Collectanea Mathematica (electronic version): http://www.collectanea.ub.edu

*Collect. Math.* **61**, 2 (2010), 191–204


# Weak convergence towards two independent Gaussian processes from a unique Poisson process

Xavier Bardina* and David Bascompte

*Departament de Matemàtiques, Universitat Autònoma de Barcelona*
*08193 Bellaterra (Spain)*

E-mail: bardina@mat.uab.cat bascompte@mat.uab.cat

Received July 31, 2009

Abstract

We consider two independent Gaussian processes that admit a representation in terms of a stochastic integral of a deterministic kernel with respect to a standard Wiener process. In this paper we construct two families of processes, from a unique Poisson process, the finite dimensional distributions of which converge in law towards the finite dimensional distributions of the two independent Gaussian processes.

As an application of this result we obtain families of processes that converge in law towards fractional Brownian motion and sub-fractional Brownian motion.

## 1. Introduction and preliminaries

Let $f(t,\cdot)$ and $g(t,\cdot)$ be functions of $\mathrm{L}^2(\mathbb{R}^+)$ for all $t \in [0,T]$, $T>0$, and consider the processes given by

$$Y^f = \left\{ \int_0^\infty f(t,s) \mathrm{d}W_s, t \in [0,T] \right\} \tag{1.1}$$

and

$$\tilde{Y}^g = \left\{ \int_0^\infty g(t,s) \mathrm{d}\tilde{W}_s, t \in [0,T] \right\}, \tag{1.2}$$

* Corresponding author
The authors are partially supported by MEC-Feder Grant MTM2006-06427.

*Keywords:* Weak convergence; Gaussian processes; Poisson process; Sub-fractional Brownian motion; Fractional Brownian motion.

*MSC2000:* 60F17; 60G15.

where $W = \{W_s, s \geq 0\}$ and $\tilde{W} = \{\tilde{W}_s, s \geq 0\}$ are independent standard Brownian motions.

The aim of this paper is to construct two families of processes, from a unique Poisson process, that converge, in the sense of the finite dimensional distributions, to the processes $Y^f$ and $\tilde{Y}^g$. We will use this result later in order to prove weak convergence results towards different kinds of processes such as fractional Brownian motion and sub-fractional Brownian motion.

It is well known the result by Stroock (see [11]) where it is shown that the family of processes

$$\Big\{x_\varepsilon(t) = \frac{1}{\varepsilon}\int_0^t (-1)^{N_{s/\varepsilon^2}}\,\mathrm{d}s, \quad t \in [0,T]\Big\},$$

defined from the kernels $\theta_\varepsilon = \frac{1}{\varepsilon}(-1)^{N_{s/\varepsilon^2}}$, converges in law in $\mathcal{C}([0,T])$ to a standard Brownian motion, where $N = \{N_s, s \geq 0\}$ is a standard Poisson process. This kind of processes were introduced by Kac in [8] in order to write the solution of telegrapher's equation in terms of Poisson process.

On the other hand, Delgado and Jolis (see [6]) extend this result to processes represented by a stochastic integral, with respect to a standard Wiener process, of a deterministic kernel that satisfies some regularity conditions.

A generalization of Stroock's result can be found in [1], where it is proved that the family

$$\Big\{x_\varepsilon^\theta(t) = \frac{2}{\varepsilon}\int_0^t e^{i\theta N_{2s/\varepsilon^2}}\,\mathrm{d}s, \quad t \in [0,T]\Big\} \tag{1.3}$$

converges in law in $\mathcal{C}([0,T])$ to a complex Brownian motion, for $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Particularly, the real part and the imaginary part of (1.3) tend to independent standard Brownian motions.

In this paper, given $\{N_s, s \geq 0\}$ a standard Poisson process and $\theta \in (0,\pi)\cup(\pi,2\pi)$, we consider the following families of approximating processes

$$Y_\varepsilon^f = \Big\{\frac{2}{\varepsilon}\int_0^\infty f(t,s)\cos\big(\theta N_{2s/\varepsilon^2}\big)\mathrm{d}s, \quad t \in [0,T]\Big\} \tag{1.4}$$

and

$$\tilde{Y}_\varepsilon^g = \Big\{\frac{2}{\varepsilon}\int_0^\infty g(t,s)\sin\big(\theta N_{2s/\varepsilon^2}\big)\mathrm{d}s, \quad t \in [0,T]\Big\}. \tag{1.5}$$

The main result of this paper is the proof that the finite dimensional distributions of the processes $Y_\varepsilon^f$ and $\tilde{Y}_\varepsilon^g$ converge in law to the finite dimensional distributions of the processes $Y^f$ and $\tilde{Y}^g$ given by (1.1) and (1.2), respectively.

It is important to note that the processes $Y_\varepsilon^f$ and $\tilde{Y}_\varepsilon^g$ are both functionally dependent. Nevertheless, integrating and taking limits, we obtain two independent processes.

As an application of this result it can be obtained approximations for different examples of centered Gaussian processes, among others, fractional Brownian motion and sub-fractional Brownian motion.

Recall that *fractional Brownian motion* (fBm for short) $B^H = \{B^H(t), t \geq 0\}$ is a centered Gaussian process with covariance function

$$\mathrm{Cov}(B_t^H, B_s^H) = \frac{1}{2}\Big(s^H + t^H - |s-t|^H\Big) \tag{1.6}$$

where $H \in (0,2)$. Usually fBm is defined with Hurst parameter belonging to the interval $(0,1)$ with the corresponding covariance, but in order to compare it with sub-fBm we use the stated representation with $H \in (0,2)$.

On the other hand, *sub-fractional Brownian motion* (sub-fBm for brevity) $S^H = \{S^H(t), t \geq 0\}$ is a centered Gaussian process with covariance function

$$\mathrm{Cov}(S_t^H, S_s^H) = s^H + t^H - \frac{1}{2}\Big[(s+t)^H + |s-t|^H\Big] \tag{1.7}$$

where $H \in (0,2)$.

This process was introduced by Bojdecki *et al.* in 2004 (see [3]) as an intermediate process between standard Brownian motion and fractional Brownian motion. Note that both fBm and sub-fBm are standard Brownian motions for $H = 1$.

