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b-weighted dyadic BMO from dyadic BMO and associated $T(b)$ theorems

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Abstract

Given a function b, and using adapted Haar wavelets, we define a BMO-type norm which is dependent on b. In both global and local cases, we find the dependence of the bounds on $\|f\|_{\mathrm{BMO}}$ by the bounds on the b-weighted BMO norm of f. We show that the dependence is sharp in the global case. Multiscale analysis is used in the local case. We formulate as corollaries global and local dyadic $T(b)$ theorems whose hypotheses include a bound on the b-weighted BMO-norm of $T^*(1)$.

1. Introduction

In their 1984 paper [6], David and Journé give a necessary and sufficient condition for a singular integral operator to be bounded on the space $L^2(\mathbb{R}^n)$. Both the properties of the operator T and cancellation properties of its associated kernel K are considered. In 1985, David, Journé, and Semmes [7] further develop the theory by by replacing the constant function 1 on which the operator T is evaluated by a function b whose mean is bounded away from zero. These $T(b)$ theorems have been studied in several contexts (see [3, 4, 12]) and in their 2002 paper [2], Auscher, Hofmann, Muscalu, Tao, and Thiele prove several $T(b)$ theorems in a dyadic setting in the context of Carleson measures and trees. In 2003, Tolsa [15] used the non-doubling $T(b)$ theorem in [12] in his answer to the Painlevé problem and his proof of the semiadditivity of analytic capacity of a compact set in $\mathbb{C}$.

As part of a program to understand the dependency of the bounds on the operator T by the bounds on b in several formulations of the dyadic $T(b)$ theorems, we formulate a b-weighted BMO-norm and compare it to the standard BMO norm. In

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effect, this changes the theorem from a b-input case (where T is evaluated at a function b) to a b-output case, where $T^*(1)$ is evaluated by a norm which depends on b. This is a natural extension of the previous work, in particular in [2] where the dyadic $T(b)$ theorems are proven and in [13], where sharp growth bounds for b-input theorems are proven. Our approach utilizes b-adapted Haar wavelets as in [5]. Associated b-output $T(b)$ theorems in both local and global cases follow as corollaries from theorems which compare b-weighted dyadic BMO with the dyadic BMO developed in [9]. In addition to formulating and proving these theorems, we look at the dependence of the bounds on the b-weighted BMO-norm on the bounds on b and the usual dyadic BMO-norm, and provide an example in the global case to show that the dependence is sharp. In the local case, sharpness is an open question.

Though the notation will be developed rigorously in the following sections, we state the major results of this paper here and direct the reader to Section 2 for definitions and notation, and Sections 3 and 5 for proofs.

Comparison of $\mathrm{BMO}(b)$ **and** BMO **for a globally defined** b. *Consider a function* b *defined on* $[0,1)$ *such that*

$$\frac{1}{|I|}\left|\int_I b dx\right| = |[b]_I| \geq 1$$

for all $I \in \Delta$ *and such that there exists a* $0 < \gamma \ll 1$ *with*

$$\|b\|_{\mathrm{BMO}} \leq \frac{1}{\gamma}.$$

Let $f \in L^2_{\mathrm{loc}}$, *such that* $\|f\|_{\mathrm{BMO}(b)} \leq G$. *Then*

$$\|f\|_{\mathrm{BMO}} \leq C\gamma^{-1}\|f\|_{\mathrm{BMO}(b)}.$$

Furthermore, the power on γ *is sharp.*

The dyadic b**-output global** $T(b)$ **theorem**. *Let* b *be a function on* $\mathbb{R}$ *such that there exists a* $C > 0$ *and a* $0 < \gamma \ll 1$ *such that* $\frac{1}{C} \leq |[b]_I|$ *for all* $I \in \Delta$ *and* $\|b\|_{\mathrm{BMO}} \leq \gamma^{-1}$. *Let* T *be a dyadic singular integral operator such that*

$$|\langle T\psi_I, \psi_I\rangle| \leq 1$$
$$\|T(1)\|_{\mathrm{BMO}} \leq G_1$$
$$\|T^*(1)\|_{\mathrm{BMO}(b)} \leq G_2.$$

Then T *is bounded on* L^2 *and*

$$\|Tf\|_2 \leq C\gamma^{-1}\|f\|_2.$$

The constant C in the conclusion is dependent on G_1 and G_2, but we note that C is independent of γ.

Comparison of $\mathrm{BMO}(b)$ **and** BMO **for a set of locally defined** $\{b_I\}$. *Let* $f \in L^2_{\mathrm{loc}}$ *such that for all* $J \in \Delta$, *and for a fixed* $G \in \mathbb{Z}^+$,

$$|\langle f, \psi_J\rangle| \leq G|J|^{1/2} < \infty.$$

Let $0 < \gamma \ll 1$. *For a fixed interval* I, *let* $\{b_I\}_{I\in\Delta}$ *be a collection of functions such that for each* I, b_I *satisfies*

$$\operatorname{supp}(b_I) \subseteq I$$
$$|[b_I]_I| \geq \frac{1}{C}, \quad |[b_I]_{I_l}| \geq \frac{1}{C}, \quad |[b_I]_{I_r}| \geq \frac{1}{C}$$
$$\|b_I\|_{L^2(I)} \leq \frac{|I|^{1/2}}{\gamma}.$$

Then

$$\|f\|_{\mathrm{BMO}} \leq C\gamma^{-5}\|f\|_{\mathrm{BMO}((b_I)_{I\in\Delta})}.$$

The dyadic b-output local $T(b)$ theorem. *Let* T *be a dyadic singular integral operator such that for all* $J \in \Delta$, *and for a fixed* $G \in \mathbb{Z}^+$,

$$|\langle T^*(1), \psi_J\rangle| \leq G|J|^{1/2} < \infty,$$

and such that

$$\|T(1)\|_{\mathrm{BMO}} \leq G_1$$
$$\|T^*(1)\|_{\mathrm{BMO}((b_I)_{I\in\Delta})} \leq G_2.$$

Let $0 < \gamma \ll 1$. *Suppose that for every interval* $I \in \Delta$, *there exists a function* b_I *satisfying*

$$\operatorname{supp}(b_I) \subseteq I$$
$$|[b_I]_I| \geq \frac{1}{C}, \quad |[b_I]_{I_l}| \geq \frac{1}{C}, \quad |[b_I]_{I_r}| \geq \frac{1}{C}$$
$$\|b_I\|_{L^2(I)} \leq \frac{|I|^{1/2}}{\gamma}.$$

Then T *is bounded on* L^2 *and in particular*

$$\|Tf\|_2 \leq C\gamma^{-5}\|f\|_2.$$

2. Definitions and notation

The following is a summary of definitions and notation which will be used in the following. We use the same conventions as in [13]. For a more detailed explanation of these preliminaries, see [13, Section 2].

We consider the real line $\mathbb{R}$ decomposed into dyadic intervals, I.

Definition 2.1 $I \subset \mathbb{R}$ is a *dyadic interval* if it is of the form $[j2^k, (j+1)2^k)$, for some $j, k \in \mathbb{Z}$.

Given a dyadic interval $I \subset \mathbb{R}$, we use the notation $|I|$ to denote the Lebesgue measure of I. We adopt the convention that the left side of the interval is closed and the right side is open for two reasons: first, we would like to have intervals partition $\mathbb{R}$, and second, we would like dyadic intervals to nest nicely.

Proposition 2.1 (Nesting property of dyadic intervals)

Given two dyadic intervals I and J, one of the following situations occurs:

- $I = J$
- $I \cap J = \emptyset$
- $I \subset J$
- $J \subset I$.

Notice that given any collection of dyadic intervals, the subset of intervals which are maximal with respect to inclusion are disjoint. This property will be used heavily in the following.

