to satisfy the following Lipschitz type condition.

(Q) There exists some $q \in L^\infty(0,\infty)$ such that for all $u, v \in L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$:

$$\mathcal{N}_p\big(Q(u) - Q(v)\,;\, r\big) \le q(r)\,\mathcal{N}_p\big(u - v\,;\, r\big), \quad r > 0, \tag{37}$$

where

$$\mathcal{N}_p(u\,;\, r) = \left(\frac{1}{r^N}\int_{r<|x|<2r} |u(x)|^p\,dx\right)^{1/p}, \quad r > 0. \tag{38}$$

Equations of this type occur, for example, when one solves boundary value problems for partial differential equations with nonlinearities in the boundary condition. An example of such an operator Q is $Q(u)(x) = F(u,x)$, where F satisfies a Lipschitz-Carathéodory condition: the function $F(u,\,\cdot\,)$ is measurable for every $u \in \mathbb{R}$ and there exists some l in $L^\infty(\mathbb{R}^N)$ such that

$$|F(u,x) - F(v,x)| \le l(x)|u-v|, \quad u,v \in \mathbb{R} \text{ and } x \in \mathbb{R}^N.$$

We also assume that $F(0,\,\cdot\,) \in L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$. Then Q maps $L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ into itself and satisfies (Q) with $q(r) = \operatorname{ess\,sup}_{r<|x|<2r} l(x)$.

Since the symbol $a(\xi)$ is invertible, the operator S has an inverse, which we denote by T, with symbol $1/a(\xi)$. Formally applying T to (36), we arrive at

$$u(x) = TQ(u)(x), \quad x \in \mathbb{R}^N. \tag{39}$$

Under natural assumptions, we will establish existence and uniqueness of solutions to (39). The operator T can also be represented as a singular integral operator:

$$Tv(x) = (\text{V.P.}) \int_{\mathbb{R}^N} K(x-y)v(y)\,dy, \quad x \in \mathbb{R}^N, \tag{40}$$

where the kernel K is positively homogenous of order $-N$, infinitely differentiable outside the origin, and satisfies a cancellation condition; see Stein [9, p. 26]. We let $\mu(\rho) = \rho^N$ for $0 \le \rho < 1$ and $\mu(\rho) = 1$ for $\rho \ge 1$. The operator T is defined for functions $v \in L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ such that

$$\int_0^\infty \mu(\rho)\,\mathcal{N}_p(v\,;\,\rho)\,\frac{d\rho}{\rho} < \infty. \tag{41}$$

In fact, the following inequality holds:

$$\mathcal{N}_p(Tu\,;\,r) \le C_T \int_0^\infty \mu\left(\frac{\rho}{r}\right) \mathcal{N}_p(u\,;\,\rho)\,\frac{d\rho}{\rho}, \quad r > 0, \tag{42}$$

where the constant C_T only depends on N, p, and K. In Kozlov, Thim, and Turesson [7], this result is proved for the Riesz transform, but it holds with the obvious modifications for all operators of the type given in (40).

In what follows, we require that

$$\sup_{r>0} q(r) < \frac{N}{4C_T}, \tag{43}$$

and put $\omega(t) = NC_T q(e^t)$ for $t \in \mathbb{R}$. In order to formulate our results for (39), let us introduce an auxiliary differential equation:

$$-(\partial_t + N)\partial_t w(t) - \omega(t)w(t) = h(t), \quad t \in \mathbb{R}, \tag{44}$$

where $h(t) = NC_T\mathcal{N}_p(Q(0)\,;\,e^t)$, $t \in \mathbb{R}$. A Green's function for the differential operator $-(\partial_t + N)\partial_t$ is given by the function $g(t) = N^{-1}\mu(e^{-t})$, $t \in \mathbb{R}$. As before, the Green's function $g_\omega(t,\tau)$ for the differential operator $-(\partial_t + N)\partial_t - \omega$ may be represented as the Neumann series in (26) with g as above. To give another representation for g_ω, we introduce two positive solutions $v_\pm$ to the to (44) corresponding homogeneous equation such that

(i) $v_\pm(t) > 0$ for $t \in \mathbb{R}$, $v_\pm(t) = o(e^{-Nt/2})$ as $t \to \pm\infty$, $e^{Nt/2}v_\pm(t) \to \infty$ as $t \to \mp\infty$.

One can show that $v_\pm$ satisfy

(ii) $|v_\mp(t)| + |\partial_t v_\mp(t)| \le Ce^{-Nt/2 \pm Nt/2}$ for $t \gtrless 0$.