For $H \neq 1$, sub-fBm preserves some of the main properties of fBm, such as long-range dependence, but its increments are not stationary; they are more weakly correlated on non-overlapping intervals than fBm ones, and their covariance decays polynomially at a higher rate as the distance between the intervals tends to infinity. For a more detailed discussion of sub-fBm and its properties we refer the reader to [3]. Some properties of this process have also been studied in [13] and [12]. On the other hand there is an extension of sub-fBm in [4].

In [10] (see Theorem 3.3 below) the authors obtain a decomposition of the sub-fBm in terms of fBm and another process with absolutely continuous trajectories, $X^H = \{X_t^H, t \geq 0\}$, which is defined by Lei and Nualart in [9] by

$$X_t^H = \int_0^\infty (1 - e^{-rt}) r^{-(1+H)/2} \, \mathrm{d}W_r \tag{1.8}$$

where $W$ is a standard Brownian motion. Lei and Nualart introduce this process in order to obtain a decomposition of bifractional Brownian motion into the sum of a transformation of $X_t^H$ and a fBm.

The decomposition is different for $H \in (0,1)$ and $H \in (1,2)$. In the first case, sub-fBm is obtained as a sum of two independent processes, a fBm and the process defined by (1.8), while for $H \in (1,2)$ is fBm that is decomposed into the sum of the process (1.8) and a sub-fBm, being these independents.

The paper is organized as follows. In Section 2 we will prove the general result of weak convergence, in the sense of the finite dimensional distributions, towards integrals of functions of $L^2(\mathbb{R}^+)$ with respect to two independent standard Brownian motions. This theorem permits us to obtain, in Section 3, results of convergence in law, in the space $\mathcal{C}([0,T])$, towards fBm, the process defined in (1.8) and, finally, sub-fBm with parameter $H \in (0,1)$ using the decomposition of this process as a sum of two independent processes.

Positive constants, denoted by $C$, with possible subscripts indicating appropriate parameters, may vary from line to line.

## 2. General convergence result

In this section we prove the main result of weak convergence in the sense of the finite dimensional distributions. We will use this result later in order to prove weak convergence results towards fractional Brownian motion and sub-fractional Brownian motion.

**Theorem 2.1**

*Let $f(t,\cdot)$ and $g(t,\cdot)$ be functions of $\mathrm{L}^2(\mathbb{R}^+)$ for all $t \in [0,T]$, $T>0$, let $\{N_s, s \geq 0\}$ be a standard Poisson process and $\theta \in (0,\pi)\cup(\pi,2\pi)$. Define the processes $Y^f$ and $\tilde{Y}^g$, which are given by $Y^f = \{\int_0^\infty f(t,s)\mathrm{d}W_s, t \in [0,T]\}$ and $\tilde{Y}^g = \{\int_0^\infty g(t,s)\mathrm{d}\tilde{W}_s, t \in [0,T]\}$ and where $W=\{W_s, s\geq 0\}$ and $\tilde{W}=\{\tilde{W}_s, s \geq 0\}$ are independent standard Brownian motions. We also define the following processes*

$$Y_\varepsilon^f = \Big\{\frac{2}{\varepsilon}\int_0^\infty f(t,s)\cos\big(\theta N_{2s/\varepsilon^2}\big)ds, \quad t\in[0,T]\Big\} \tag{2.1}$$

*and*

$$\tilde{Y}_\varepsilon^g = \Big\{\frac{2}{\varepsilon}\int_0^\infty g(t,s)\sin\big(\theta N_{2s/\varepsilon^2}\big)ds, \quad t\in[0,T]\Big\}. \tag{2.2}$$

*Then, the finite dimensional distributions of the processes $\{Y_\varepsilon^f\}$ and $\{\tilde{Y}_\varepsilon^g\}$ converge in law to the finite dimensional distributions of the processes $Y^f$ and $\tilde{Y}^g$.*

*Proof.* Taking into account that the proof is valid for any fixed $t\in[0,T]$, by abuse of notation we will write $f(s)$ instead of $f(t,s)$. Slightly modifying the [6, proof of Theorem 1], in order to prove the weak convergence, in the sense of the finite dimensional distributions, it suffices to show that

$$\mathbb{E}\big[(Y_\varepsilon^f)^2\big] \leq C\Big(\int_0^\infty f^2(s)\,\mathrm{d}s\Big), \quad \mathbb{E}\big[(\tilde{Y}_\varepsilon^g)^2\big] \leq C\Big(\int_0^\infty g^2(s)\,\mathrm{d}s\Big). \tag{2.3}$$

Observe that defining

$$Z_\varepsilon^f = Y_\varepsilon^f + i\tilde{Y}_\varepsilon^f = \frac{2}{\varepsilon}\int_0^\infty f(s)e^{i\theta N_{2s/\varepsilon^2}}\mathrm{d}s$$

we have $\mathbb{E}\big[Z_\varepsilon^f \bar{Z}_\varepsilon^f\big] = \mathbb{E}\big[(Y_\varepsilon^f)^2 + (\tilde{Y}_\varepsilon^f)^2\big]$. Therefore if we prove $\mathbb{E}\big[Z_\varepsilon^f\bar{Z}_\varepsilon^f\big] \leq C\|f\|_2^2$, where $\|\cdot\|_2$ is the $L^2(\mathbb{R}^+)$ norm, the stated convergence follows.

$$\begin{aligned}
\mathbb{E}\big[Z_\varepsilon^f\bar{Z}_\varepsilon^f\big] &= \mathbb{E}\Big[\frac{2}{\varepsilon}\int_0^\infty f(s)e^{i\theta N_{2s/\varepsilon^2}}\mathrm{d}s\frac{2}{\varepsilon}\int_0^\infty f(r)e^{-i\theta N_{2r/\varepsilon^2}}\mathrm{d}r\Big] \\
&= \frac{4}{\varepsilon^2}\mathbb{E}\Big[\int_0^\infty\int_0^\infty f(s)f(r)e^{i\theta\left(N_{2s/\varepsilon^2}-N_{2r/\varepsilon^2}\right)}\mathrm{d}s\,\mathrm{d}r\Big] \\
&= \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{I}_{\{r\leq s\}}f(s)f(r)\mathbb{E}\Big[e^{i\theta\left(N_{2s/\varepsilon^2}-N_{2r/\varepsilon^2}\right)}\Big]\mathrm{d}r\,\mathrm{d}s \\
&\quad + \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{I}_{\{s\leq r\}}f(s)f(r)\mathbb{E}\Big[e^{-i\theta\left(N_{2r/\varepsilon^2}-N_{2s/\varepsilon^2}\right)}\Big]\mathrm{d}s\,\mathrm{d}r.
\end{aligned}$$