Given a dyadic interval I, we refer to the left and right halves of I, denoted I_l and I_r respectively, as the *children* of I. Each dyadic interval has exactly two children, four grandchildren, eight great-grandchildren, and so on. It also has a unique *parent*, of which it is either a *left or right child*.

As in [2, 13], we restrict ourselves to a finite set of dyadic intervals on the half-line. We fix a large M, and let

$$\Delta_M = \big\{ I = [j2^k, (j+1)2^k) : j, k \in \mathbb{Z}, -M \le k \le M, \text{ and } I \subseteq [0, 2^M) \big\}.$$

Our estimates will be independent of M, and so we freely take $\Delta_M = \Delta$. As such, we can use a standard translation and limiting argument to get bounds over the non-truncated dyadic line.

We will also use the language of trees as developed in [2] in our lemmas and their proofs.

DEFINITION 2.2 A *dyadic tree* (henceforth abbreviated *tree*) is a collection of dyadic intervals $\mathcal{T} \subseteq \Delta$ with a top interval (called the *top* of the tree), denoted $I_{\mathcal{T}}$, which is the unique dyadic interval in $\mathcal{T}$ such that for all $J \in \mathcal{T}$, we have that $J \subseteq I_{\mathcal{T}}$.

A tree $\mathcal{T}$ is said to be *complete* if $J \in \mathcal{T}$ for all $J \subseteq I_{\mathcal{T}}$. We let $\mathrm{Tree}(I)$ denote the complete tree with top I.

Given a function f defined on an interval I, we let

$$[f]_I = \frac{1}{|I|} \int_I f dx$$

be the mean value of f on I.

We study operators of a particular type, following the notation from [4]. By *singular integral operator*, we mean an operator which is defined as an integral against a kernel which is in some way singular. This definition may be formal, as

$$Tf(x) = \int K(x, y) f(y) dy \tag{1}$$

may not be finite for all values of x. The kernel K is a function from $(\mathbb{R}\times\mathbb{R})\setminus\{x=y\}$ to $\mathbb{R}$ which is integrable off the diagonal. For reasons of simplicity, we limit ourselves to one-dimensional analysis. To sharpen the formal definition (1), we use the following from [4]:

Definition 2.3 A kernel K on $\mathbb{R}$ is said to satisfy *standard estimates* if there exist $\delta > 0$ and $C < \infty$ such that for all distinct $x, y \in \mathbb{R}$ and all z such that $|x-z| < \frac{|x-y|}{2}$:

(1) $|K(x,y)| \leq \frac{C}{|x-y|}$

(2) $|K(x,y) - K(z,y)| \leq C\frac{|x-z|^\delta}{|x-y|^{1+\delta}}$

(3) $|K(y,x) - K(y,z)| \leq C\frac{|x-z|^\delta}{|x-y|^{1+\delta}}$.

We will refer to a function satisfying the above estimates as a *standard kernel.*

We define a dyadic metric on $\mathbb{R}$ in the following manner: given $x, y \in \mathbb{R}$, let $|x-y|_{\text{dyadic}}$ be the length of the smallest dyadic interval containing both x and y. We adapt the definition of a standard kernel to this metric, normalizing so that the constant $C = 1$.

Definition 2.4 A kernel K on $(\mathbb{R}\times\mathbb{R})\setminus\{x=y\}$ is said to satisfy *dyadic standard estimates* if

(1) For all $(x,y) \in (\mathbb{R}\times\mathbb{R})\setminus\{x=y\}$,

$$|K(x,y)| \leq \frac{1}{|x-y|_{\text{dyadic}}},$$

(2) For all $x, x' \in I$ and $y \in J$ for sibling dyadic intervals I and J,

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| = 0.$$

A dyadic singular integral operator, T, is an operator which is defined as integration against such a kernel.

Given $f \in C_0^\infty$, T a singular integral operator, and provided that x is not in the support of f, we can define $Tf(x)$ as in (1) above without sacrificing rigor.

As in [6], the adjoint operator T^* is defined by $\langle T^*f, g\rangle = \langle f, Tg\rangle$ and is associated to a related standard kernel given by $L(x,y) = K(y,x)$.

In the continuous case in [6], defining the action of T on the constant function 1 is problematic and must be done carefully using distributions. In the truncated dyadic case defined earlier, we look only at a finite number of scales contained in one large dyadic interval, so these problems cease to exist and we may, without losing rigor, refer to $T(1)$ without confusion. (Similarly, $T^*(1)$ is defined rigorously and intuitively on our space.)

In our analysis, we use Haar wavelets,

$$\psi_I = \frac{1}{|I|^{1/2}}(\chi_{I_l} - \chi_{I_r}),$$

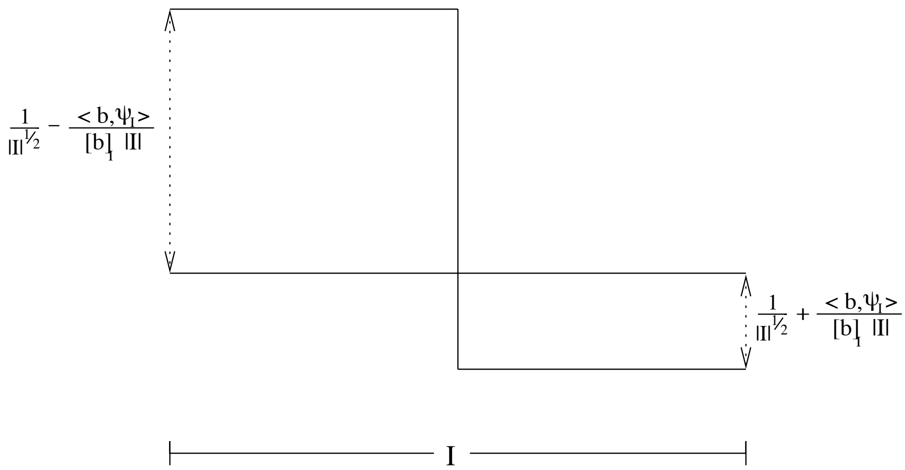

Figure 1: A diagram of ψ_I^b.

where $I \in \Delta$, and where χ_J is the characteristic function on the interval J. We also use an L^2-normalized characteristic function $\widetilde{\chi_I} = \frac{1}{|I|^{1/2}}\chi_I$. The set $\{\psi_I\}_{I\in\Delta} \cup \chi_{[0,1)}$ forms an orthonormal basis for $L^2([0,1))$.

For our purposes, we will deal only with functions defined on $\mathbb{R}$, and will use the common notation BMO = BMO($\mathbb{R}$). Furthermore, we will look only at *dyadic* BMO, a norm for which is defined below for locally integrable functions. For more information on dyadic BMO, see [9].

DEFINITION 2.5 A locally integrable function f will be said to belong to BMO if

$$\|f\|_{\mathrm{BMO}} = \sup_{J\in\Delta} \frac{1}{|J|^{1/2}} \left(\int_J |f - [f]_J|^2 dx \right)^{1/2} = \sup_{J\in\Delta} \frac{1}{|J|^{1/2}} \left(\sum_{I\subseteq J} |\langle f, \psi_I\rangle|^2 \right)^{1/2} < \infty.$$

We develop a new norm which is similar to the BMO-norm defined above, but our norm differs in that it depends on a function b in the global case and on a set of functions $\{b_I\}_{I\in\Delta}$ in the local case. In order to define this norm, we must look at Haar wavelets which are adapted to the function b as in [2, 5].

