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Jacobi transplantations and Weyl integrals

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Abstract

We prove that transplantations for Jacobi polynomials can be derived from representation of a special integral operator as fractional Weyl's integral. Furthermore, we show that, in a sense, Jacobi transplantation can be reduced to transplantations for ultraspherical polynomials. As an application of these results, we obtain transplantation theorems for Jacobi polynomials in $\mathrm{Re}\, H^1$ and BMO. The paper gives an extension of the results obtained for ultraspherical polynomials by the first named author (MR2148530 (2006a:42045)).

1. Introduction

Let $P_n^{\alpha,\beta}(z)$ be the Jacobi polynomial of degree n and order (α, β), where $\alpha, \beta > -1$ (see [14, 4.1]). The Jacobi polynomials are orthogonal on $(-1, 1)$ with respect to the

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measure $(1-z)^\alpha(1+z)^\beta\,dz$ and

$$\int_{-1}^{1}[P_n^{(\alpha,\beta)}(z)]^2(1-z)^\alpha(1+z)^\beta\,dz$$
$$=\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}=[s_n^{(\alpha,\beta)}]^2$$

(see [14, (4.3.3), p.68]). The functions

$$\varphi_n^{(\alpha,\beta)}(y)=\frac{2^{(\alpha+\beta+1)/2}P_n^{(\alpha,\beta)}(\cos y)(\sin y/2)^{\alpha+1/2}(\cos y/2)^{\beta+1/2}}{s_n^{(\alpha,\beta)}} \tag{1.1}$$

form an orthonormal system on $(0,\pi)$ with respect to Lebesgue measure. For $\alpha=\beta$ we obtain the system of ultraspherical polynomials

$$u_n^{(\lambda)}(y)=\varphi_n^{(\alpha,\alpha)}(y),\quad \lambda=\alpha+\frac{1}{2}.$$

Transplantations of coefficients from one orthonormal system to another have a long history. One of the first transplantation theorems was obtained by Askey and Wainger [2] for ultraspherical polynomials. This theorem was extended by Askey [1] to general Jacobi series. Later on, the boundedness of the Jacobi transplantation operator in weighted L^p–spaces $(1<p<\infty)$ was studied in [8] and [13] (see also [4]). It is well known that this operator fails to be bounded in L^1 and L^∞ (see [3]). Thus, it is natural to ask whether transplantation holds in the spaces $\mathrm{Re}\,H^1$ and BMO. This problem was first studied in [9, 11] for ultraspherical polynomials. Afterwards, sharp results on Jacobi transplantation for weighted Hardy spaces were obtained in [12]. We mention also the work [10] in which a transplantation theorem for the Hankel transform in the space $\mathrm{Re}\,H^1(\mathbb{R})$ was proved.

Recall that the real Hardy space $\mathrm{Re}\,H^1$ is the space of all 2π–periodic functions $\varphi\in L^1[0,2\pi]$ for which the conjugate function $\widetilde{\varphi}$ also belongs to $L^1[0,2\pi]$. The norm in $\mathrm{Re}\,H^1$ is defined by

$$\|\varphi\|_{\mathrm{Re}\,H^1}=\|\varphi\|_{L^1}+\|\widetilde{\varphi}\|_{L^1}.$$

The space BMO consists of all 2π–periodic functions $f\in L^1[0,2\pi]$ such that

$$\|f\|_*\equiv\sup_I\frac{1}{|I|}\int_I|f(x)-f_I|\,dx<\infty,\qquad f_I=\int_I f(t)\,dt,$$

where the supremum is taken over all intervals $I\subset\mathbb{R}$. We set

$$\|f\|_{\mathrm{BMO}}=\left|\int_0^{2\pi}f(x)\,dx\right|+\|f\|_*. \tag{1.2}$$

By the Spanne–Stein theorem [15, p. 207], for any function $f\in\mathrm{BMO}$ its conjugate function $\widetilde{f}$ also belongs to BMO, and $\|\widetilde{f}\|_*\le c\|f\|_*$.

In [11], there were proved two-sided transplantation theorems in the spaces $\mathrm{Re}\,H^1$ and BMO betweeen ultraspherical series and trigonometric series. With the use of

Mehler's integral representation of ultraspherical polynomials, the proofs were reduced to representation of a special integral operator as fractional Weyl's integral.

In this paper we show that the approach developed in [11] can be extended to Jacobi polynomials. We shall give a brief description of our main results.

Let $f \in L^1[0,\pi]$ and $\alpha,\beta > -1/2$. The Fourier-Jacobi coefficients of f are defined by

$$a_n^{(\alpha,\beta)}(f) = \int_0^\pi f(y)\varphi_n^{(\alpha,\beta)}(y)\,dy. \tag{1.3}$$

We will apply a Mehler type formula for Jacobi polynomials, to obtain that

$$a_n^{(\alpha,\beta)}(f) = t_n^{(\alpha,\beta)} \int_0^\pi K_{\alpha,\beta}(f;x)\cos(n+\gamma)x\,dx \quad (\gamma = (\alpha+\beta+1)/2),$$

where

$$K_{\alpha,\beta}(f;x) = \int_{|x|}^\pi f(y)J_{\alpha,\beta}(x,y)\,dy \quad (|x| \le \pi) \tag{1.4}$$

and $J_{\alpha,\beta}(x,y)$ is a special kernel. In the case $\alpha = \beta$ this kernel has a comparatively simple form,

$$J_{\alpha,\alpha}(x,y) = \left(\frac{\sin y}{\cos x - \cos y}\right)^{1/2-\alpha}, \quad 0 \le x < y \le \pi.$$

We show explicitly that under quite weak conditions on a function space X, the transplantation from Jacobi polynomials to the trigonometric system in X can be immediately derived from representation of the operator (1.4) as fractional Weyl's integral (see Theorem 3.1).