Since $\mathbb{E}[e^{i\theta X}] = e^{-\lambda(1-e^{i\theta})}$ and $\mathbb{E}[e^{-i\theta X}] = e^{-\lambda(1-e^{-i\theta})}$, being $X$ a Poisson random variable of parameter $\lambda$, we obtain

$$\begin{aligned}\mathbb{E}\big[Z_\varepsilon^f \bar{Z}_\varepsilon^f\big] &= \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{r\le s\}} f(s)f(r)e^{-2(s-r)(1-e^{i\theta})/\varepsilon^2}\mathrm{d}r\,\mathrm{d}s\\
&\quad+\frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{s\le r\}} f(s)f(r)e^{-2(r-s)(1-e^{-i\theta})/\varepsilon^2}\mathrm{d}r\,\mathrm{d}s\\
&\le \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{r\le s\}} |f(s)f(r)|\, e^{-2(s-r)(1-\cos\theta)/\varepsilon^2}\mathrm{d}r\,\mathrm{d}s\\
&\quad+\frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{s\le r\}} |f(s)f(r)|\, e^{-2(r-s)(1-\cos\theta)/\varepsilon^2}\mathrm{d}r\,\mathrm{d}s.\end{aligned}$$

Using the inequality $|f(s)f(r)| \le \frac{1}{2}\big(f^2(s)+f^2(r)\big)$ and noting that, by means of a change of variables, the last two integrals are the same we have that

$$\begin{aligned}\mathbb{E}\big[Z_\varepsilon^f \bar{Z}_\varepsilon^f\big] &\le \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty \mathbb{1}_{\{s\le r\}}\big(f^2(s)+f^2(r)\big)\, e^{-2(r-s)(1-\cos\theta)/\varepsilon^2}\mathrm{d}r\,\mathrm{d}s\\
&= \frac{4}{\varepsilon^2}\Big(\int_0^\infty f^2(s)\int_s^\infty e^{-2(r-s)(1-\cos\theta)/\varepsilon^2}\mathrm{d}r\,\mathrm{d}s\\
&\quad+\int_0^\infty f^2(r)\int_0^r e^{-2(r-s)(1-\cos\theta)/\varepsilon^2}\mathrm{d}s\,\mathrm{d}r\Big)\\
&= 2\Big(\int_0^\infty f^2(s)\Big(\frac{1}{1-\cos\theta}\Big)\mathrm{d}s + \int_0^\infty f^2(r)\Big(\frac{1-e^{-2r(1-\cos\theta)/\varepsilon^2}}{1-\cos\theta}\Big)\mathrm{d}r\Big)\\
&\le \frac{4}{1-\cos\theta}\int_0^\infty f^2(s)\,\mathrm{d}s.\end{aligned}$$

Then, the convergence of the finite dimensional distributions has been proved and it remains to prove the independence of the limit processes. We begin by proving that the family $\{Y_\varepsilon^f \tilde{Y}_\varepsilon^g\}_{\varepsilon>0}$ is uniformly integrable. Indeed, we will prove that $\sup_{\varepsilon>0}\mathbb{E}\big[(Y_\varepsilon^f \tilde{Y}_\varepsilon^g)^2\big]<\infty$. Using Hölder's inequality we have

$$\sup_{\varepsilon>0}\mathbb{E}\big[(Y_\varepsilon^f \tilde{Y}_\varepsilon^g)^2\big] \le \sup_{\varepsilon>0}\Big(\mathbb{E}\big[(Y_\varepsilon^f)^4\big]\Big)^{1/2}\Big(\mathbb{E}\big[(\tilde{Y}_\varepsilon^g)^4\big]\Big)^{1/2}.$$

In order to prove that the last expression is finite, we will show that

$$\mathbb{E}\big[(Y_\varepsilon^f)^4\big] \le C\Big(\int_0^\infty f^2(s)\,\mathrm{d}s\Big)^2, \quad \mathbb{E}\big[(\tilde{Y}_\varepsilon^g)^4\big] \le C\Big(\int_0^\infty g^2(s)\,\mathrm{d}s\Big)^2. \tag{2.4}$$

Being $Z_\varepsilon^f$ like before, we can prove (2.4) showing that $\mathbb{E}\big[(Z_\varepsilon^f \bar{Z}_\varepsilon^f)^2\big] \le C\|f\|_2^4$.

$$\begin{aligned}&\mathbb{E}\big[(Z_\varepsilon^f \bar{Z}_\varepsilon^f)^2\big]\\
&\quad= \frac{16}{\varepsilon^4}\mathbb{E}\Big[\int_{[0,\infty)^4} f(s_1)\cdots f(s_4)e^{i\theta\big(N_{2s_1/\varepsilon^2}+N_{2s_2/\varepsilon^2}-N_{2s_3/\varepsilon^2}-N_{2s_4/\varepsilon^2}\big)}\mathrm{d}s_1\cdots\mathrm{d}s_4\Big]\\
&\quad= \frac{64}{\varepsilon^4}\int_{[0,\infty)^4}\mathbb{1}_{\{s_1\le\cdots\le s_4\}} f(s_1)\cdots f(s_4)\mathbb{E}\left[E_1+\cdots+E_6\right]\mathrm{d}s_1\cdots\mathrm{d}s_4\end{aligned}$$

where

$$\begin{aligned}
E_1 &= e^{i\theta\left(N_{2s_1/\varepsilon^2}+N_{2s_2/\varepsilon^2}-N_{2s_3/\varepsilon^2}-N_{2s_4/\varepsilon^2}\right)} \\
&= e^{-i\theta\left(N_{2s_4/\varepsilon^2}-N_{2s_3/\varepsilon^2}+2\left(N_{2s_3/\varepsilon^2}-N_{2s_2/\varepsilon^2}\right)+N_{2s_2/\varepsilon^2}-N_{2s_1/\varepsilon^2}\right)}, \\
E_2 &= e^{-i\theta\left(N_{2s_4/\varepsilon^2}-N_{2s_3/\varepsilon^2}+N_{2s_2/\varepsilon^2}-N_{2s_1/\varepsilon^2}\right)}, \\
E_3 &= e^{i\theta\left(N_{2s_4/\varepsilon^2}-N_{2s_3/\varepsilon^2}-\left(N_{2s_2/\varepsilon^2}-N_{2s_1/\varepsilon^2}\right)\right)},
\end{aligned}$$