DEFINITION 2.6 Let $\mathbb{I} \subseteq \Delta$ be a collection of intervals and b a function which has nonzero mean on all intervals in $\mathbb{I}$. For each $I \in \mathbb{I}$, we define the *b-adapted Haar wavelet*

$$\psi_I^b = \psi_I - \frac{\langle b, \psi_I\rangle}{[b]_I} \frac{\chi_I}{|I|}. \tag{2}$$

See Figure 1. Note that ψ_I^b is defined only on intervals where $[b]_I \neq 0$. Though we do make this assumption on b in the global case, it becomes an issue in the local case. The wavelet ψ_I^b does not have mean 0, but it does have b-weighted mean 0:

$$\int b\,\psi_I^b = 0.$$

As a result, for all $I \neq J$,

$$\int \psi_I^b \, b \, \psi_J^b = 0.$$

As an alternate definition, we may use

$$\psi_I^b = |I|^{-1/2} \frac{[b]_{I_r}}{[b]_I} \chi_{I_l} - |I|^{-1/2} \frac{[b]_{I_l}}{[b]_I} \chi_{I_r}.$$

We note that

$$\int \psi_I^b \, b \, \psi_I^b = \frac{[b]_{I_r}[b]_{I_l}}{[b]_I} = \frac{2}{[b]_{I_l}^{-1} + [b]_{I_r}^{-1}}.$$

In particular, if $[b]_I > 1, [b]_{I_l} > 1$, and $[b]_{I_r} > 1$, then

$$\left| \int \psi_I^b \, b \, \psi_I^b \right| = \left| \frac{2}{[b]_{I_r}^{-1} + [b]_{I_l}^{-1}} \right| \geq 1.$$

We define the dual to ψ_I^b in the following way:

Definition 2.7 Given an adapted Haar wavelet ψ_I^b, we define the *dual b-adapted Haar wavelet* by

$$\rho_I^b = \frac{\psi_I^b \, b}{\int \psi_I^b \, b \, \psi_I^b}.$$

Note that for those I on which ρ_I^b is defined,

$$\frac{1}{|I|} \int_I \rho_I^b = 0.$$

It will be useful to express ρ_I^b as

$$\rho_I^b = \alpha_I b \chi_{I_l} + \beta_I b \chi_{I_r} \tag{3}$$

where

$$\alpha_I = \frac{1}{|I|^{1/2}} \frac{1}{[b]_{I_l}} \quad \text{and} \quad \beta_I = -\frac{1}{|I|^{1/2}} \frac{1}{[b]_{I_r}}.$$

We also have $\langle \rho_I^b, \psi_J^b \rangle = \delta_{IJ}$, the Kronecker delta, and we have the representation formula,

$$f = \sum_{I \in \Delta} \langle f, \psi_I^b \rangle \rho_I^b,$$

for all $f \in$ Span$\{\rho_I^b, I \in \Delta\}$, provided $[b]_I \neq 0$ for all I. We characterize this span as the co-dimension 1 subspace of $L^2(I)$ where

$$f - \frac{[f]_I}{[b]_I} b = \sum_{J \in \text{Tree}(I)} \langle f, \psi_J^b \rangle \rho_J^b. \tag{4}$$

In order to state and prove a global half-sided output theorem, we must define the b-dependent norm.

DEFINITION 2.8 Given b, a function with $|[b]_I| \geq 1$ for all $I \in \Delta$, and given the associated b-adapted Haar wavelets ψ_I^b and ρ_I^b, we define the BMO(b) semi-norm of a function as follows:

$$\|f\|_{\mathrm{BMO}(b)} = \sup_J \frac{1}{|J|^{1/2}} \left(\sum_{I \subseteq J} |\langle f, \rho_I^b \rangle|^2 \right)^{1/2}.$$

We say that $f \in \mathrm{BMO}(b)$ if and only if $\|f\|_{\mathrm{BMO}(b)} < \infty$.

It is clear that $\|cf\|_{\mathrm{BMO}(b)} = |c| \|f\|_{\mathrm{BMO}(b)}$ for any constant c. Furthermore, the triangle inequality holds, so this is a semi-norm. For any constant function C

$$\begin{aligned} \|C\|_{\mathrm{BMO}(b)} &= \sup_J \frac{1}{|J|^{1/2}} \left(\sum_{I \subseteq J} |\langle C, \rho_I^b \rangle|^2 \right)^{1/2} \\ &= 0 \end{aligned}$$

as $\langle C, \rho_I^b \rangle = 0$ for all I, so this is not a norm. However, if we define this up to additive constants (as the BMO-"norm" is defined) then this is a norm on the equivalence classes of functions $f \sim g$ if and only if $f - g = C$.

In particular, when T is a dyadic singular integral operator, and $f = T^*(1)$, we get

$$\begin{aligned} \|T^*(1)\|_{\mathrm{BMO}(b)} &= \sup_J \frac{1}{|J|^{1/2}} \left(\sum_{I \subseteq J} |\langle T^*(1), \rho_I^b \rangle|^2 \right)^{1/2} \\ &= \sup_J \frac{1}{|J|^{1/2}} \left(\sum_{I \subseteq J} |\langle 1, T\rho_I^b \rangle|^2 \right)^{1/2} \\ &= \sup_J \frac{1}{|J|^{1/2}} \left(\sum_{I \subseteq J} |\langle \chi_I, T\rho_I^b \rangle|^2 \right)^{1/2} \qquad \text{(as } T\rho_I^b \text{ is supported on } I.) \\ &= \sup_J \frac{1}{|J|^{1/2}} \left(\sum_{I \subseteq J} |\langle T^*\chi_I, \rho_I^b \rangle|^2 \right)^{1/2}. \end{aligned}$$

We note further that if $b = C$, a constant function, then $\rho_I^b = C\psi_I$ and

$$\|f\|_{\mathrm{BMO}(C)} = |C| \sup_J \frac{1}{|J|^{1/2}} \left(\sum_{I \subseteq J} |\langle f, \psi_I \rangle|^2 \right)^{1/2} = |C| \|f\|_{\mathrm{BMO}},$$

so our definition of BMO(C) matches the standard definition of a dyadic BMO norm.

3. Comparing BMO(b) and BMO for a globally defined b

We formulate and prove a theorem comparing the BMO(b) norm of a local L^2 function and its dyadic BMO norm. We track the powers of γ, the constant associated to the norm of the function b, and we provide an example to show that this power is sharp. A dyadic global b-output $T(b)$ theorem follows as an easy corollary.

Theorem 3.1 (Comparison of BMO(b) and BMO for a globally defined b)
Consider a function b defined on $[0,1)$ such that

$$|[b]_I| \geq 1$$

for all $I \in \Delta$ and such that there exists a $0 < \gamma \ll 1$ with

$$\|b\|_{\mathrm{BMO}} \leq \frac{1}{\gamma}.$$

Let $f \in L^2_{\mathrm{loc}}$, such that $\|f\|_{\mathrm{BMO}(b)} \leq G$. Then

$$\|f\|_{\mathrm{BMO}} \leq C\gamma^{-1}\|f\|_{\mathrm{BMO}(b)}.$$

Furthermore, the power on γ is sharp.