It is clear that if $\operatorname{supp} f \subset [0,2\pi/3]$, then the value of β does not have an essential influence on the behaviour of $a_n^{(\alpha,\beta)}(f)$. Roughly speaking, one can expect that in this case the Jacobi coefficients $a_n^{(\alpha,\beta)}(f)$ behave as ultraspherical coefficients $a_n^{(\alpha,\alpha)}(f)$. We give a quantitative form of this phenomenon in terms of $J_{\alpha,\beta}(x,y)$. Namely, we prove that for $0 \le x < y \le 2\pi/3$ the ultraspherical kernel $J_{\alpha,\alpha}(x,y)$ gives a good approximation of the Jacobi kernel $J_{\alpha,\beta}(x,y)$ (see Proposition 2.3 below).

We show that these results and representations of the operator (1.4) (with $\alpha = \beta$) proved in [11] readily yield transplantation theorems in $\operatorname{Re} H^1$ and BMO for Jacobi polynomials in the range $\alpha,\beta \in (-1/2,1/2)$ (see Theorems 4.6 and 4.7 below).

2. Dirichlet–Mehler type formula and Jacobi kernels

We will use a Dirichlet–Mehler type integral representation for Jacobi polynomials. This representation was first found in [7].

Let $F(a,b;c;z)$ be the Gauss hypergeometric function (see [14, Chapter 4]),

$$F(a,b;c;z) = 1 + \sum_{n=1}^\infty \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{2.1}$$

where $(\mu)_n = \mu(\mu+1)\cdots(\mu+n-1)$. Note that for $\mu > 0$ and $n \in \mathbb{N}$ we have

$$(\mu)_n = \frac{\Gamma(n+\mu)}{\Gamma(\mu)} \quad \text{and} \quad \Gamma(n+\mu) = \Gamma(n)[n^\mu + O(n^{\mu-1})]. \tag{2.2}$$

Lemma 2.1

For $\alpha > -1/2$, $\beta > -1$ and $0 < y < \pi$, we have the integral representation

$$\frac{P_n^{(\alpha,\beta)}(\cos y)}{P_n^{(\alpha,\beta)}(1)} = \frac{2^{(\alpha+\beta+1)/2}\Gamma(\alpha+1)}{\Gamma(1/2)\Gamma(\alpha+1/2)}(1-\cos y)^{-\alpha}\int_0^y \cos(n+(\alpha+\beta+1)/2)x$$
$$\times \frac{(\cos x-\cos y)^{\alpha-1/2}}{(1+\cos x)^{(\alpha+\beta)/2}}F\left(\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta}{2};\alpha+\frac{1}{2};\frac{\cos x-\cos y}{1+\cos x}\right)dx.$$

Note that (see [14, (4.1.1), p. 58])

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

Thus, the functions (1.1) can be written as

$$\varphi_n^{(\alpha,\beta)}(y) = t_n^{(\alpha,\beta)}\int_0^y J_{\alpha,\beta}(x,y)\cos\left(n+\frac{\alpha+\beta+1}{2}\right)x\,dx, \tag{2.3}$$

where

$$J_{\alpha,\beta}(x,y) = \left(\frac{\sin y}{\cos x-\cos y}\right)^{1/2-\alpha} G_{\alpha,\beta}(x,y), \quad |x|<|y|\le\pi, \tag{2.4}$$

$$G_{\alpha,\beta}(x,y) = \left(\frac{1+\cos y}{1+\cos x}\right)^{\alpha+\beta/2} F\left(\frac{\alpha+\beta+1}{2},\frac{\alpha+\beta}{2};\alpha+\frac{1}{2};\frac{\cos x-\cos y}{1+\cos x}\right),$$

and

$$t_n^{(\alpha,\beta)} = c_{\alpha,\beta}n^{\alpha+1/2}+O(n^{\alpha-1/2}). \tag{2.5}$$

If $\alpha=\beta$, denote $\lambda=\alpha+1/2$. In this case (2.3) is the Mehler formula for ultraspherical polynomials (see [5, p. 177]). We set for $\lambda>0$,

$$U_\lambda(x,y) = \left(\frac{\sin y}{\cos x-\cos y}\right)^{1-\lambda}, \quad |x|<|y|\le\pi. \tag{2.6}$$

It follows from (2.1) that $G_{\alpha,\alpha}(x,y)=1$ and $J_{\alpha,\alpha}(x,y)=U_{\alpha+1/2}(x,y)$.

In what follows we denote

$$\Omega = \big\{(x,y): 0\le x<y\le 2\pi/3\big\}.$$

Set also

$$D_{\alpha,\beta}(x,y) = J_{\alpha,\beta}(x,y)-U_{\alpha+1/2}(x,y), \quad |x|<|y|\le\pi. \tag{2.7}$$

Our main result in this section is that the derivatives of $D_{\alpha,\beta}(x,y)$ with respect to x in Ω are comparatively small. First we prove the following lemma.

Lemma 2.2

Let $\lambda>0$ and let ν be the least integer such that $\nu\ge\lambda$. Set

$$\psi(x,y) = (\cos x-\cos y)^\lambda(\sin y)^{1-\lambda}, \quad (x,y)\in\Omega.$$

Then

$$\left|\frac{\partial^r\psi}{\partial x^r}(x,y)\right| \le C_\lambda y(y-x)^{\lambda-r} \tag{2.8}$$

for any $(x,y)\in\Omega$ and $0\le r\le\nu$.

