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Involutions on surfaces with $p_g = q = 1$

Carlos Rito

Departamento de Matemática, Ed. de Ciências Florestais,
Quinta de Prados, Apartado 1013
5001-801 Vila Real - Portugal
E-mail: crito@utad.pt

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Abstract

In this paper some numerical restrictions for surfaces with an involution are obtained. These formulas are used to study surfaces of general type S with $p_g = q = 1$ having an involution i such that S/i is a non-ruled surface and such that the bicanonical map of S is not composed with i. A complete list of possibilities is given and several new examples are constructed, as bidouble covers of surfaces. In particular the first example of a minimal surface of general type with $p_g = q = 1$ and $K^2 = 7$ having birational bicanonical map is obtained.

1. Introduction

Several authors have studied surfaces of general type with $p_g = q = 1$ ([6, 7, 8, 9, 10, 17, 18, 16, 15]), but these surfaces are still not completely understood.

In [19], the author gives several new examples of double planes of general type with $p_g = q = 1$ having bicanonical map ϕ_2 composed with the corresponding involution. The case S/i non-ruled and ϕ_2 composed with i is considered in [20].

In this paper we study the case ϕ_2 not composed with i. More precisely, we consider surfaces of general type S with $p_g = q = 1$ having an involution i such that S/i is a non-ruled surface and ϕ_2 is not composed with i. A list of possibilities is given and new examples are obtained for each value of the birational invariants of S/i (only the existence of the case $\mathrm{Kod}(S/i) = 2$, $\chi(S/i) = 1$ and $q(S/i) = 0$ remains an open problem).

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The paper is organized as follows. In Section 2.2 we obtain formulas which, for a surface S with an involution i, relate the invariants of S and S/i with the branch locus of the cover $S \to S/i$, its singularities and the number of nodes of S/i. Section 2.3 contains a description of the action of the involution i on the Albanese fibration of S. In Section 3 we apply the numerical formulas of Section 2.2 to the case $p_g = q = 1$, obtaining a list of possibilities (Theorems 7, 8 and 9). Results of Miyaoka and Sakai on the maximal number of disjoint smooth rational or elliptic curves on a surface are also used here. Finally Section 4 contains the construction of examples, as bidouble covers of surfaces: the surfaces constructed in Sections 4.1, 4.2 and 4.3 are Du Val double planes (*cf.* [19]) which have other interesting involutions; Section 4.4 contains the construction of a surface with $K^2 = 4$, Albanese fibration of genus $g = 2$ and $\deg(\phi_2) = 2$ (thus it is not the example in [7], for which ϕ_2 is composed with the three involutions associated to the bidouble cover); in Section 4.5 a new surface with $K^2 = 8$ is obtained (it is not a standard isotrivial fibration); Sections 4.6 and 4.7 contain the construction of new surfaces with $K^2 = 7, 6$ and $\deg(\phi_2) = 1$ (it is the first example with $p_g = q = 1$ and $K^2 = 7$ having birational bicanonical map); bidouble covers of irregular ruled surfaces give interesting examples in Sections 4.8 and 4.9.

Some branch curves for these bidouble cover examples are computed in Appendix A.2, using the *Computational Algebra System Magma* ([4]).

Notation and conventions

We work over the complex numbers; all varieties are assumed to be projective algebraic. For a projective smooth surface S, the *canonical class* is denoted by K, the *geometric genus* by $p_g := h^0(S, \mathcal{O}_S(K))$, the *irregularity* by $q := h^1(S, \mathcal{O}_S(K))$ and the *Euler characteristic* by $\chi = \chi(\mathcal{O}_S) = 1 + p_g - q$.

An $(-n)$-*curve* C on a surface is a curve isomorphic to $\mathbb{P}^1$ such that $C^2 = -n$. We say that a curve singularity is *negligible* if it is either a double point or a triple point which resolves to at most a double point after one blow-up. An $(m_1, m_2, \ldots)$-point, or point of order $(m_1, m_2, \ldots)$, is a point of multiplicity m_1, which resolves to a point of multiplicity m_2 after one blow-up, etc.