Proof. Fix $I \in \Delta$. Let $f \in L^2_{\mathrm{loc}}$. We wish to bound $\|f\|_{\mathrm{BMO}}$. Take $g \in \mathrm{Span}\{\rho^b_J : J \subseteq I\}$. We know that $[g]_I = 0$ and so by the representation formula, we can write

$$g = \sum_{J \subseteq I} \langle g, \psi^b_J \rangle \rho^b_J.$$

We pair the functions f and g to find the necessary bound. In the following calculation, we make use of the Carleson Embedding Theorem [14] with $p = 2$.

$$\begin{aligned}
|\langle f, g\rangle| &= \left| \sum_{J \subseteq I} \langle g, \psi^b_J \rangle \langle f, \rho^b_J \rangle \right| \\
&\leq \left(\sum_{J \subseteq I} |\langle g, \psi^b_J \rangle|^2 \right)^{1/2} \left(\sum_{J \subseteq I} |\langle f, \rho^b_J \rangle|^2 \right)^{1/2} \qquad \text{(by Cauchy-Schwarz)} \\
&\leq \left(\sum_{J \subseteq I} |\langle g, \psi^b_J \rangle|^2 \right)^{1/2} |I|^{1/2} \|f\|_{\mathrm{BMO}(b)} \\
&\leq |I|^{1/2}\|f\|_{\mathrm{BMO}(b)} \left[\Big(\sum_{J \subseteq I} |\langle g, \psi_J \rangle|^2 \Big)^{1/2} + \Big(\sum_{J \subseteq I} \frac{|\langle b, \psi_J \rangle|^2}{|[b]_J|^2} |[g]_J|^2 \Big)^{1/2} \right] \\
&\leq |I|^{1/2}\|f\|_{\mathrm{BMO}(b)} \left[\|g\|_2 + C\|g\|_2 \Big(\sup_K \frac{1}{|K|} \sum_{J \subseteq K} |\langle b, \psi_J \rangle|^2 \Big)^{1/2} \right] \\
&\leq C|I|^{1/2}\|f\|_{\mathrm{BMO}(b)}\|g\|_2\|b\|_{\mathrm{BMO}} \\
&\leq C\gamma^{-1}|I|^{1/2}\|f\|_{\mathrm{BMO}(b)}\|g\|_2.
\end{aligned}$$

By duality, we get that

$$\|f\|_{\mathrm{BMO}} \leq C\gamma^{-1}\|f\|_{\mathrm{BMO}(b)},$$

as desired.

Next, we demonstrate functions f and b for which this power on γ is sharp.

Fix $\gamma \in (0,1)$, and let

$$f = \frac{1}{\gamma}\psi_{[0,1)} - \sqrt{2}\psi_{[0,1/2)}. \tag{5}$$

Then $\|f\|_{\mathrm{BMO}} = \frac{1}{\gamma}$.

We now look at $\|f\|_{\mathrm{BMO}(b)}$ for a general b. Later, we will pick a b that satisfies the hypotheses of the theorem and that makes $\|f\|_{\mathrm{BMO}(b)}$ bounded independent of γ. By definition of f in (5), and by the definition of the BMO(b) seminorm,

$$\begin{aligned}\|f\|_{\mathrm{BMO}(b)} &= \sup_J \frac{1}{|J|^{1/2}}\left(\sum_{I\subseteq J} |\langle f, \rho_I^b\rangle|^2\right)^{1/2} \\ &= \sup_J \frac{1}{|J|^{1/2}}\left(\sum_{I\subseteq J} \left|\frac{1}{\gamma}\langle \psi_{[0,1)}, \rho_I^b\rangle - \sqrt{2}\langle \psi_{[0,1/2)}, \rho_I^b\rangle\right|^2\right)^{1/2}.\end{aligned}$$

We note that all of the summands are of the form

$$\langle \psi_K, \rho_I^b\rangle$$

for some pair of dyadic intervals I and K. If $I \cap K = \emptyset$, then

$$\langle \psi_K, \rho_I^b\rangle = 0,$$

as the functions have disjoint supports. If $I \subsetneq K$, then by the nesting property of dyadic intervals, $I \subseteq K_l$ or $I \subseteq K_r$. We know that for any b with $[b]_{I_l} \neq 0$ and $[b]_{I_r} \neq 0$, ρ_I^b is supported and mean zero on I, so for $I \subsetneq K$,

$$\langle \psi_K, \rho_I^b\rangle = \frac{1}{|K|^{1/2}}\left(\int_{K_l} \rho_I^b - \int_{K_r} \rho_I^b\right) = 0.$$

If $I = K$, then

$$\langle \psi_I, \rho_I^b\rangle = \langle \psi_I^b, \rho_I^b\rangle + C\langle \chi_I, \rho_I^b\rangle = 1.$$

This leaves the case where $K \subsetneq I$. In fact, in our example, $K = [0,1)$ which is contained in no larger intervals in our space, or $K = [0, \frac{1}{2})$ which is only contained in $[0,1)$. We need only to calculate the inner product when $K = [0, \frac{1}{2})$ and $I = [0,1)$. We recall from (3) that for any I on which $[b]_{I_l} \neq 0$ and $[b]_{I_r} \neq 0$,

$$\rho_I^b = \alpha_I b\chi_{I_l} + \beta_I b\chi_{I_r}$$

where

$$\alpha_I = \frac{1}{|I|^{1/2}}\frac{1}{[b]_{I_l}}, \quad \text{and} \quad \beta_I = -\frac{1}{|I|^{1/2}}\frac{1}{[b]_{I_r}}.$$

Therefore

$$\begin{aligned}\left\langle \psi_{[0,1/2)}, \rho_{[0,1)}^b\right\rangle &= \alpha_{[0,1)}\left\langle \psi_{[0,1/2)}, b\chi_{[0,1/2)}\right\rangle + \beta_{[0,1)}\left\langle \psi_{[0,1/2)}, b\chi_{[1/2,1)}\right\rangle \\ &= \frac{1}{[b]_{[0,1/2)}}\left\langle \psi_{[0,1/2)}, b\right\rangle.\end{aligned}$$

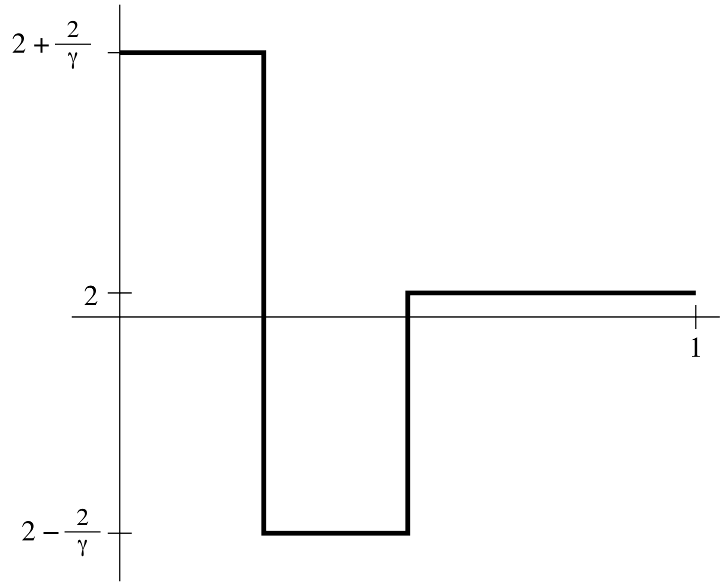

Figure 2: The function b, supported on $[0, 1)$.

We now use these calculations to find $\|f\|_{\mathrm{BMO}(b)}$.

We know that for any b,

$$\begin{aligned}\|f\|_{\mathrm{BMO}(b)} &= C\Big(|\langle f, \rho^b_{[0,1/2)}\rangle|^2 + |\langle f, \rho^b_{[0,1)}\rangle|^2\Big)^{1/2} \\ &= C\Bigg(\Big|\frac{1}{\gamma}\langle \psi_{[0,1)}, \rho^b_{[0,1/2)}\rangle - \sqrt{2}\langle \psi_{[0,1/2)}, \rho^b_{[0,1/2)}\rangle\Big|^2 \\ &\qquad + \Big|\frac{1}{\gamma}\langle \psi_{[0,1)}, \rho^b_{[0,1)}\rangle - \sqrt{2}\langle \psi_{[0,1/2)}, \rho^b_{[0,1)}\rangle\Big|^2\Bigg)^{1/2} \\ &= C\Bigg(2 + \Big|\frac{1}{\gamma} - \sqrt{2}\frac{1}{[b]_{[0,1/2)}}\langle \psi_{[0,1/2)}, b\rangle\Big|^2\Bigg)^{1/2}\end{aligned}$$

where $C = 1$ if the supremum is attained on $[0, 1)$ and $C = \sqrt{2}$ if the supremum is attained on $[0, \frac{1}{2})$. We want $\|f\|_{\mathrm{BMO}(b)}$ to be independent of γ, and so we pick the function b accordingly.