$E_4 = \overline{E_3}$, $E_5 = \overline{E}_2$, $E_6 = \overline{E}_1$. To obtain the last expression note that we can arrange $s_1, s_2, s_3, s_4$ in 24 different ways and due to the symmetry between $s_1$ and $s_2$ and between $s_3$ and $s_4$ we have 6 possible different situations, $E_1, \ldots, E_6$, each one repeated 4 times. By means of the properties of Poisson process we have

$$\|\mathbb{E}[E_1]\|, \|\mathbb{E}[E_2]\|, \|\mathbb{E}[E_3]\| \le e^{-2(s_4-s_3)(1-\cos\theta)/\varepsilon^2} e^{-2(s_2-s_1)(1-\cos\theta)/\varepsilon^2}$$

and we can conclude

$$\begin{aligned}
\mathbb{E}\big[(Z_\varepsilon^f \bar{Z}_\varepsilon^f)^2\big] &\le \frac{384}{\varepsilon^4} \int_{[0,\infty)^4} \mathbb{I}_{\{s_1\le\cdots\le s_4\}} |f(s_1)\cdots f(s_4)| \\
&\qquad \times e^{-2(s_4-s_3)(1-\cos\theta)/\varepsilon^2} e^{-2(s_2-s_1)(1-\cos\theta)/\varepsilon^2} \mathrm{d}s_1\cdots\mathrm{d}s_4 \\
&\le \frac{384}{2\varepsilon^2}\Big( \int_{[0,\infty)^2} \mathbb{I}_{\{s_1\le s_2\}} |f(s_1)f(s_2)| e^{-2(s_2-s_1)(1-\cos\theta)/\varepsilon^2} \mathrm{d}s_1\mathrm{d}s_2\Big)^2 \\
&\le 3\Big(\frac{4}{1-\cos\theta}\int_0^\infty f^2(s)\,\mathrm{d}s\Big)^2.
\end{aligned}$$

Then the family $\{Y_\varepsilon^f \tilde{Y}_\varepsilon^g\}_{\varepsilon>0}$ is uniformly integrable and consequently

$$\mathbb{E}\big[\lim_{\varepsilon\to0} Y_\varepsilon^f(t)\tilde{Y}_\varepsilon^g(s)\big] = \lim_{\varepsilon\to0}\mathbb{E}\big[Y_\varepsilon^f(t)\tilde{Y}_\varepsilon^g(s)\big].$$

Since $Y^f$ and $\tilde{Y}^g$ are centered Gaussian processes, in order to prove their independence it suffices to show that the last limit converges to zero as $\varepsilon$ tends to zero. To deal with this limit, we observe that

$$\begin{aligned}
\mathbb{E}\big[Y_\varepsilon^f \tilde{Y}_\varepsilon^g\big] &= \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty f(s)g(r)\mathbb{E}\left[\cos(\theta N_{2s/\varepsilon^2})\sin(\theta N_{2r/\varepsilon^2})\right]\mathrm{d}s\,\mathrm{d}r \\
&= \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty f(s)g(r)\mathbb{I}_{\{s\le r\}}\mathbb{E}\left[\cos(\theta N_{2s/\varepsilon^2})\sin(\theta N_{2r/\varepsilon^2})\right]\mathrm{d}s\,\mathrm{d}r \\
&\quad + \frac{4}{\varepsilon^2}\int_0^\infty\int_0^\infty f(s)g(r)\mathbb{I}_{\{r\le s\}}\mathbb{E}\left[\cos(\theta N_{2s/\varepsilon^2})\sin(\theta N_{2r/\varepsilon^2})\right]\mathrm{d}s\,\mathrm{d}r \\
&= I_1^\varepsilon + I_2^\varepsilon.
\end{aligned}$$

Applying the formula $2\sin a\cos b = \sin(a+b)+\sin(a-b) = \sin(a+b)-\sin(b-a)$ we have

$$\begin{aligned}
I_1^\varepsilon &= \frac{2}{\varepsilon^2}\int_0^\infty\int_0^\infty f(s)g(r)\mathbb{1}_{\{s\le r\}} \\
&\qquad \mathbb{E}\Big[\sin\big(\theta(N_{2s/\varepsilon^2}+N_{2r/\varepsilon^2})\big)+\sin\big(\theta(N_{2r/\varepsilon^2}-N_{2s/\varepsilon^2})\big)\Big]\mathrm{d}s\,\mathrm{d}r \\
&= I_{1,1}^\varepsilon + I_{1,2}^\varepsilon, \\
I_2^\varepsilon &= \frac{2}{\varepsilon^2}\int_0^\infty\int_0^\infty f(s)g(r)\mathbb{1}_{\{r\le s\}} \\
&\qquad \mathbb{E}\Big[\sin\big(\theta(N_{2s/\varepsilon^2}+N_{2r/\varepsilon^2})\big)-\sin\big(\theta(N_{2s/\varepsilon^2}-N_{2r/\varepsilon^2})\big)\Big]\mathrm{d}s\,\mathrm{d}r \\
&= I_{2,1}^\varepsilon - I_{2,2}^\varepsilon.
\end{aligned}$$

We proceed to show that $I_{1,1}^\varepsilon$ and $I_{2,1}^\varepsilon$ converges to zero as $\varepsilon$ tends to zero and that $I_{1,2}^\varepsilon$ and $I_{2,2}^\varepsilon$ have the same (finite) limit, thus obtaining the stated result. We note that

$$\begin{aligned}
I_{1,1}^\varepsilon &= \mathrm{Im}\Big(\frac{2}{\varepsilon^2}\int_0^\infty\int_0^\infty f(s)g(r)\mathbb{1}_{\{s\le r\}}\mathbb{E}\big[e^{i\theta(N_{2r/\varepsilon^2}-N_{2s/\varepsilon^2})}e^{2i\theta N_{2s/\varepsilon^2}}\big]\mathrm{d}s\,\mathrm{d}r\Big) \\
&= \mathrm{Im}(A^\varepsilon).
\end{aligned}$$