An *involution* of a surface S is an automorphism of S of order 2. We say that a map is *composed with an involution* i of S if it factors through the double cover $S \to S/i$.

The rest of the notation is standard in Algebraic Geometry.

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2. Results on involutions

2.1 General facts

Let S be a smooth minimal surface of general type with an involution i. Since S is minimal of general type, this involution is biregular. The fixed locus of i is the union of a smooth curve R'' (possibly empty) and of $t \geq 0$ isolated points $P_1, \dots, P_t$. Let S/i be the quotient of S by i and $p : S \to S/i$ be the projection onto the quotient. The surface S/i has nodes at the points $Q_i := p(P_i)$, $i = 1, \dots, t$, and is smooth elsewhere. If $R'' \neq \emptyset$, the image via p of R'' is a smooth curve B'' not containing the singular points Q_i, $i = 1, \dots, t$. Let now $h : V \to S$ be the blow-up of S at $P_1, \dots, P_t$ and set $R' = h^*(R'')$. The involution i induces a biregular involution $\widetilde{i}$ on V whose fixed locus is $R := R' + \sum_1^t h^{-1}(P_i)$. The quotient $W := V/\widetilde{i}$ is smooth and one has a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{h} & S \\ \pi \downarrow & & \downarrow p \\ W & \xrightarrow{g} & S/i \end{array}$$

where $\pi : V \to W$ is the projection onto the quotient and $g : W \to S/i$ is the minimal desingularization map. Notice that

$$A_i := g^{-1}(Q_i), \quad i = 1, \dots, t,$$

are (-2)-curves and $\pi^*(A_i) = 2 \cdot h^{-1}(P_i)$.

Set $B' := g^*(B'')$. Since π is a double cover with branch locus $B' + \sum_1^t A_i$, it is determined by a line bundle L on W such that

$$2L \equiv B := B' + \sum_1^t A_i.$$

It is well known that (*cf.* [1, Chapter V, Section 22]):

$$\begin{aligned} p_g(S) = p_g(V) &= p_g(W) + h^0\big(W, \mathcal{O}_W(K_W + L)\big), \\ q(S) = q(V) &= q(W) + h^1\big(W, \mathcal{O}_W(K_W + L)\big) \end{aligned} \tag{1}$$

and

$$\begin{aligned} K_S^2 - t = K_V^2 &= 2(K_W + L)^2, \\ \chi(\mathcal{O}_S) = \chi(\mathcal{O}_V) &= 2\chi(\mathcal{O}_W) + \tfrac{1}{2}L(K_W + L). \end{aligned} \tag{2}$$

Denote by ϕ_2 the bicanonical map of S (given by $|2K|$). From the papers [11] and [5],

$$\phi_2 \text{ is composed with } i \text{ if and only if } h^0\big(W, \mathcal{O}_W(2K_W + L)\big) = 0.$$

2.2 Numerical restrictions

Let P be a minimal model of the resolution W of S/i and $\rho : W \to P$ be the corresponding projection. Denote by $\overline{B}$ the projection $\rho(B)$ and by δ the "projection" of L.

Remark 1 If $\overline{B}$ is singular, there are exceptional divisors E_i and numbers $r_i \in 2\mathbb{N}$ such that

$$\begin{aligned} E_i^2 &= -1, \\ K_W &\equiv \rho^*(K_P) + \sum E_i, \\ 2L \equiv B &= \rho^*(\overline{B}) - \sum r_i E_i \equiv \rho^*(2\delta) - \sum r_i E_i. \end{aligned}$$

Proposition 2

With the previous notation, if S is a surface of general type then:

a) $\chi(\mathcal{O}_P) - \chi(\mathcal{O}_S) = K_P(K_P + \delta) + \frac{1}{2}\sum(r_i - 2) - h^0\big(W, \mathcal{O}_W(2K_W + L)\big)$;

b) $\delta^2 = -2\chi(\mathcal{O}_P) - 2K_P^2 - 3K_P\delta + \frac{1}{4}\sum(r_i - 2)(r_i - 4) + 2h^0\big(W, \mathcal{O}_W(2K_W + L)\big)$.