Let

$$b(x) = \begin{cases} 2\left(1 + \frac{1}{\gamma}\right) & x \in [0, \frac{1}{4}) \\ 2\left(1 - \frac{1}{\gamma}\right) & x \in [\frac{1}{4}, \frac{1}{2}) \\ 2 & x \in [\frac{1}{2}, 1). \end{cases}$$

See Figure 2. Then

$$\begin{aligned} |[b]_I| &\geq 2 \quad \text{for all } I \subseteq [0,1), \text{ and} \\ \|b\|_{\mathrm{BMO}} &\leq \frac{C}{\gamma}, \end{aligned}$$

so b satisfies the hypotheses of the theorem. Furthermore,

$$\begin{aligned} [b]_{[0,1/2)} &= 2 \\ \langle \psi_{[0,1/2)}, b\rangle &= \sqrt{2}\left(\int_0^{1/4} b - \int_{1/4}^{1/2} b\right) \\ &= \sqrt{2}\left(\frac{1}{2}\Big(1+\frac{1}{\gamma}\Big) - \frac{1}{2}\Big(1-\frac{1}{\gamma}\Big)\right) \\ &= \sqrt{2}\,\frac{1}{\gamma}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|f\|_{\mathrm{BMO}(b)} &= C\left(2 + \left|\frac{1}{\gamma} - \sqrt{2}\frac{1}{[b]_{[0,1/2)}}\langle \psi_{[0,1/2)}, b\rangle\right|^2\right)^{1/2} \\ &= C\left(2 + \left|\frac{1}{\gamma} - \frac{1}{\gamma}\right|^2\right)^{1/2} \\ &= C\sqrt{2}. \end{aligned}$$

We may take C to be $\sqrt{2}$ as the supremum is attained on $[0, \frac{1}{2})$. Therefore

$$\|f\|_{\mathrm{BMO}(b)} = 2.$$

This establishes the sharpness of the power on γ. $\square$

We apply this result to the particular case to get an associated dyadic global b-output $T(b)$ theorem.

Theorem 3.2 (Dyadic global b-output $T(b)$ theorem)

Let b be a function on $\mathbb{R}$ such that there exists a $C > 0$ and a $0 < \gamma \ll 1$ with $\frac{1}{C} \leq |[b]_I|$ for all $I \in \Delta$ and $\|b\|_{\mathrm{BMO}} \leq \gamma^{-1}$. Let T be a dyadic singular integral operator such that

$$\begin{aligned} |\langle T\psi_I, \psi_I\rangle| &\leq 1 \\ \|T(1)\|_{\mathrm{BMO}} &\leq G_1 \\ \|T^*(1)\|_{\mathrm{BMO}(b)} &\leq G_2. \end{aligned}$$

Then T is bounded on L^2 and

$$\|Tf\|_2 \leq C\gamma^{-1}\|f\|_2.$$

Furthermore, the power on γ is sharp.

The constant C in the conclusion is dependent on G_1 and G_2, but we note that C is independent of γ.

4. Modified definitions for the local case

In the previous section, we develop a global b-output $T(b)$ theorem in which we have $|[b]_I| \geq 1$ for all $I \in \Delta$. Because the mean of b is large on all dyadic intervals, ρ_I^b is defined on all intervals and we can define $\|f\|_{\mathrm{BMO}(b)}$.

In order to develop a corresponding local theorem, we need to make some modifications to the definition of this norm.

For each $I \in \Delta$ suppose we have a function b_I, with $\mathrm{supp}(b_I) \subseteq I$, $|[b_I]_I| \geq 1$, and $\|b_I\|_{L^2(I)} \leq \frac{|I|^{1/2}}{\gamma}$. Though the absolute value of the mean of b_I on I is greater than one, it is possible that the mean of b_I could be small, even zero, when taken over a subinterval J of I. If we have J such that $[b_I]_J = 0$, then $\psi_J^{b_I}$ is not defined, nor is $\rho_{\tilde{J}}^{b_I}$, where $\tilde{J}$ is the parent of J. Therefore, to form a local definition of the $\mathrm{BMO}((b_I)_{I\in\Delta})$ seminorm, for each interval I, we exclude those intervals J where $\rho_J^{b_I}$ is not defined. So that we can define $\rho_I^{b_I}$, we must also require that for all I,

$$|[b_I]_{I_l}| \geq 1 \quad \text{and} \quad |[b_I]_{I_r}| \geq 1.$$

Fix an interval $I \in \Delta$ and a function b_I as above. We decompose the complete tree of intervals contained in I into three disjoint collections of intervals. This allows us to run a more explicit stopping-time argument. We let $\mathbb{I}_I$ be the tree of subintervals where b_I is "good" in the sense that its mean is large, it's L^2-norm is small, and the means of b_I on the children of intervals in $\mathbb{I}_I$ are large. Both $\psi_J^{b_I}$ and $\rho_J^{b_I}$ are defined for $J \in \mathbb{I}_I$. We quantify this below, with the following caveat: in the decomposition of I in the stopping-time argument, we "stop" (i.e. label as "bad") those intervals where the mean is small or the L^2-norm is large. We also label as "bad" all future generations of such intervals, *even if* the mean is large after the initial stop.

Definition 4.1 Let $\mathbb{I}_I$ be the subset of dyadic intervals contained in I such that for any interval $J \in \mathbb{I}_I$, the following hold:

$$|[b_I]_J| \geq \frac{1}{C}, \quad |[b_I]_{J_r}| \geq \frac{1}{C}, \quad |[b_I]_{J_l}| \geq \frac{1}{C}$$

$$\|b_I\|_{L^2(J)} \leq \frac{C|J|^{1/2}}{\gamma^2}$$

and such that J is not contained in any interval where the above four properties do not hold.

We know that $I \in \mathbb{I}_I$ because of our additional assumption that the means of b_I on both children of I are large. Note that if $|[b_I]_J| > 1$ for all $J \subseteq I$ as in the global theorem, then $\mathbb{I}_I = \mathrm{Tree}(I)$.

Definition 4.2 For $f \in L^2_{\mathrm{loc}}$, and b_I and $\mathbb{I}_I$ as above, let

$$\Pi_I f = \sum_{K\in\mathbb{I}_I} \langle f, \psi_K \rangle \psi_K.$$

We can now redefine the local dyadic BMO(b) norm as follows:

DEFINITION 4.3 Let $f \in L^2_{\mathrm{loc}}$ and let $\{b_I\}_{I\in\Delta}$ be a collection of functions such that for each I, b_I is supported on I, $|[b_I]_I| \geq 1, |[b_I]_{I_l}| \geq 1, |[b_I]_{I_r}| \geq 1$, and $\|b_I\|_{L^2(I)} \leq \frac{|I|^{1/2}}{\gamma}$. We define

$$\|f\|_{\mathrm{BMO}((b_I)_{I\in\Delta})} = \sup_I \frac{1}{|I|^{1/2}} \left(\sum_{J\in\mathbb{I}_I} \left|\langle \Pi_I f, \rho_J^{b_I}\rangle\right|^2 \right)^{1/2}.$$

We prove that $\|\cdot\|_{\mathrm{BMO}((b_I)_{I\in\Delta})}$ is a semi-norm on L^2_{loc}, and a norm on the set of equivalence classes of functions where $f \sim g$ when $f - g \equiv c$, a constant. It is clear that for any f, $\|f\|_{\mathrm{BMO}((b_I)_{I\in\Delta})} \geq 0$, and that for a constant C, $\|Cf\|_{\mathrm{BMO}((b_I)_{I\in\Delta})} = |C|\|f\|_{\mathrm{BMO}((b_I)_{I\in\Delta})}$.