To find the limit of $I_{1,1}^\varepsilon$ we see that $\|A^\varepsilon\|$ converges to zero as $\varepsilon$ tends to zero.

$$\begin{aligned}
\|A^\varepsilon\| &\le \frac{2}{\varepsilon^2}\int_0^\infty\int_0^\infty |f(s)g(r)|\mathbb{1}_{\{s\le r\}}e^{-2(r-s)(1-\cos\theta)/\varepsilon^2}e^{-2s(1-\cos 2\theta)/\varepsilon^2}\mathrm{d}s\,\mathrm{d}r \\
&\le \frac{1}{\varepsilon^2}\int_0^\infty\int_0^\infty \big(f^2(s)+g^2(r)\big)\,\mathbb{1}_{\{s\le r\}}e^{-2r(1-\cos\theta)/\varepsilon^2}e^{2s(\cos 2\theta-\cos\theta)/\varepsilon^2}\mathrm{d}s\,\mathrm{d}r \\
&= \frac{1}{\varepsilon^2}\int_0^\infty f^2(s)e^{2s(\cos 2\theta-\cos\theta)/\varepsilon^2}\int_s^\infty e^{-2r(1-\cos\theta)/\varepsilon^2}\mathrm{d}r\,\mathrm{d}s \\
&\quad + \frac{1}{\varepsilon^2}\int_0^\infty g^2(r)e^{-2r(1-\cos\theta)/\varepsilon^2}\int_0^r e^{2s(\cos 2\theta-\cos\theta)/\varepsilon^2}\mathrm{d}s\,\mathrm{d}r \\
&= A_1^\varepsilon + A_2^\varepsilon.
\end{aligned}$$

When $\cos\theta = \cos 2\theta$ it is easy to check the convergence to zero. Otherwise, we integrate obtaining

$$\begin{aligned}
A_1^\varepsilon &= \frac{1}{2(1-\cos\theta)}\int_0^\infty f^2(s)e^{-2s(1-\cos 2\theta)/\varepsilon^2}\mathrm{d}s, \\
A_2^\varepsilon &= \frac{1}{2(\cos 2\theta-\cos\theta)}\int_0^\infty g^2(r)e^{-2r(1-\cos\theta)/\varepsilon^2}\Big(e^{2r(\cos 2\theta-\cos\theta)/\varepsilon^2}-1\Big)\,\mathrm{d}r \\
&= \frac{1}{2(\cos 2\theta-\cos\theta)}\int_0^\infty g^2(r)\Big(e^{-2r(1-\cos 2\theta)/\varepsilon^2}-e^{-2r(1-\cos\theta)/\varepsilon^2}\Big)\,\mathrm{d}r
\end{aligned}$$

which concludes, as the convergence to zero is easily seen by dominated convergence. In the same manner we can see that $I^\varepsilon_{2,1}$ converges to zero.

With respect to the term $I^\varepsilon_{1,2}$ we observe that

$$I^\varepsilon_{1,2} = \operatorname{Im}\Big(\frac{2}{\varepsilon^2}\int_0^\infty\int_0^\infty f(s)g(r)\mathbb{1}_{\{s\le r\}}e^{-2(r-s)(1-e^{i\theta})/\varepsilon^2}\mathrm{d}s\,\mathrm{d}r\Big).$$

Since $\frac{2}{\varepsilon^2}(1-e^{i\theta})e^{-2(r-s)(1-e^{i\theta})/\varepsilon^2}$ is an approximation of the identity, we have that $I^\varepsilon_{1,2}$ converges, as $\varepsilon$ tends to zero, to $\operatorname{Im}\big(\frac{1}{1-e^{i\theta}}\int_0^\infty f(s)g(s)\mathrm{d}s\big) < \infty$. Clearly the same result is obtained for $I^\varepsilon_{2,2}$. This finishes the proof. □

*Remark 2.2* We can use this result to approximate two independent processes of many kinds, such as processes with a Gousart kernel (see for instance [6]), the Holmgren-Riemann-Liouville fractional integral ([6]), fractional Brownian motion and sub-fractional Brownian motion. In the next section we study weak convergence, in the space $\mathcal{C}([0,T])$, towards these two last processes.

## 3. Weak approximation of fractional and sub-fractional Brownian motion

In this section we apply the main theorem to prove weak convergence results towards fractional Brownian motion and sub-fBm. Let us begin with fBm and later on we will study the convergence for sub-fBm.

### 3.1 Weak approximation of fractional Brownian motion

We are going to prove a result of weak convergence in $\mathcal{C}([0,T])$ towards fBm, applying Theorem 2.1. In order to do so, we use the following representation of the fBm as the integral of a deterministic kernel with respect to standard Brownian motion (see for instance [7])

$$B^H_t = \int_0^t \tilde{K}^H(t,s)\,\mathrm{d}W_s, \tag{3.1}$$

where $H\in(0,2)$, $\tilde{K}^H(t,s)$ is defined on the set $\{0<s<t\}$ and is given by

$$\tilde{K}^H(t,s) = d^H(t-s)^{(H-1)/2} + d^H\Big(\frac{1-H}{2}\Big)\int_s^t (u-s)^{(H-3)/2}\Big(1-\Big(\frac{s}{u}\Big)^{(1-H)/2}\Big)\mathrm{d}u, \tag{3.2}$$

where the normalizing constant $d^H$ is

$$d^H = \Big(\frac{H\Gamma(\frac{3-H}{2})}{\Gamma(\frac{H+1}{2})\Gamma(2-H)}\Big)^{1/2}.$$

Since in this section the domain of fBm is restricted to the interval $t\in[0,T]$, we can rewrite the integral representation as

$$B^H_t = \int_0^t \tilde{K}^H(t,s)\,\mathrm{d}W_s = \int_0^T K^H(t,s)\,\mathrm{d}W_s,$$

where $K^H(t,s) = \tilde{K}^H(t,s)\mathbb{1}_{[0,t]}(s)$.