Proof. We have that

$$\psi(x,y) = 2^\lambda \left(\sin \frac{y-x}{2}\right)^\lambda \left(\sin \frac{y+x}{2}\right)^\lambda (\sin y)^{1-\lambda}.$$

For $r = 0$, (2.8) is obvious. Set $\omega(t) = (\sin t)^\lambda$. Then

$$\omega(t) = t^\lambda \big(1 + g(t)\big)^\lambda,$$

where

$$g(t) = \sum_{k=1}^{\infty} (-1)^k \frac{t^{2k}}{(2k+1)!}.$$

It is clear that $g \in C^\infty[0, 2\pi/3]$. Moreover,

$$1 + g(t) = \frac{\sin t}{t} \geq \frac{1}{4}, \quad 0 \leq t \leq \frac{2\pi}{3}.$$

It follows easily that

$$\left|\frac{d^r}{dt^r}\omega(t)\right| \leq c_\lambda t^{\lambda - r}, \quad 0 < t \leq 2\pi/3, \tag{2.9}$$

for any $1 \leq r \leq \nu$. Applying (2.9) and the Leibniz formula, we obtain

$$\begin{aligned}\left|\frac{\partial^r \psi}{\partial x^r}(x,y)\right| &\leq c_\lambda y^{1-\lambda} \sum_{k=0}^{r} \binom{r}{k} (y-x)^{\lambda-k}(y+x)^{\lambda-r+k} \\ &\leq 2^r c_\lambda y^{1-\lambda}(y-x)^{\lambda-r}(y+x)^\lambda \leq c'_\lambda y(y-x)^{\lambda-r}\end{aligned}$$

for any $(x,y) \in \Omega$ and any $1 \leq r \leq \nu$. □

Let $D_{\alpha,\beta}(x,y)$ be defined by (2.7).

Proposition 2.3

Let $\alpha, \beta > -1/2$. Suppose that ν is the least integer such that $\nu \geq \alpha + 1/2$. Then

$$\left|\frac{\partial^r D_{\alpha,\beta}}{\partial x^r}(x,y)\right| \leq c_{\alpha,\beta}\; y(y-x)^{\alpha - r + 1/2}, \tag{2.10}$$

for any $0 \leq r \leq \nu$ and $(x,y) \in \Omega$.

Proof. The functions $(1-z)^{(\alpha+\beta)/2}$ and

$$F_{\alpha,\beta}(z) = F\left(\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta}{2}; \alpha + \frac{1}{2}; z\right)$$

are both analytic in $|z| < 1$. Set $g(z) = (1-z)^{(\alpha+\beta)/2} F_{\alpha,\beta}(z)$. Then

$$g(z) = 1 + \sum_{k=1}^{\infty} \gamma_k(\alpha,\beta) z^k, \tag{2.11}$$

where the series at the right hand side has radius of convergence at least 1. Thus, by the Cauchy-Hadamard formula,

$$\overline{\lim_{k\to\infty}} \sqrt[k]{|\gamma_k(\alpha,\beta)|} \le 1. \tag{2.12}$$

Denote

$$\zeta(x,y) = \frac{\cos x - \cos y}{1+\cos x}, \quad (x,y)\in\Omega.$$

Then

$$0 \le \zeta(x,y) \le \frac{3}{4}, \quad (x,y)\in\Omega. \tag{2.13}$$

Further, we have $G_{\alpha,\beta}(x,y) = g(\zeta(x,y))$. Thus, by (2.11) and (2.13), for any $(x,y)\in\Omega$

$$G_{\alpha,\beta}(x,y) = 1 + \sum_{k=1}^{\infty} \gamma_k(\alpha,\beta)\zeta(x,y)^k.$$

Now, we get

$$\begin{aligned} D_{\alpha,\beta}(x,y) &= U_{\alpha+1/2}(x,y)(G_{\alpha,\beta}(x,y)-1) \\ &= U_{\alpha+1/2}(x,y)\sum_{k=1}^{\infty}\gamma_k(\alpha,\beta)\zeta^k(x,y) \\ &= (\sin y)^{1/2-\alpha}(\cos x - \cos y)^{\alpha+1/2}\Phi_{\alpha,\beta}(x,y), \end{aligned}$$

where

$$\Phi_{\alpha,\beta}(x,y) = (1+\cos x)^{-1}\sum_{k=1}^{\infty}\gamma_k(\alpha,\beta)\zeta^{k-1}(x,y).$$

It is obvious that

$$\left|\frac{\partial^r}{\partial x^r}\zeta(x,y)\right| \le c_\nu, \quad (x,y)\in\Omega,$$

for any $1\le r\le\nu$, where c_ν depends only on ν. It easily follows from this estimate that

$$\left|\frac{\partial^r}{\partial x^r}\zeta^k(x,y)\right| \le c'_\nu, \quad (x,y)\in\Omega,$$

for any $1\le k\le\nu$, and

$$\left|\frac{\partial^r}{\partial x^r}\zeta^k(x,y)\right| \le c'_\nu k^r\zeta^{k-r}, \quad (x,y)\in\Omega,$$

for any $k>\nu$, where $1\le r\le\nu$. Thus, applying (2.12) and (2.13), we obtain that

$$\left|\frac{\partial^r}{\partial x^r}\Phi_{\alpha,\beta}(x,y)\right| \le c_{\alpha,\beta}, \quad (x,y)\in\Omega,$$

for any $0\le r\le\nu$. Using this estimate and Lemma 2.2, we get (2.10). □

3. Weyl fractional integrals

We recall the definition of Weyl fractional integrals. Let $0 < \lambda < 1$. Set

$$(ik)^\lambda = |k|^\lambda \exp\left(\frac{\lambda\pi i}{2}\operatorname{sign} k\right) \quad (k \in \mathbb{Z}, k \neq 0).$$

Further, denote

$$\Psi_\lambda(t) = \sum_{n\in\mathbb{Z}, n\neq 0} \frac{e^{int}}{(in)^\lambda} = 2\sum_{k=1}^{\infty} \frac{\cos(kt - \lambda\pi/2)}{k^\lambda}. \tag{3.1}$$

These series converge for all $t \in (0, 2\pi)$ and $\Psi_\lambda \in L^1[0, 2\pi]$; moreover,

$$|\Psi_\lambda(t)| \leq c_\lambda |t|^{\lambda-1}, \quad 0 < |t| \leq \pi, \tag{3.2}$$

(see [16, Chapter 12, § 8]).