We use the definition of $\Pi_I f$ and that for $J \subsetneq K$ and $J \cap K = \emptyset$, $\langle \rho_J^{b_I}, \psi_K\rangle = 0$ to get

$$\|f\|_{\mathrm{BMO}((b_I)_{I\in\Delta})} = \sup_I \frac{1}{|I|^{1/2}} \left(\sum_{J\in\mathbb{I}_I} \Big| \sum_{K\in\mathbb{I}_I} \langle f, \psi_K\rangle\langle \rho_J^{b_I}, \psi_K\rangle \Big|^2 \right)^{1/2}$$

Therefore for f and g in L^2_{loc}

$$\begin{aligned}
\|f+g\|_{\mathrm{BMO}((b_I)_{I\in\Delta})} &= \sup_I \frac{1}{|I|^{1/2}} \left(\sum_{J\in\mathbb{I}_I} \Big| \sum_{\substack{K\subseteq J\\ K\in\mathbb{I}_I}} \langle f+g, \psi_K\rangle\langle \rho_J^{b_I}, \psi_K\rangle \Big|^2 \right)^{1/2} \\
&= \sup_I \frac{1}{|I|^{1/2}} \left(\sum_{J\in\mathbb{I}_I} \Big| \sum_{\substack{K\subseteq J\\ K\in\mathbb{I}_I}} \langle f, \psi_K\rangle\langle \rho_J^{b_I}, \psi_K\rangle + \langle g, \psi_K\rangle\langle \rho_J^{b_I}, \psi_K\rangle \Big|^2 \right)^{1/2} \\
&\leq \sup_I \frac{1}{|I|^{1/2}} \left(\sum_{J\in\mathbb{I}_I} \Big| \sum_{\substack{K\subseteq J\\ K\in\mathbb{I}_I}} \langle f, \psi_K\rangle\langle \rho_J^{b_I}, \psi_K\rangle \Big|^2 \right)^{1/2} \\
&\quad + \sup_I \frac{1}{|I|^{1/2}} \left(\sum_{J\in\mathbb{I}_I} \Big| \sum_{\substack{K\subseteq J\\ K\in\mathbb{I}_I}} \langle g, \psi_K\rangle\langle \rho_J^{b_I}, \psi_K\rangle \Big|^2 \right)^{1/2} \\
&= \|f\|_{\mathrm{BMO}((b_I)_{I\in\Delta})} + \|g\|_{\mathrm{BMO}((b_I)_{I\in\Delta})}.
\end{aligned}$$

Furthermore, functions which are equal modulo an additive constant have the same value under the above seminorm. As the constant function $c \sim 0$, the zero function, and as $\|0\|_{\mathrm{BMO}((b_I)_{I\in\Delta})} = 0$, we have a norm on equivalence classes.

5. Comparing $\mathrm{BMO}(b_I)$ and BMO for a set of locally defined $\{b_I\}$

In order to prove the local theorem, we must use a stopping-time argument which relies on the following two lemmas from [2]. The first tells us that to bound the maximal size of a function on a tree $\mathcal{T}$, it suffices to do so outside a set of intervals $\mathbb{J}$ where

$$\sum_{I\in\mathbb{J}} |I| \leq (1-\eta) I_{\mathcal{T}}$$

for some $\eta > 0$. The second lemma makes a claim of the same flavor about functions of large mean. Specifically, if a function b has large mean on the top of a tree, then $|[b]_J| < \frac{1}{4}$ on a non-trivial set of intervals.

Lemma 5.1

Suppose $\mathbb{I} \subseteq \Delta$ is a collection of dyadic intervals and $a : \mathbb{I} \to \mathbb{R}^+$ is a function. Suppose also that we have constants $A > 0$ and $0 < \eta < 1$ such that for every tree $\mathcal{T}$ we have

$$\frac{1}{|I_{\mathcal{T}} \setminus \bigcup_{\mathcal{T}' \in \mathbb{T}_{\mathcal{T}}} I_{\mathcal{T}'}|} \sum_{I \in \mathcal{T} \setminus \bigcup_{\mathcal{T}' \in \mathbb{T}_{\mathcal{T}}} \mathcal{T}'} a(I) \leq A$$

for some collection $\mathbb{T}_{\mathcal{T}}$ of trees in $\mathcal{T}$ whose tops cover at most $(1-\eta)$ of $I_{\mathcal{T}}$, i.e.

$$\sum_{\mathcal{T}' \in \mathbb{T}_{\mathcal{T}}} |I_{\mathcal{T}'}| \leq (1-\eta)|I_{\mathcal{T}}|.$$

Then we have

$$\mu = \sup_{\mathcal{T} \in \mathbb{I}} \frac{1}{|I_{\mathcal{T}}|} \sum_{I \in \mathcal{T}} a(I) \leq \frac{A}{\eta}.$$

Note that while we allow $\mathbb{T}_{\mathcal{T}}$ to depend on $\mathcal{T}$, we will use the shorthand notation $\mathbb{T}_{\mathcal{T}} = \mathbb{T}$.

In the application, we will take $\eta = \gamma^2$.

Lemma 5.2

Let $\mathcal{T}_0 \subseteq \Delta$ be a convex tree and let b be a function such that

$$\left\| \sum_{I \in \mathcal{T}_0} \langle b, \psi_I \rangle \psi_I \right\|_2 \leq \frac{C_0 |I_{\mathcal{T}_0}|^{1/2}}{\gamma}$$

for some $C_0 > 0$, and $0 < \gamma \ll 1$ and such that $|[b]_{I_{\mathcal{T}_0}}| > 1$. Then there exists a family $\mathbb{T}$ of disjoint convex subtrees of $\mathcal{T}_0$ such that $|[b]_I| > \frac{1}{4}$ for all $I \in \mathcal{T}_0 \setminus \bigcup_{\mathcal{T} \in \mathbb{T}} \mathcal{T}$ and the tops of the trees in $\mathbb{T}$ cover at most $(1 - \frac{\gamma^2}{2C_0^2})$ of $I_{\mathcal{T}_0}$. Furthermore $|[b]_{I_{\mathcal{T}}}| \leq \frac{1}{4}$ for all $\mathcal{T} \in \mathbb{T}$.

Using the above lemma and definitions from the previous section, we state and prove the following theorem while tracking the powers of γ:

Theorem 5.3 (Comparison of $\mathrm{BMO}(b_I)$ and BMO for a set of locally defined $\{b_I\}$)

Let $f \in L^2_{\mathrm{loc}}$ such that for all $J \in \Delta$, and for a fixed $G \in \mathbb{Z}^+$,

$$|\langle f, \psi_J \rangle| \leq G|J|^{1/2} < \infty.$$

Let $0 < \gamma \ll 1$. For a fixed interval I, let $\{b_I\}_{I \in \Delta}$ be a collection of functions such *that for each I, b_I satisfies*

$$\begin{gathered}\operatorname{supp}(b_I) \subseteq I\\ |[b_I]_I| \geq \frac{1}{C},\ |[b_I]_{I_l}| \geq \frac{1}{C},\ |[b_I]_{I_r}| \geq \frac{1}{C}\\ \|b_I\|_{L^2(I)} \leq \frac{|I|^{1/2}}{\gamma}.\end{gathered}$$

Then

$$\|f\|_{\mathrm{BMO}} \leq C\gamma^{-5}\|f\|_{\mathrm{BMO}((b_I)_{I\in\Delta})}.$$

Proof. Fix an interval $I \in \Delta$. We begin by decomposing the interval I. So that we may apply Lemma 5.2, let $C \geq 4$. This constant C is also dependent on C_0 from Lemma 5.2.