Applying this representation, since $K^H(t,\cdot)\in \mathrm{L}^2(\mathbb{R}^+)$, the following result is a corollary of Theorem 2.1

**Corollary 3.1**

*Let $K^H(t,s) = \tilde{K}^H(t,s)\mathbb{1}_{[0,t]}(s)$, where $\tilde{K}^H(t,s)$ is defined by (3.2), let $\{N_s, s \geq 0\}$ be a standard Poisson process and let $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Then the processes*

$$B_\varepsilon^H = \left\{ \frac{2}{\varepsilon} \int_0^T K^H(t,s) \cos\left(\theta N_{2s/\varepsilon^2}\right) ds, \quad t \in [0,T] \right\} \tag{3.3}$$

*and*

$$\tilde{B}_\varepsilon^H = \left\{ \frac{2}{\varepsilon} \int_0^T K^H(t,s) \sin\left(\theta N_{2s/\varepsilon^2}\right) ds, \quad t \in [0,T] \right\} \tag{3.4}$$

*converge in law, in the sense of the finite dimensional distributions, towards two independent fractional Brownian motions.*

We now proceed to prove the continuity and the tightness of the families of processes defined by (3.3) and (3.4), and consequently, proving the weak convergence in the space $\mathcal{C}([0,T])$.

**Theorem 3.2**

*Under the hypothesis of Corollary 3.1, if moreover one of the following conditions is satisfied:*

(1) $H \in (\frac{1}{2}, 2)$,
(2) $H \in (0, \frac{1}{2}]$ *and* $\theta$ *satisfies* $\cos((2i+1)\theta) \neq 1$ *for all* $i \in \mathbb{N}$ *such that* $i \leq \frac{1}{2}\left[\frac{1}{H}\right]$,

*then the processes $B_\varepsilon^H$ and $\tilde{B}_\varepsilon^H$ converge in law in $\mathcal{C}([0,T])$ towards two independent fractional Brownian motions.*

*Proof.* We first observe that the processes $B_\varepsilon^H$ and $\tilde{B}_\varepsilon^H$ are continuous. In fact, $B_\varepsilon^H$ and $\tilde{B}_\varepsilon^H$ are continuous for all $H \in (0,2)$ and absolutely continuous if $H \in (1,2)$, since it can be proved that (see [2, Lemma 2.1])

$$|B_\varepsilon^H(t) - B_\varepsilon^H(s)| \leq C_H (t-s)^{((H+1)/2)\wedge 1}$$

and

$$|\tilde{B}_\varepsilon^H(t) - \tilde{B}_\varepsilon^H(s)| \leq C_H (t-s)^{((H+1)/2)\wedge 1}.$$

It only remains to prove the tightness of the families of processes defined by (3.3) and (3.4). Since $B_\varepsilon^H(0) = 0$, using Billingsley's criterion (see for instance [5]) it is enough to check that for some $m > 0$ and $\alpha > 1$

$$\mathbb{E}[|B_\varepsilon^H(t) - B_\varepsilon^H(s)|^m] \leq C(F(t) - F(s))^\alpha,$$

where $F$ is a nondecreasing continuous function.

On the other hand, it is known that

$$\int_0^T \left(K^H(t,r) - K^H(s,r)\right)^2 \mathrm{d}r = \mathbb{E}\left[(B_t^H - B_s^H)^2\right] = (t-s)^H,$$

and then it is sufficient to show that

$$\mathbb{E}\left[(y_\varepsilon^f)^m\right] \leq C_m \Big( \int_0^T f^2(r)\,\mathrm{d}r \Big)^{m/2}, \quad \mathbb{E}\left[(\tilde{y}_\varepsilon^f)^m\right] \leq C_m \Big( \int_0^T f^2(r)\,\mathrm{d}r \Big)^{m/2} \tag{3.5}$$

holds for some $m$ satisfying the condition $Hm/2 > 1$, where

$$f(r) := K^H(t,r) - K^H(s,r),\ y_\varepsilon^f = \frac{2}{\varepsilon}\int_0^T f(r)\cos(\theta N_{2r/\varepsilon^2})\,\mathrm{d}r$$

and

$$\tilde{y}_\varepsilon^f = \frac{2}{\varepsilon}\int_0^T f(r)\sin(\theta N_{2r/\varepsilon^2})\,\mathrm{d}r.$$

Then, in the case (1), it is sufficient to prove (3.5) for $m = 4$, which can be seen proving that $\mathbb{E}[(z_\varepsilon^f \bar{z}_\varepsilon^f)^2] \le C\|f\|_2^4$, where $\|\cdot\|_2$ is the $L^2[0,T]$ norm and $z_\varepsilon^f = y_\varepsilon^f + i\tilde{y}_\varepsilon^f$. If we extend $f$ to $\mathbb{R}^+$ for zeros, i.e., if we consider $F(r) := f(r)\mathbb{1}_{[0,T]}(r)$, we have proved in Theorem 2.1 that

$$\mathbb{E}\big[(Z_\varepsilon^F \bar{Z}_\varepsilon^F)^2\big] \le 3\Big(\frac{4}{1-\cos\theta}\int_0^\infty F^2(s)\,\mathrm{d}s\Big)^2.$$

Then,

$$\begin{aligned}\mathbb{E}\big[(z_\varepsilon^f \bar{z}_\varepsilon^f)^2\big] &= \mathbb{E}\big[(Z_\varepsilon^F \bar{Z}_\varepsilon^F)^2\big] \\ &\le 3\Big(\frac{4}{1-\cos\theta}\int_0^\infty F^2(s)\,\mathrm{d}s\Big)^2 = 3\Big(\frac{4}{1-\cos\theta}\int_0^T f^2(s)\,\mathrm{d}s\Big)^2.\end{aligned}$$

To prove the result under the hypothesis (2) we can show that (3.5) is satisfied for some even $m$ such that $\frac{Hm}{2} > 1$. If we proceed in the same way as in case (1) we obtain an expression that depends on $1 - \cos((2i+1)\theta)$ for all $i = 0, 1, \dots, \big[\frac{1}{2H}\big]$ and the constant $C_m$ depends on $\max_{i=0,1,\dots,[\frac{1}{2H}]} \frac{1}{1-\cos((2i+1)\theta)}$. □

### 3.2 Convergence towards sub-fractional Brownian motion

In order to obtain the convergence to sub-fractional Brownian motion, we will apply a decomposition result due to Ruiz de Chávez and Tudor in [10]. In this paper, they use the process $X^H$ introduced by Lei and Nualart in [9] and defined in (1.8) by the equation

$$X_t^H = \int_0^\infty (1 - e^{-rt}) r^{-(1+H)/2}\,\mathrm{d}W_r,$$

where $W$ is a standard Brownian motion. It can be proved (see [9, 10]) that its covariance function is

$$\mathrm{Cov}(X_t^H, X_s^H) = \begin{cases} \frac{\Gamma(1-H)}{H}\left[t^H + s^H - (t+s)^H\right] & \text{if } H \in (0,1), \\ \frac{\Gamma(2-H)}{H(H-1)}\left[(t+s)^H - t^H - s^H\right] & \text{if } H \in (1,2), \end{cases} \tag{3.6}$$

and that $X^H$ has a version with absolutely continuous trajectories on $[0,\infty)$.