Let $\varphi \in L^1[0, 2\pi]$ be a 2π–periodic function. The Weyl fractional integral of order $\lambda > 0$ of the function φ is defined by the equality

$$I_\lambda\varphi(x) = \frac{1}{2\pi}\int_0^{2\pi} \Psi_\lambda(x - t)\varphi(t)\,dt.$$

The last integral converges absolutely for almost all x. Moreover, if $\varphi \in L^p[0, 2\pi]$ $(1 \leq p \leq \infty)$, then by (3.2) and Minkowski inequality,

$$\|I_\lambda\varphi\|_{L^p} \leq c\|\varphi\|_{L^p}.$$

It follows from (3.1) that

$$I_\lambda(I_\mu\varphi) = I_{\lambda+\mu}\varphi \quad (\lambda, \mu > 0). \tag{3.3}$$

Let X be a linear normed space of $2\pi-$periodic functions. In what follows we assume that X satisfies the following conditions:

(i) $X \subset L^1[0, 2\pi]$ and there exists a constant c_X such that

$$||f||_{L^1} \leq c_X||f||_X \quad \text{for any} \quad f \in X;$$

(ii) if $f \in X$ and $g(x) = f(-x)$, then $g \in X$ and $||g||_X \leq c_X||f||_X$;

(iii) if $\gamma = \{\gamma_n\}_{n\in\mathbb{N}}$ is a bounded sequence, and

$$\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \tag{3.4}$$

is the Fourier series of a function $f \in X$, then the series

$$\sum_{n=1}^{\infty} \frac{\gamma_n}{n}(a_n \cos nx + b_n \sin nx) \tag{3.5}$$

is the Fourier series of some function $g \in X$, and $||g||_X \leq c||f||_X$, where c depends only on X and γ.

We observe that the spaces $L^p[0,2\pi]$ $(1 \leq p \leq \infty)$, $\mathrm{Re}\, H^1$, BMO satisfy these conditions. Indeed, properties (i) and (ii) are obvious. Further, let $||\gamma||_{L^\infty} = \sup_n |\gamma_n|$. If $f \in L^p[0,2\pi]$ $(1 \leq p < 2)$, and (3.4) is the Fourier series of f, then

$$\left(\sum_{n=1}^{\infty}(a_n^2+b_n^2)\frac{\gamma_n^2}{n^2}\right)^{1/2} \leq ||\gamma||_{L^\infty}||f||_{L^1}.$$

Thus, (3.5) is the Fourier series of some function $g \in L^2[0,2\pi]$ and $||g||_{L^2} \leq ||\gamma||_{L^\infty}||f||_{L^1}$. In particular, this implies that $||g||_{\mathrm{Re}\,H^1} \leq c||\gamma||_{L^\infty}||f||_{L^1}$. Further, if $f \in L^p[0,2\pi]$ $(2 \leq p \leq \infty)$, then

$$\sum_{n=1}^{\infty}\big(|a_n|+|b_n|\big)\frac{\gamma_n}{n} \leq c||\gamma||_{L^\infty}||f||_{L^2} \leq c'||\gamma||_{L^\infty}||f||_{L^p}.$$

Hence, (3.5) is the Fourier series of some function $g \in L^\infty[0,2\pi]$ and $||g||_{L^\infty} \leq c||\gamma||_{L^\infty}||f||_{L^p}$. This implies also that $||g||_\infty \leq c||\gamma||_{L^\infty}||f||_{\mathrm{BMO}}$.

For any $2\pi-$periodic function $f \in L^1[-\pi,\pi]$ we henceforth denote

$$a_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad b_n(f) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$(n = 0,1,...)$.

For $\lambda > 0$, we denote by $L^\lambda(X)$ the class of all functions $f \in L^1[0,2\pi]$ that can be represented a.e. in the form

$$f(x) = \frac{a_0(f)}{2} + I_\lambda\varphi(x),$$

where $\varphi \in X$ and $a_0(\varphi) = 0$. The function φ is defined uniquely. We denote $\varphi = D^\lambda f$. Set also

$$\|f\|_{L^\lambda(X)} = \frac{|a_0(f)|}{2} + \|\varphi\|_X.$$

If $X = L^p[-\pi,\pi]$ $\quad(1 \leq p \leq \infty)$, then we write $L^\lambda(X) = L_p^\lambda$.

The following theorem shows that transplantation theorems follow directly from representation of the operator (1.4) as Weyl's integral. We use notations (1.3) and (2.4).

Theorem 3.1

Let $\alpha,\beta > -1/2$, $\alpha + 1/2 \notin \mathbb{N}$, and let $\gamma = (\alpha+\beta+1)/2$. Let a space X satisfy the conditions (i) - (iii). Let $f \in X$. Set

$$F(x) = \int_{|x|}^{\pi} f(y)J_{\alpha,\beta}(x,y)\,dy \quad (|x| \leq \pi),$$
$$F_1(x) = F(x)\cos\gamma x, \quad F_2(x) = F(x)\sin\gamma x \quad (-\pi < x \leq \pi)$$

and extend F_1 and F_2 to the whole line with the period 2π. Assume that F_1 and F_2 belong to $L^{\alpha+1/2}(X)$. Then the series

$$\sum_{n=1}^{\infty} a_n^{(\alpha,\beta)}(f)\cos nx \quad \text{and} \quad \sum_{n=1}^{\infty} a_n^{(\alpha,\beta)}(f)\sin nx \tag{3.6}$$

are the Fourier series of some functions $\varphi_1, \varphi_2 \in X$ such that

$$||\varphi_1||_X + ||\varphi_2||_X \le c\big(||D^{\alpha+1/2}F_1||_X + ||D^{\alpha+1/2}F_2||_X\big).$$