Let $\hat{I}$ be the set of all intervals $J \subseteq I$ such that J is the union of intervals K on which

$$|[b_I]_K| < \frac{1}{C} \quad \text{or} \quad \frac{1}{|K|}\int_K |b_I|^2 > \frac{2C}{\gamma^4}.$$

Let $\hat{I}_{\max}$ be those intervals in $\hat{I}$ which are maximal with respect to inclusion. By construction, $\hat{I}_{\max}$ is sibling-free (i.e. if I_l is in $\hat{I}_{\max}$ then I_r cannot be, and vice versa), and there is a $C > 0$ such that $\hat{I}_{\max}$ is a $(1 - \frac{1}{C}\gamma^2)$-cover of I. That is,

$$\sum_{J\in\hat{I}_{\max}} |J| \leq \Big(1 - \frac{1}{C}\gamma^2\Big)|I|.$$

To prove this, let S be the set of intervals in $\hat{I}_{\max}$ for which $\frac{1}{|K|}\int_K |b_I|^2 > \frac{2C}{\gamma^4}$. Then

$$\begin{aligned}\sum_{K\in S} |K| &\leq \sum_{K\in S} \frac{\gamma^4}{2C}\int_K |b_I|^2\\ &\leq \frac{\gamma^4}{2C}\|b_I\|^2_{L^2(K)}\\ &\leq \frac{\gamma^4}{2C}\cdot\frac{|I|}{\gamma^2}\\ &= \frac{\gamma^2}{2C}|I|.\end{aligned}$$

We know that the intervals on which the mean is small is a $(1 - \frac{1}{C}\gamma^2)$ cover of I as well, so together, they form a $(1 - \frac{1}{C}\gamma^2)$ cover of I.

Let $\mathbb{T}_I$ be the collection of complete trees whose tops are the intervals in $\hat{I}_{\max}$. That is

$$\mathbb{T}_I = \{\mathrm{Tree}(J) : J \in \hat{I}_{\max}\}.$$

We will say (abusing notation) that $K \in \mathbb{T}_I$ if $K \in \mathrm{Tree}(J)$ for some $J \in \hat{I}_{\max}$.

Let $\mathbb{I}_{\text{buffer}}$ be those intervals $J \in \mathrm{Tree}(I) \setminus \mathbb{T}_I$ such that exactly one child of J is in $\hat{I}_{\max}$, and therefore the top of a tree in $\mathbb{T}_I$. Note that both of the children of

$J \in \text{Tree}(I) \setminus \mathbb{T}_I$ cannot be in $\mathbb{T}_I$ by construction. Because of the disjointness of the intervals in $\hat{I}_{\max}$, we have that

$$\sum_{J \in \mathbb{I}_{\text{buffer}}} |J| \leq 2|I|.$$

Let

$$\mathbb{I}_I = (\text{Tree}(I) \setminus \mathbb{T}_I) \setminus \mathbb{I}_{\text{buffer}},$$

so that

$$\text{Tree}(I) = \mathbb{I}_I \cup \mathbb{I}_{\text{buffer}} \cup \mathbb{T}_I.$$

Let $g \in \text{span}\{\rho_J^{b_I} : J \in \mathbb{I}_I\}$. Then $[g]_I = 0$ and by the representation formula, we can write

$$g = \sum_{J \in \mathbb{I}_I} \langle g, \psi_J^{b_I} \rangle \rho_J^{b_I}.$$

Recall that

$$\Pi_I f = \sum_{K \in \mathbb{I}_I} \langle f, \psi_K \rangle \psi_K,$$

the projection of f onto the space of functions which are constant on dyadic intervals in $\mathbb{I}_{\text{buffer}}$, and in particular on intervals in $\hat{I}_{\max}$.

We first consider

$$\begin{aligned} |\langle \Pi_I f, g \rangle| &= \Big| \sum_{J \in \mathbb{I}_I} \langle g, \psi_J^{b_I} \rangle \langle \Pi_I f, \rho_J^{b_I} \rangle \Big| \\ &\leq \left(\sum_{J \in \mathbb{I}_I} |\langle g, \psi_J^{b_I} \rangle|^2 \right)^{1/2} \left(\sum_{J \in \mathbb{I}_I} |\langle \Pi_I f, \rho_J^{b_I} \rangle|^2 \right)^{1/2}, \end{aligned}$$

by Cauchy-Schwarz. By definition, we have

$$\frac{1}{|I|^{1/2}} \left(\sum_{J \in \mathbb{I}_I} |\langle \Pi_I f, \rho_J^{b_I} \rangle|^2 \right)^{1/2} \leq \|f\|_{\text{BMO}((b_I)_{I \in \Delta})}.$$

On the other hand, we can bound the first sum as we did in the global case using the definition of $\psi_I^{b_I}$ and the Carleson Embedding Theorem [14]. Recall that for all $J \in \mathbb{I}_I$, we know $\|b_I\|_{L^2(J)} \leq \frac{C|J|^{1/2}}{\gamma^2}$.

$$\begin{aligned} \left(\sum_{J \in \mathbb{I}_I} |\langle g, \psi_J^{b_I} \rangle|^2 \right)^{1/2} &\leq \left(\sum_{J \in \mathbb{I}_I} |\langle g, \psi_J \rangle|^2 \right)^{1/2} + \left(\sum_{J \in \mathbb{I}_I} \frac{|\langle b_I, \psi_J \rangle|^2}{|[b_I]_J|^2} |[g]_J|^2 \right)^{1/2} \\ &\leq \|g\|_{L^2(I)} + \left(\sum_{J \in \mathbb{I}_I} \frac{|\langle b_I, \psi_J \rangle|^2}{|[b_I]_J|^2} |[g]_J|^2 \right)^{1/2} \\ &\leq \|g\|_{L^2(I)} + C\|g\|_{L^2(I)} \sup_{\mathcal{T} \subseteq \mathbb{I}_I} \frac{1}{|I_{\mathcal{T}}|^{1/2}} \left(\sum_{J \in \mathcal{T}} |\langle b_I, \psi_J \rangle|^2 \right)^{1/2} \\ &\leq C\|g\|_{L^2(I)} \gamma^{-2}. \end{aligned}$$

Putting this together, we get that

$$\begin{aligned}|\langle \Pi_I f, g\rangle| &\le \left(\sum_{J\in\mathbb{I}_I} |\langle g, \psi_J^{b_I}\rangle|^2\right)^{1/2} \left(\sum_{J\in\mathbb{I}_I} \left|\langle \Pi_I f, \rho_J^{b_I}\rangle\right|^2\right)^{1/2} \\ &\le \left(\|g\|_{L^2(I)} + C\|g\|_{L^2(I)}\cdot \frac{1}{\gamma^2}\right)\|f\|_{\mathrm{BMO}((b_I)_{I\in\Delta})}|I|^{1/2} \\ &= \|g\|_{L^2(I)}\Big(1+\frac{C}{\gamma^2}\Big)|I|^{1/2}\|f\|_{\mathrm{BMO}((b_I)_{I\in\Delta})}.\end{aligned}$$

We need to bound

$$\sup_{\substack{h\in\mathrm{span}\{\psi_J : J\in\mathbb{I}_I\}\\ \|h\|_2=1}} |\langle \Pi_I f, h\rangle|,$$

and we have found that

$$\sup_{\substack{g\in\mathrm{span}\{\rho_J^{b_I} : J\in\mathbb{I}_I\}\\ \|g\|_2=1}} |\langle \Pi_I f, g\rangle| \le \Big(1+\frac{C}{\gamma^2}\Big)|I|^{1/2}\|f\|_{\mathrm{BMO}((b_I)_{I\in\Delta})}.$$

We therefore need to compare the first supremum to the second to find a bound over functions in the span of the orthonormal basis of Haar functions.