The decomposition result can be stated and proved as follows:

**Theorem 3.3** (Decomposition of sub-fBm)

*Let $B^H$ be a fBm, $S^H$ a sub-fBm and $W = \{W_t, t \ge 0\}$ a standard Brownian motion. Let $X^H$ be the process given by (1.8). If for $H \in (0,1)$ we suppose that*

*$B^H$ and $W$ are independents, then the processes $\{Y_t^H = C_1 X_t^H + B_t^H, t \geq 0\}$ and $\{S_t^H, t \geq 0\}$ have the same law, where $C_1 = \sqrt{\frac{H}{2\Gamma(1-H)}}$. If for $H \in (1,2)$ we suppose that $S^H$ and $W$ are independents, then the processes $\{Y_t^H = C_2 X_t^H + S_t^H, t \geq 0\}$ and $\{B_t^H, t \geq 0\}$ have the same law, where $C_2 = \sqrt{\frac{H(H-1)}{2\Gamma(2-H)}}$.*

*Proof.* It is clear that the process $Y^H$ is centered and Gaussian in both cases. For $H \in (0,1)$, from (1.6), (3.6) and using the independence of $X^H$ and $B^H$ we have

$$\begin{aligned}\mathrm{Cov}(Y_t^H, Y_s^H) &= C_1^2 \mathrm{Cov}[X_t^H, X_s^H] + \mathrm{Cov}[B_t^H, B_s^H] \\ &= s^H + t^H - \frac{1}{2}\left[(s+t)^H + |s-t|^H\right],\end{aligned}$$

which completes the proof in this case, and for $H \in (1,2)$, from (1.7), (3.6) and using the independence of $X^H$ and $S^H$ we have

$$\begin{aligned}\mathrm{Cov}(Y_t^H, Y_s^H) &= C_2^2 \mathrm{Cov}[X_t^H, X_s^H] + \mathrm{Cov}[S_t^H, S_s^H] \\ &= \frac{1}{2}\left(s^H + t^H - |s-t|^H\right),\end{aligned}$$

which completes the proof. □

In order to apply the main theorem to prove weak convergence to sub-fBm, we have to prove weak convergence to fBm and the process $X^H$ introduced by Lei and Nualart. Then, applying the decomposition theorem and the independence of the limit laws, we can state the weak convergence to sub-fBm for $H \in (0,1)$.

So, it just remains to prove for the process $X^H$ defined by (1.8) the same results we have obtained for fBm.

**Corollary 3.4**

*Let $X^H$ be the process defined by (1.8), let $\{N_s, s \geq 0\}$ be a standard Poisson process and let $\theta \in (0,\pi) \cup (\pi, 2\pi)$. Then the processes*

$$X_\varepsilon^H = \left\{\frac{2}{\varepsilon}\int_0^\infty (1-e^{-st}) s^{-(1+H)/2} \cos\left(\theta N_{2s/\varepsilon^2}\right) ds, \quad t \in [0,T]\right\} \tag{3.7}$$

*and*

$$\tilde{X}_\varepsilon^H = \left\{\frac{2}{\varepsilon}\int_0^\infty (1-e^{-st}) s^{-(1+H)/2} \sin\left(\theta N_{2s/\varepsilon^2}\right) ds, \quad t \in [0,T]\right\} \tag{3.8}$$

*converge in law, in the sense of the finite dimensional distributions, towards two independent processes with the same law that $X^H$.*

**Theorem 3.5**

*Under the hypothesis of Corollary 3.4 the processes defined by (3.7) and (3.8) converge in law, in $\mathcal{C}([0,T])$, towards two independent processes with the same law that the process defined by (1.8).*

*Proof.* We first need to show that the processes $X_\varepsilon^H$ and $\tilde{X}_\varepsilon^H$ are continuous. In fact, they are absolutely continuous. Let us consider for all $r > 0$ the process

$$Y_r = \frac{2}{\varepsilon}\int_0^\infty s^{(1-H)/2} e^{-sr} \cos\big(\theta N_{2s/\varepsilon^2}\big)\, \mathrm{d}s.$$

This integral exists because, using inequality (2.3), we have

$$\mathbb{E}[Y_r^2] \le C\Big(\int_0^\infty s^{1-H} e^{-2sr} \mathrm{d}s\Big) = C r^{H-2}\Gamma(2-H).$$

On the other hand,

$$\mathbb{E}\Big[\int_0^t |Y_r| \mathrm{d}r\Big] \le \int_0^t (\mathbb{E}[Y_r^2])^{1/2}\mathrm{d}r \le C\int_0^t r^{(H-2)/2}\mathrm{d}r < \infty$$

since $H \in (0,2)$.

Let us now observe that $X_\varepsilon^H = \int_0^t Y_r \mathrm{d}r$. Indeed, applying Fubini's theorem,

$$\begin{aligned}\int_0^t Y_r \mathrm{d}r &= \frac{2}{\varepsilon}\int_0^\infty s^{(1-H)/2}\Big(\int_0^t e^{-sr}\mathrm{d}r\Big)\cos\big(\theta N_{2s/\varepsilon^2}\big)\,\mathrm{d}s\\ &= \frac{2}{\varepsilon}\int_0^\infty s^{-(1+H)/2}(1-e^{-st})\cos\Big(\theta N_{2s/\varepsilon^2}\Big)\mathrm{d}s\\ &= X_\varepsilon^H.\end{aligned}$$

The same proof shows that the process $\tilde{X}_\varepsilon^H$ is continuous.

Next, we prove the convergence only for (3.7). For (3.8) the result is proved similarly.