Proof. From (1.3) and (2.3), for any $n \in \mathbb{N}$,

$$\begin{aligned} a_n^{(\alpha,\beta)}(f) &= t_n^{(\alpha,\beta)} \int_0^{\pi} F(x)\cos(n+\gamma)x\,dx \\ &= \frac{\pi}{2} t_n^{(\alpha,\beta)} \big[a_n(F_1) - b_n(F_2)\big]. \end{aligned} \tag{3.7}$$

Let $\lambda = \alpha + 1/2$. By our assumption, there exist functions $g_1, g_2 \in X$ such that

$$F_1(x) = \frac{a_0(F_1)}{2} + \frac{1}{2\pi}\int_0^{2\pi} \Psi_\lambda(x-t) g_1(t)\,dt,$$

$$F_2(x) = \frac{1}{2\pi}\int_0^{2\pi} \Psi_\lambda(x-t) g_2(t)\,dt,$$

and $a_0(g_1) = a_0(g_2) = 0$.

Since F_1 is even, from (3.1) we obtain that

$$n^\lambda a_n(F_1) = a_n(g_1)\cos(\lambda\pi/2) - b_n(g_1)\sin(\lambda\pi/2)$$

and

$$b_n(g_1)\cos(\lambda\pi/2) + a_n(g_1)\sin(\lambda\pi/2) = 0.$$

Thus,

$$a_n(F_1) = \frac{a_n(g_1)}{n^\lambda \cos(\lambda\pi/2)}, \quad n \in \mathbb{N}.$$

Similarly, since F_2 is odd, we have that

$$b_n(F_2) = \frac{a_n(g_2)}{n^\lambda \sin(\lambda\pi/2)}, \quad n \in \mathbb{N}.$$

Set

$$g_0(x) = \frac{1}{2}\left(\frac{g_1(x) + g_1(-x)}{\cos(\lambda\pi/2)} - \frac{g_2(x) + g_2(-x)}{\sin(\lambda\pi/2)}\right).$$

Then g_0 is an even function and $a_0(g_0) = 0$,

$$a_n(g_0) = n^\lambda\big[a_n(F_1) - b_n(F_2)\big], \quad n \in \mathbb{N}.$$

Moreover, by the property (ii), $g_0 \in X$ and

$$||g_0||_X \le c\big(||g_1||_X + ||g_2||_X\big). \tag{3.8}$$

By (3.7) and (2.5),

$$a_n^{(\alpha,\beta)}(f) = \frac{\pi}{2} t_n^{(\alpha,\beta)} n^{-\lambda} a_n(g_0) = \left(c_{\alpha,\beta} + \frac{\gamma_n}{n}\right) a_n(g_0), \quad n \in \mathbb{N},$$

where $\{\gamma_n\}$ is a bounded sequence. Applying property (iii) and (3.8), we obtain that the first series in (3.6) is the Fourier series of a function $\varphi_1 \in X$ such that

$$\|\varphi_1\|_X \le c\big(||g_1||_X + ||g_2||_X\big).$$

Similarly, using the equalities

$$a_n(F_1) = -\frac{b_n(g_1)}{n^\lambda \sin(\lambda\pi/2)}, \quad b_n(F_2) = \frac{b_n(g_2)}{n^\lambda \cos(\lambda\pi/2)}, \quad n \in \mathbb{N},$$

we obtain that the second series in (3.6) is the Fourier series of a function $\varphi_2 \in X$ for which

$$\|\varphi_2\|_X \le c\big(||g_1||_X + ||g_2||_X\big). \qquad \square$$

Remark 3.2 The statement of Theorem 3.1 may fail if $\alpha + 1/2 \in \mathbb{N}$. However, it remains true if we assume, in addition, that the conjugate function operator is bounded in X.

Lemma 3.3

Let $0 \le \sigma < 1$, $1 \le p < q \le \infty$, and $0 \le 1/p - 1/q < 1 - \sigma$. Assume that $f \in L^p[-\pi,\pi]$ and set

$$h(x) = \int_{-\pi}^{\pi} f(y) \frac{dy}{|y-x|^\sigma}, \quad x \in [-\pi,\pi].$$

Then $h \in L^q[-\pi,\pi]$ and

$$\|h\|_{L^q} \le c\|f\|_{L^p}.$$

This lemma is well known and follows immediately from the Young inequality for convolutions.

The definition of the fractional integral immediately implies that if ψ is a $2\pi-$periodic absolutely continuous function, then

$$\psi(x) = \frac{a_0(\psi)}{2} + I_1\psi'(x). \tag{3.9}$$

Let $D_{\alpha,\beta}(x,y)$ be defined by (2.7) (note that $D_{\alpha,\beta}(x,y)$ is even in each of variables x, y).

Lemma 3.4

Let $-1/2 < \alpha < 1/2$ and $\beta > -1/2$. Assume that $f \in L^1[-\pi,\pi]$ is a $2\pi-$periodic function such that

$$f(x) = 0 \quad \textit{for} \quad 2\pi/3 \le |x| \le \pi.$$

Set

$$H(x) = \int_{|x|}^{\pi} f(y) D_{\alpha,\beta}(x,y)\, dy, \quad |x| \le \pi.$$

Suppose that η is a $2\pi-$periodic function, differentiable in $(-\pi,\pi)$, and

$$|\eta(x)| \le 1,\ |\eta'(x)| \le 1 \quad \textit{for all} \quad x \in (-\pi,\pi). \tag{3.10}$$

Set $\psi = \eta H$ and extend ψ with the period 2π to the whole line. Then:

(1) *if $f \in L^p[-\pi,\pi]$ $(1 \le p < \infty)$, $p \le q \le \infty$, and $1/p - 1/q < 1$, then $\psi \in L_q^{\alpha+1/2}$, and*

$$\|D^{\alpha+1/2}\psi\|_{L^q} \le c\|f\|_{L^p}; \tag{3.11}$$

(2) *$\psi \in L^{\alpha+1/2}(\operatorname{Re} H^1)$, and*

$$\|D^{\alpha+1/2}\psi\|_{\operatorname{Re} H^1} \le c\|f\|_{L^1}; \tag{3.12}$$

(3) *if $f \in \mathrm{BMO}$, then $\psi \in L_\infty^{\alpha+1/2}$, and*

$$\|D^{\alpha+1/2}\psi\|_{L^\infty} \le c\|f\|_{\mathrm{BMO}}. \tag{3.13}$$

Proof. Denote $\lambda = \alpha + 1/2$; then $0 < \lambda < 1$. By virtue of Proposition 2.3,

$$|D_{\alpha,\beta}(x,y)| \le c \quad \text{and} \quad \left|\frac{\partial D_{\alpha,\beta}}{\partial x}(x,y)\right| \le c(y-x)^{\lambda-1} \tag{3.14}$$

for $(x,y) \in \Omega$. This implies that H is absolutely continuous on $[-\pi,\pi]$. We have also that $H(x) = 0$ if $2\pi/3 \le |x| \le \pi$. Thus, taking into account (3.10), we obtain that ψ is absolutely continuous on $[-\pi,\pi]$. By Proposition 2.3, $D_{\alpha,\beta}(x,x) = 0$ for $|x| \le 2\pi/3$. Hence,

$$\psi'(x) = \eta'(x)H(x) + \eta(x)\operatorname{sign} x \int_{|x|}^{\pi} f(y)\frac{\partial D_{\alpha,\beta}}{\partial x}(x,y)\,dy$$

for almost all $x \in [-\pi,\pi]$. By (3.10) and (3.14), we get

$$|\psi'(x)| \le c\int_{|x|}^{\pi} |f(y)|\frac{dy}{(y-|x|)^{1-\lambda}}, \quad |x| \le \pi. \tag{3.15}$$

Further, by (3.9) and (3.3)

$$\psi - \frac{a_0(\psi)}{2} = I_1\psi' = I_\lambda\varphi, \quad \text{where} \quad \varphi = I_{1-\lambda}\psi' \quad (a_0(\varphi) = 0).$$

Thus, we have $\varphi = D^{\alpha+1/2}\psi$.

Assume that $f \in L^p[-\pi,\pi]$ $(1 \le p < \infty)$. Let $0 \le 1/p - 1/q < 1$. Choose such $r \in [p,q]$ that $1/p - 1/r < \lambda$ and $1/r - 1/q < 1 - \lambda$. By (3.15) and Lemma 3.3,

$$\|\psi'\|_{L^r} \le c\|f\|_{L^p}. \tag{3.16}$$

Further, by (3.2),

$$|\varphi(x)| = |I_{1-\lambda}\psi'(x)| \le c\int_{-\pi}^{\pi} |\psi'(y)|\frac{dy}{|y-x|^\lambda}.$$

Thus, applying Lemma 3.3 and (3.16), we get

$$\|\varphi\|_{L^q} \le c\|\psi'\|_{L^r} \le c'\|f\|_{L^p}.$$

This implies (3.11).

Since $\|\varphi\|_{\mathrm{Re}\, H^1} \le c||\varphi||_q$ for any $q > 1$, we obtain also (3.12).

Finally, if $f \in \mathrm{BMO}$, then $f \in L^p$ for any $p < \infty$, and

$$\|f\|_{L^p} \le c\|f\|_{\mathrm{BMO}}$$

(see [6, p. 226]). Let $1 < p < \infty$. Then, by (3.11),

$$\|\varphi\|_{L^\infty} \le c||f||_{L^p} \le c'\|f\|_{\mathrm{BMO}}.$$

This estimate implies (3.13). □

4. Transplantation theorems

In this section we obtain transplantation theorems for Jacobi polynomials in $\mathrm{Re}\, H^1$ and BMO. We derive these theorems using the approach applied in [11] to ultraspherical polynomials. The core of this approach is made up of the following lemmas proved therein.

Lemma 4.1

Let $f \in \mathrm{Re}\, H^1$, $0 < \lambda < 1$, and

$$G(x) = \int_{|x|}^{\pi} f(y) \left(\frac{\sin y}{\cos x - \cos y} \right)^{1-\lambda} dy, \quad |x| \le \pi. \tag{4.1}$$

Suppose that η is a $2\pi-$periodic function, differentiable in $(-\pi, \pi)$, and

$$|\eta(x)| \le 1, \ |\eta'(x)| \le 1 \quad \textit{for all} \quad x \in (-\pi, \pi). \tag{4.2}$$

Then the function $\chi = \eta G$ belongs to $L^\lambda(\mathrm{Re}\, H^1)$ and

$$||D^\lambda \chi||_{\mathrm{Re}\, H^1} \le c||f||_{\mathrm{Re}\, H^1}.$$

Lemma 4.2

Let $f \in \mathrm{BMO}$ be an odd function such that

$$f(x) = 0 \quad \textit{for} \quad 2\pi/3 \le |x| \le \pi.$$

Let $0 < \lambda < 1$ and let G be defined by (4.1). Suppose that η is a $2\pi-$periodic function, differentiable in $(-\pi, \pi)$ and satisfying (4.2). Then the function $\chi = \eta G$ belongs to $L^\lambda(\mathrm{BMO})$ and

$$||D^\lambda \chi||_* \le c||f||_*.$$

We observe that even for $\lambda < 1$ the proofs of Lemmas 4.1 and 4.2 are rather long and complicated. Using these lemmas, we shall extend the results obtained in [11] for ultraspherical polynomials to the Jacobi polynomials $\varphi_n^{(\alpha,\beta)}$ for the values $\alpha, \beta \in (-1/2, 1/2)$.