We partition I into intervals K which are the maximal subintervals of I on which $\Pi_I f$ is forced to be constant. Since

$$\Pi_I f = \sum_{J\in\mathbb{I}_I} \langle f, \psi_J\rangle \psi_J,$$

this means that $K\in\mathbb{I}_{\mathrm{buffer}}$. If such intervals K do not partition because their disjoint union is not all of I we may add in those intervals of smallest scale (i.e. $|K| = 2^{-M}$) and fill in with these small intervals. Call this partition $\mathcal{P}$. So

$$\bigcup_{K\in\mathcal{P}} K = I.$$

Let h_0 be the function on which

$$\sup_{\substack{h\in\mathrm{span}\{\psi_J : J\in\mathbb{I}_I\}\\ \|h\|_2=1}} |\langle \Pi_I f, h\rangle|$$

is attained. There exists $g_0\in\mathrm{span}\{\rho_J^{b_I} : J\in\mathbb{I}_I\}$ such that $[g_0]_K = [h_0]_K$ for all $K\in\mathcal{P}$, and therefore

$$|\langle \Pi_I f, g_0\rangle| = |\langle \Pi_I f, h_0\rangle|.$$

Recall that

$$g_0 = \sum_{J\in\mathbb{I}_I} \langle g_0, \psi_J^{b_I}\rangle \rho_J^{b_I}.$$

For $K\in\mathcal{P}\cap\mathbb{I}_{\mathrm{buffer}}$, we know that

$$g_0|_K = \alpha b_I|_K$$

for some constant α. Therefore

$$\|g_0\|_{L^2(K)}^2 = \alpha^2 \int_K b_I^2 \leq \frac{C|K|}{\gamma^4}.$$

If $K \in \mathcal{P}$ is of the smallest scale, so $|K| = 2^{-M}$, and is used to "fill in the gaps" of the partition, then we know that on no ancestor of K is $\Pi_I f$ forced to be constant. Therefore no ancestor of K is in $\mathbb{I}_{\text{buffer}}$, so such a K is in $\mathbb{I}_I$.

By the stopping-time argument, we know that $|[b_I]_K| \geq \frac{1}{C}$ for all $K \in \mathcal{P}$. Therefore $|[g_0]_K| \geq \frac{1}{C}$ for all $K \in \mathcal{P}$. This gives us the following:

$$\begin{aligned}
\|g_0\|_2^2 &= \sum_{K \in \mathcal{P}} \int_K g_0^2 \\
&\leq C\gamma^{-4} \sum_{K \in \mathcal{P}} |K| [g_0]_K^2 \\
&= C\gamma^{-4} \sum_{K \in \mathcal{P}} |K| [h_0]_K^2 \\
&\leq C\gamma^{-4} \|h_0\|_{L^2(I)}^2 \\
&= C\gamma^{-4}.
\end{aligned}$$

Recall that we know that

$$|\langle \Pi_I f, g_0 \rangle| \leq C \|g_0\|_2 |I|^{1/2} \|f\|_{\text{BMO}((b_I)_{I\in\Delta})} \gamma^{-2}.$$

Then we know that

$$\begin{aligned}
\|\Pi_I f\|_{L^2(\mathbb{I}_I)}^2 &= \sup_{\substack{h \in \text{span}\{\psi_J : J \in \mathbb{I}_I\} \\ \|h\|_2 = 1}} |\langle \Pi_I f, h \rangle|^2 \\
&= |\langle \Pi_I f, h_0 \rangle|^2 \\
&= |\langle \Pi_I f, g_0 \rangle|^2 \\
&\leq C \|g_0\|_2^2 |I| \|f\|_{\text{BMO}((b_I)_{I\in\Delta})}^2 \gamma^{-4} \\
&\leq C\gamma^{-4} \|h_0\|_2^2 |I| \|f\|_{\text{BMO}((b_I)_{I\in\Delta})}^2 \gamma^{-4} \\
&= C\gamma^{-8} |I| \|f\|_{\text{BMO}((b_I)_{I\in\Delta})}^2.
\end{aligned}$$

By the assumption that $|\langle f, \psi_J \rangle| \leq M|J|^{1/2}$ for all J, we know that

$$\sum_{J \in \mathbb{I}_{\text{buffer}}} |\langle f, \psi_J \rangle|^2 \leq G^2 \sum_{J \in \mathbb{I}_{\text{buffer}}} |J| \leq 2G^2 |I|.$$

Now, we may use Lemma 5.1 to get the result we want. We know that

$$\begin{aligned}
\frac{1}{|I|}\sum_{J\in \mathrm{Tree}(I)\setminus \mathbb{T}_I} |\langle f,\psi_J\rangle|^2 &= \frac{1}{|I|}\left(\sum_{J\in \mathbb{I}_I}|\langle f,\psi_J\rangle|^2 + \sum_{J\in \mathbb{I}_{\mathrm{buffer}}}|\langle f,\psi_J\rangle|^2\right)\\
&\le \frac{1}{|I|}\left(\sum_{J\in \mathbb{I}_I}|\langle f,\psi_J\rangle|^2 + 2G^2|I|\right)\\
&\le \frac{1}{|I|}\left(C\gamma^{-8}|I|\|f\|^2_{\mathrm{BMO}((b_I)_{I\in\Delta})} + 2G^2|I|\right)\\
&\le (C\gamma^{-8}\|f\|^2_{\mathrm{BMO}((b_I)_{I\in\Delta})} + 2G^2)\\
&= C\gamma^{-8}\|f\|^2_{\mathrm{BMO}((b_I)_{I\in\Delta})} + 2\gamma^{-2}G^2.
\end{aligned}$$

Then, applying Lemma 5.2, we get that

$$\sup_{J\subseteq I}\frac{1}{|J|}\sum_{K\subseteq J}|\langle f,\psi_K\rangle|^2 \le C\gamma^{-10}\|f\|^2_{\mathrm{BMO}((b_I)_{I\in\Delta})} + 2\gamma^{-4}G^2$$

and therefore

$$\|f\|_{\mathrm{BMO}} \le C\gamma^{-5}\|f\|_{\mathrm{BMO}((b_I)_{I\in\Delta})}.$$

Of course, the constant C above is dependent on G, but not on γ.

The local version of the b-output $T(b)$ theorem also follows as an immediate corollary if we take $f = T^*(1)$. We will need to control $\|T^*(1)\|_{\mathrm{BMO}}$, but given that we control $\|T^*(1)\|_{\mathrm{BMO}((b_I)_{I\in\Delta}}$, we can use the theorem above to compare the two norms.

Theorem 5.4 (Dyadic local b-output $T(b)$ theorem)

Let T be a dyadic singular integral operator such that for all $J\in\Delta$, and for a fixed $G\in\mathbb{Z}^+$,

$$|\langle T^*(1),\psi_J\rangle| \le G|J|^{1/2} < \infty,$$

and such that

$$\begin{aligned}
\|T(1)\|_{\mathrm{BMO}} &\le G_1\\
\|T^*(1)\|_{\mathrm{BMO}((b_I)_{I\in\Delta}} &\le G_2.
\end{aligned}$$

Let $0<\gamma\ll 1$. Suppose that for every interval $I\in\Delta$, there exists a function b_I satisfying

$$\begin{aligned}
\mathrm{supp}(b_I) &\subseteq I\\
|[b_I]_I| &\ge \frac{1}{C}\\
|[b_I]_{I_l}| &\ge \frac{1}{C}\\
|[b_I]_{I_r}| &\ge \frac{1}{C}\\
\|b_I\|_{L^2(I)} &\le \frac{|I|^{1/2}}{\gamma}.
\end{aligned}$$

Then T is bounded on L^2 and in particular

$$\|Tf\|_2 \leq C\gamma^{-5}\|f\|_2.$$

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