It suffices to prove the tightness of the family $\{X_\varepsilon^H\}_\varepsilon$. Since $X_\varepsilon^H(0) = 0$, using Billingsley's criterion we only need to prove that

$$\mathbb{E}\left[|X_\varepsilon^H(t) - X_\varepsilon^H(s)|^4\right] \le |F(t) - F(s)|^2$$

where $F$ is a continuous, non-decreasing function. We observe that

$$\mathbb{E}\left[|X_\varepsilon^H(t) - X_\varepsilon^H(s)|^4\right] = \mathbb{E}\left[\frac{2}{\varepsilon}\int_0^\infty \big(\Phi^H(t,r) - \Phi^H(s,r)\big)\cos(\theta N_{2r/\varepsilon^2})\,\mathrm{d}r\right]^4$$

where $\Phi^H(t,r) = (1-e^{-rt})r^{-(1+H)/2} \in \mathrm{L}^2(\mathbb{R}^+)$.

Since $\Phi^H \in \mathrm{L}^2(\mathbb{R}^+)$, applying the bound (2.4), which is proved in Theorem 2.1, we obtain

$$\begin{aligned}\mathbb{E}\left[|X_\varepsilon^H(t) - X_\varepsilon^H(s)|^4\right] &\le C\Big(\int_0^\infty \big(\Phi^H(t,r) - \Phi^H(s,r)\big)^2\,\mathrm{d}r\Big)^2\\ &= C\Bigg(\int_0^\infty \Big((1-e^{-rt})^2 r^{-(1+H)} + (1-e^{-rs})^2 r^{-(1+H)}\\ &\qquad - 2(1-e^{-rt})(1-e^{-rs})r^{-(1+H)}\Big)\mathrm{d}r\Bigg)^2.\end{aligned}$$

Using (3.6) and assuming $s < t$ we obtain for $H \in (0, 1)$

$$\begin{aligned}\mathbb{E}\left[|X_\varepsilon^H(t) - X_\varepsilon^H(s)|^4\right] &\leq C\left(2(t+s)^H - (2t)^H - (2s)^H\right)^2 \\ &\leq C\left((2t)^H - (2s)^H\right)^2,\end{aligned}$$

since $s + t < 2t$. In the same way, if $H \in (1, 2)$,

$$\begin{aligned}\mathbb{E}\left[|X_\varepsilon^H(t) - X_\varepsilon^H(s)|^4\right] &\leq C\left((2t)^H + (2s)^H - 2(t+s)^H\right)^2 \\ &\leq C\left((2t)^H - (2s)^H\right)^2,\end{aligned}$$

since $s + t > 2s$. In both cases we have proved the result with $F(x) = (2x)^H$. □

Finally, we obtain the result of weak convergence to sub-fractional Brownian motion, as a direct conclusion of the previous results.

**Theorem 3.6**

*Let $H \in (0, 1)$, let $\{X_\varepsilon^H(t), t \in [0, T]\}$ be the family of processes defined by (3.7), let $\{\tilde{B}_\varepsilon^H(t), t \in [0, T]\}$ be the family of processes defined by (3.4) and $C_1 = \sqrt{\frac{H}{2\Gamma(1-H)}}$. Let us assume $\theta \in (0, \pi) \cup (\pi, 2\pi)$ and, for $H \in (0, \frac{1}{2}]$, that $\theta$ is such that $\cos((2i+1)\theta) \neq 1$ for all $i \in \mathbb{N}$ such that $i \leq \frac{1}{2}\left[\frac{1}{H}\right]$. Then, $\{Y_\varepsilon^H(t) = C_1 X_\varepsilon^H(t) + B_\varepsilon^H(t), t \in [0, T]\}$ weakly converges in $\mathcal{C}([0, T])$ to a sub-fractional Brownian motion.*

*Proof.* Applying Theorems 3.2 and 3.5 we know that, respectively, the processes $\tilde{B}_\varepsilon^H$ and $X_\varepsilon^H$ converge in law in $\mathcal{C}([0, T])$ towards a fBm and the process defined by (1.8). Moreover, applying Theorem 2.1, we know that the limit laws are independent. Hence, we are under the hypothesis of Theorem 3.3, which proves the stated result. □

*Remark 3.7* Obviously we can also obtain the same result using the families of processes defined by (3.8) and (3.3).

## References


1. X. Bardina, The complex Brownian motion as a weak limit of processes constructed from a Poisson process, *Stochastic analysis and related topics, VII (Kusadasi, 1998)*, 149–158, Progr. Probab., Birkhäuser Boston, Boston, MA, 2001.
2. X. Bardina and C. Florit, Approximation in law to the $d$-parameter fractional Brownian sheet based on the functional invariance principle, *Rev. Mat. Iberoamericana* **21** (2005), 1037–1052.
3. T. Bojdecki, L.G. Gorostiza, and A. Talarczyk, Sub-fractional Brownian motion and its relation to occupation times, *Statist. Probab. Lett.* **69** (2004), 405–419.
4. T. Bojdecki, L.G. Gorostiza, and A. Talarczyk, Some extensions of fractional Brownian motion and sub-fractional Brownian motion related to particle systems, *Electron. Comm. Probab.* **12** (2007), 161–172.
5. P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons Inc., New York, 1968.
6. R. Delgado and M. Jolis, Weak approximation for a class of Gaussian processes, *J. Appl. Probab.* **37** (2000), 400–407.
7. L. Decreusefond and A.S. Üstünel, Stochastic analysis of the fractional Brownian motion, *Potential Anal.* **10** (1999), 177–214.

8. M. Kac, A stochastic model related to the telegrapher's equation, *Rocky Mountain J. Math.* **4** (1974), 497–509.
9. P. Lei and D. Nualart, A decomposition of the bifractional brownian motion and some applications, *Statist. Probab. Lett.* **79** (2009), 619–624.
10. J. Ruiz de Chávez and C. Tudor, A decomposition of sub-fractional Brownian motion, *Math. Rep. (Bucur.)* **61** (2009), 67–74.
11. D.W. Stroock, *Lectures on Topics in Stochastic Differential Equations*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics **68**, Tata Institute of Fundamental Research, Bombay; by Springer-Verlag, Berlin-New York, 1982.
12. C. Tudor, Some properties of the sub-fractional Brownian motion, *Stochastics* **79** (2007), 431–448.
13. C. Tudor, Inner product spaces of integrands associated to subfractional Brownian motion, *Statist. Probab. Lett.* **78** (2008), 2201–2209.