Compare (i) and (ii) with $w_\pm$ in Section 3.2. We may then represent g_ω as

$$g_\omega(t,\tau) = \begin{cases} D\, v_+(t)e^{N\tau}v_-(\tau), & t \ge \tau, \\ D\, v_-(t)e^{N\tau}v_+(\tau), & t \le \tau, \end{cases} \tag{45}$$

where D is some positive constant. Observe that $e^{Nt}v_\pm(t)$ are solutions to the to (44) corresponding homogeneous equation for the formal adjoint operator, i.e., the equation

$$-w''(t) + Nw'(t) - \omega(t)w(t) = 0.$$

Theorem 3.4

Suppose that

$$\int_0^\infty g_\omega(0, \log\rho)\, \mathcal{N}_p(Q(0)\,;\,\rho)\, \frac{d\rho}{\rho} < \infty.$$

Then there exists a solution $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ *to* (39). *This solution satisfies*

$$\mathcal{N}_p(u\,;\,r) \le \int_0^\infty g_\omega(\log r, \log\rho)\, \mathcal{N}_p(Q(0)\,;\,\rho)\, \frac{d\rho}{\rho}, \quad r > 0,$$

and

$$\mathcal{N}_p(u\,;\,r) = \begin{cases} o(v_+(\log r)) & \text{as } r \to 0^+, \\ o(v_-(\log r)) & \text{as } r \to \infty. \end{cases} \tag{46}$$

A solution in $L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ *to* (39) *that satisfies* (46) *is unique.*

Proof. Let $\mathscr{X} = L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and put $\Omega = (0,\infty)$. We let $\{\mathcal{N}_p(\,\cdot\,;\,r)\}_{r>0}$ be the seminorms on this space. We start by defining $K\colon \mathscr{D}_K \to \mathbb{R}^\Omega$ by

$$K\eta(r) = C_T \int_0^\infty \mu\left(\frac{\rho}{r}\right) \omega(\log\rho)\, \eta(\rho)\, \frac{d\rho}{\rho}, \quad r > 0,$$

where C_T is the constant in (42). The domain $\mathscr{D}_K$ of K is the set of functions η in $L^1_{\mathrm{loc}}(0,\infty)$ such that

$$\int_0^\infty \mu(\rho)\, \omega(\log\rho)\, \eta(\rho)\, \frac{d\rho}{\rho} < \infty. \tag{47}$$

Using monotone convergence, it follows that K is closed for non-negative increasing sequences. The functions $v_\pm(\log\rho)$, $\rho > 0$, both belong to $\mathscr{D}_K$; this follows by the same kind of argument that was used in the proof of Theorem 3.3, after first making the substitution $\rho = e^t$. Next, we define $\mathscr{K}$ as the right-hand side in (39).

To see that $(K2)$ is satisfied, let $h(t) = NC_T\mathcal{N}_p(Q(0)\,;\,e^t)$, $t \in \mathbb{R}$. By Theorem 6.5.2 in Kozlov and Maz'ya [4], there exists a solution w to (44), given by (33) with g_ω as in (45), and this solution satisfies

$$|w(t)| = o(v_\pm(t)) \quad \text{as } t \to \mp\infty.$$

The function $r \mapsto w(\log r)$ belongs to $\mathscr{D}_K$ since we know that $v_\pm \circ \log \in \mathscr{D}_K$. Furthermore, $g(t)$ is a Green's function for $-(\partial_t + N)\partial_t$, so after a change of variables, we see that z solves the equation

$$Kz(r) + (h * g)(\log(r)) = z(r), \quad r > 0,$$

and hence, $(K2)$ is satisfied. It is also clear that $(K4)$ is valid. Thus, $(K1)$ to $(K4)$ are all satisfied, so we may apply Lemma 2.1 to obtain a unique minimal $\sigma \in \mathscr{D}_K$ that solves (4).

Existence. We let $\mathscr{D}_{\mathscr{K}}$ be the linear space of those $x \in L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ such that $\mathcal{N}_p(u\,;\, r) \leq \sigma(r)$, $r > 0$. Inequality (42) implies that $\mathscr{K}(u) \in L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ for $u \in \mathscr{D}_{\mathscr{K}}$. From (4) it now follows that $\mathscr{K}$ maps $\mathscr{D}_{\mathscr{K},\sigma}$ into $\mathscr{D}_{\mathscr{K},\sigma}$. The condition in (Q) also shows that $(\mathscr{K}1)$ holds.

Let $\{u_n\}$ be a Cauchy sequence in $\mathscr{D}_{\mathscr{K},\sigma}$. Since $L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ is (sequentially) complete, we have $u_n \to u$ for some $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$. Furthermore, from the facts that $\sigma \in \mathscr{D}_K$ and $\mathcal{N}_p(u\,;\,\cdot\,) \leq \sigma$ we obtain that $\mathcal{N}_p(u\,;\,\cdot\,) \in \mathscr{D}_K$ by dominated convergence. Hence, $u \in \mathscr{D}_{\mathscr{K},\sigma}$. This proves that $(\mathscr{K}2)$ is satisfied. Thus, all requirements for Theorem 2.4 are satisfied and there exists a fixed point $u \in \mathscr{D}_{\mathscr{K},\sigma}$ of $\mathscr{K}$ such that the inequality $\mathcal{N}_p(u\,;\,\cdot\,) \leq \sigma$ is holds.

Uniqueness. As in the proof of uniqueness in Theorem 3.3, one can show that with $\mathscr{D}_{\mathscr{K}}$ as those functions $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ such that (46) holds, $\mathscr{K}(u)$ is defined and satisfies (46). It is now straightforward to verify that (II) and (III) are valid. After the substitution $t = \log r$ in the proof of uniqueness in Theorem 3.3, and using $v_\pm$ in the place of $w_\pm$, we obtain that $K^n\eta(r) \to 0$ as $n \to \infty$ for all $r > 0$. Thus, Theorem 2.5 is applicable, which proves that the solution is unique. □

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