As in [11], we first prove one-sided transplantation inequalities in $\operatorname{Re} H^1$ and BMO. We use the notation (1.3) for Jacobi coefficients.

Proposition 4.3

Let $f \in \operatorname{Re} H^1$ and $\alpha, \beta \in (-1/2, 1/2)$. Then the series

$$\sum_{n=1}^{\infty} a_n^{(\alpha,\beta)}(f) \cos nx \tag{4.3}$$

is the Fourier series of some function $\varphi \in \operatorname{Re} H^1$ such that

$$||\varphi||_{\operatorname{Re} H^1} \leq c||f||_{\operatorname{Re} H^1}.$$

Proof. Let $\xi(x)$ be a continuous $2\pi-$periodic function defined in $[-\pi, \pi]$ as follows:

$$\xi(x) = \begin{cases} 1 \quad \text{if} \quad |x| \leq \pi/3, \\ 0 \quad \text{if} \quad 2\pi/3 \leq |x| \leq \pi, \\ \text{linear in} \quad [\pi/3, 2\pi/3] \quad \text{and} \quad [-2\pi/3, -\pi/3]. \end{cases} \tag{4.4}$$

Then

$$|\xi(x') - \xi(x'')| \leq \frac{3}{\pi}|x' - x''|. \tag{4.5}$$

Let $u(x) = f(x)\xi(x)$. Then $u(x) = 0$ for $2\pi/3 \leq |x| \leq \pi$. Furthermore, for the conjugate function $\tilde{u}$, we have

$$\begin{aligned} |\tilde{u}(x)| &= \frac{1}{\pi}\left|\text{v. p.} \int_{-\pi}^{\pi} \frac{\xi(x+t)f(x+t)}{2\tan(t/2)}\,dt\right| \\ &\leq \frac{3}{\pi^2}\int_{-\pi}^{\pi} |f(x+t)|\,dt + \xi(x)|\tilde{f}(x)| \leq \frac{3}{\pi^2}||f||_{L^1} + |\tilde{f}(x)|. \end{aligned}$$

This implies that $u \in \operatorname{Re} H^1$ and

$$||u||_{\operatorname{Re} H^1} \leq 3||f||_{\operatorname{Re} H^1}. \tag{4.6}$$

Setting

$$F(x) = \int_{|x|}^{\pi} u(y) J_{\alpha,\beta}(x,y)\,dy \quad (|x| \leq \pi),$$

we have that $F = G + H$, where

$$G(x) = \int_{|x|}^{\pi} u(y) U_{\alpha+1/2}(x,y)\,dy \quad \text{and} \quad H(x) = \int_{|x|}^{\pi} u(y) D_{\alpha,\beta}(x,y)\,dy$$

(see (2.4), (2.6), and (2.7)). Further, let $\gamma = (\alpha+\beta+1)/2$. Denote

$$\chi(x) = G(x)\cos\gamma x \quad \text{and} \quad \psi(x) = H(x)\cos\gamma x \quad (-\pi < x \le \pi).$$

By Lemma 4.1, $\chi \in L^{\alpha+1/2}(\operatorname{Re} H^1)$ and

$$||D^{\alpha+1/2}\chi||_{\operatorname{Re} H^1} \le c||u||_{\operatorname{Re} H^1} \le 3c||f||_{\operatorname{Re} H^1}$$

(see (4.6)). Further, by Lemma 3.4, $\psi \in L^{\alpha+1/2}(\operatorname{Re} H^1)$ and

$$||D^{\alpha+1/2}\psi||_{\operatorname{Re} H^1} \le c||u||_{\operatorname{Re} H^1} \le 3c||f||_{\operatorname{Re} H^1}.$$

Using these inequalities, we obtain that the function $F_1(x) = F(x)\cos\gamma x$ belongs to $L^{\alpha+1/2}(\operatorname{Re} H^1)$ and

$$||D^{\alpha+1/2}F_1||_{\operatorname{Re} H^1} \le c||f||_{\operatorname{Re} H^1}.$$

Similarly, setting $F_2(x) = F(x)\sin\gamma x$, we have that $F_2 \in L^{\alpha+1/2}(\operatorname{Re} H^1)$ and

$$||D^{\alpha+1/2}F_2||_{\operatorname{Re} H^1} \le c||f||_{\operatorname{Re} H^1}.$$

Thus, by Theorem 3.1, the series

$$\sum_{n=1}^{\infty} a_n^{(\alpha,\beta)}(u)\cos nx$$

is the Fourier series of a function $\mu \in \operatorname{Re} H^1$ such that

$$\|\mu\|_{\operatorname{Re} H^1} \le c(||D^{\alpha+1/2}F_1||_{\operatorname{Re} H^1} + ||D^{\alpha+1/2}F_2||_{\operatorname{Re} H^1}) \le c'||f||_{\operatorname{Re} H^1}.$$

Set now $v(x) = f(x) - u(x)$. By (4.6),

$$||v||_{\operatorname{Re} H^1} \le 4||f||_{\operatorname{Re} H^1}.$$

Let $\overline{v}(x) = v(\pi - x)$. Clearly, $||\overline{v}||_{\operatorname{Re} H^1} = ||v||_{\operatorname{Re} H^1}$. Observe also that

$$\overline{v}(x) = 0 \quad \text{for} \quad 2\pi/3 \le |x| \le \pi. \tag{4.7}$$

Further, we have

$$\varphi_n^{(\alpha,\beta)}(\pi - x) = (-1)^n \varphi_n^{(\beta,\alpha)}(x)$$

(see (1.1) and [14, (4.1.3)]). Thus,

$$a_n^{(\alpha,\beta)}(v) = (-1)^n a_n^{(\beta,\alpha)}(\overline{v}). \tag{4.8}$$

Taking into account (4.7) and applying the same reasonings as above, we obtain that there exists a function $\overline{\nu} \in \operatorname{Re} H^1$ such that $\|\overline{\nu}\|_{\operatorname{Re} H^1} \le c||f||_{\operatorname{Re} H^1}$ and

$$a_n(\overline{\nu}) = a_n^{(\beta,\alpha)}(\overline{v}) \quad (n \in \mathbb{N}), \quad a_0(\overline{\nu}) = 0. \tag{4.9}$$

Set $\nu(x) = \overline{\nu}(\pi - x)$. Then $\|\nu\|_{\operatorname{Re} H^1} \le c||f||_{\operatorname{Re} H^1}$. Since $a_n(\nu) = (-1)^n a_n(\overline{\nu})$, by (4.8) and (4.9), we have

$$a_n(\nu) = a_n^{(\alpha,\beta)}(v) \quad (n \in \mathbb{N}), \quad a_0(\nu) = 0.$$

Finally, set $\varphi = \mu + \nu$. Then $\|\varphi\|_{\operatorname{Re} H^1} \le c||f||_{\operatorname{Re} H^1}$. We have $f = u + v$ and $a_n^{(\alpha,\beta)}(f) = a_n^{(\alpha,\beta)}(u) + a_n^{(\alpha,\beta)}(v)$. Thus, series (4.3) is the Fourier series of φ. $\square$

Proposition 4.4

Let $f \in \mathrm{BMO}$ be an odd function and $\alpha, \beta \in (-1/2, 1/2)$. Then the series (4.3) is the Fourier series of some function $\varphi \in \mathrm{BMO}$ such that

$$||\varphi||_* \leq c||f||_*.$$

Proof. The proof is the same as in Proposition 4.3. We need only a few additional remarks. Let ξ be defined by (4.4) and set $u = f\xi$. Since ξ is even, u is odd. Thus, we have that

$$||u||_{\mathrm{BMO}} = ||u||_* \tag{4.10}$$

(see (1.2)). Further,

$$||u||_* \leq c||f||_*. \tag{4.11}$$

Indeed, let I be an interval, $|I| \leq 2\pi$. Then, using (4.5), we have

$$\begin{aligned}
\frac{1}{|I|}\int_I |u(x) - u_I|\,dx &\leq \frac{1}{|I|^2}\int_I\int_I |u(x) - u(y)|\,dxdy \\
&\leq \frac{1}{|I|^2}\int_I\int_I |f(x) - f(y)|\,dxdy \\
&+ \frac{1}{|I|^2}\int_I\int_I |f(y)||\xi(x) - \xi(y)|\,dxdy \\
&\leq 2||f||_* + \int_{-\pi}^{\pi} |f(y)|\,dy \leq 2(\pi+1)||f||_*
\end{aligned}$$

(we have used the condition that f is odd). This implies (4.11).

Taking into account (4.10) and (4.11), and applying Lemmas 4.2 and 3.4, one can complete the proof exactly as in Proposition 4.3. □

Remark 4.5 The proofs of Propositions 4.3 and 4.4 are based on the use of Lemmas 4.1, 4.2, and 3.4. It is easy to extend Lemma 3.4 on the values $\alpha \geq 1/2$. However, the extension of Lemmas 4.1 and 4.2 to all $\lambda > 0$ is an open problem which may require complicated techniques.

Applying Propositions 4.3 and 4.4, and using duality arguments, we obtain the following theorems.

Theorem 4.6

Let $\alpha, \beta \in (-1/2, 1/2)$. Then:
(i) *if $f \in \mathrm{Re}\, H^1$, then the series*

$$\sum_{n=1}^{\infty} a_n^{(\alpha,\beta)}(f)\cos nx$$

is the Fourier series of some function $\varphi \in \mathrm{Re}\, H^1$ such that

$$||\varphi||_{\mathrm{Re}\, H^1} \leq c||f||_{\mathrm{Re}\, H^1};$$

(ii) *if* $\varphi \in \operatorname{Re} H^1$ *and*

$$\sum_{n=1}^{\infty} a_n(\varphi) \cos nx$$

is its Fourier series, then there exists an odd function $f \in \operatorname{Re} H^1$ *such that*

$$a_n^{(\alpha,\beta)}(f) = a_n(\varphi) \quad (n \in \mathbb{N})$$

and

$$||f||_{\operatorname{Re} H^1} \leq c||\varphi||_{\operatorname{Re} H^1}.$$

Theorem 4.7

Let $\alpha, \beta \in (-1/2, 1/2)$. *Then:*
(i) *if* $f \in \mathrm{BMO}$ *is an odd function, then the series*

$$\sum_{n=1}^{\infty} a_n^{(\alpha,\beta)}(f) \cos nx$$

is the Fourier series of some function $\varphi \in \mathrm{BMO}$ *such that*

$$||\varphi||_* \leq c||f||_*;$$

(ii) *if* $\varphi \in \mathrm{BMO}$ *and*

$$\sum_{n=1}^{\infty} a_n(\varphi) \cos nx$$

is the Fourier series of φ, *then there exists an odd function* $f \in \mathrm{BMO}$ *such that*

$$a_n^{(\alpha,\beta)}(f) = a_n(\varphi) \quad (n \in \mathbb{N})$$

and

$$||f||_* \leq c||\varphi||_*.$$

We omit the proofs of Theorems 4.6 and 4.7 because they are exactly the same as ones given in [11, Section 5] for ultraspherical polynomials.

We emphasize that the statement (i) in Theorem 4.7 fails if f is not odd. Indeed, failure of this statement for even functions was proved in [11] for the case $\alpha = \beta$ (see [11], Section 5, Remark 5; observe that it should be written there that the function $f - f_1$ is equal to $cu_0^{(\lambda)}$, where c is a constant). In the general case the proof is the same.

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