Proposition 3

Let t be the number of nodes of S/i. One has:

a) $t = K_S^2 + 6\chi(\mathcal{O}_W) - 2\chi(\mathcal{O}_S) - 2h^0\big(W, \mathcal{O}_W(2K_W + L)\big)$;

b) $t = K_S R'' + 8\chi(\mathcal{O}_W) - 4\chi(\mathcal{O}_S) \geq 8\chi(\mathcal{O}_W) - 4\chi(\mathcal{O}_S)$;

c) $K_S^2 \geq 2\chi(\mathcal{O}_W) - 2\chi(\mathcal{O}_S) + 2h^0\big(W, \mathcal{O}_W(2K_W + L)\big)$.

Proposition 4

With the above notation:

a) $h^0\big(W, \mathcal{O}_W(2K_W + L)\big) \leq \frac{1}{3}K_W^2 - \chi(\mathcal{O}_W) + \frac{11}{3}\chi(\mathcal{O}_S) + \frac{1}{27}K_S^2$;

b) $h^0\big(W, \mathcal{O}_W(2K_W + L)\big) \leq \frac{1}{2}K_W^2 + 5\chi(\mathcal{O}_S) + 2q(S) - 3\chi(\mathcal{O}_W) - 2q(W)$.

Proof of Proposition 2: (*cf.* [11])

a) From the Kawamata-Viehweg's vanishing theorem (see *e.g.* [12, Corollary 5.12, c)]), one has

$$h^i(W, \mathcal{O}_W(2K_W + L)) = 0, \ i = 1, 2.$$

The Riemann-Roch theorem implies

$$\chi\big(\mathcal{O}_W(2K_W + L)\big) = \chi(\mathcal{O}_W) + \frac{1}{2}L(K_W + L) + K_W(K_W + L),$$

thus, using (2),

$$h^0\big(W, \mathcal{O}_W(2K_W + L)\big) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_W) + K_W(K_W + L). \quad (3)$$

With the notation of Remark 1, we can write

$$\begin{aligned}\chi(\mathcal{O}_P) - \chi(\mathcal{O}_S) &= \frac{1}{2}K_W(2K_W + 2L) - h^0(W, \mathcal{O}_W(2K_W + L)) \\ &= \frac{1}{2}\left(\rho^*(K_P) + \sum E_i\right)\left(2\rho^*(K_P + \delta) + \sum(2 - r_i)E_i\right) \\ &\quad - h^0(W, \mathcal{O}_W(2K_W + L)) \\ &= K_P(K_P + \delta) + \frac{1}{2}\sum(r_i - 2) - h^0(W, \mathcal{O}_W(2K_W + L)).\end{aligned}$$

b) From the proof of a),

$$h^0\big(W, \mathcal{O}_W(2K_W + L)\big) = \chi\big(\mathcal{O}_W(2K_W + L)\big) = \chi(\mathcal{O}_W) + \frac{1}{2}(2K_W + L)(K_W + L).$$

Using Remark 1 this means

$$\begin{aligned}h^0(W, \mathcal{O}_W(2K_W + L)) &= \chi(\mathcal{O}_P) \\ &\quad + \frac{1}{2}\left(\rho^*(2K_P + \delta) + \frac{1}{2}\sum(4 - r_i)E_i\right) \\ &\quad \times \left(\rho^*(K_P + \delta) + \frac{1}{2}\sum(2 - r_i)E_i\right) \\ &= \chi(\mathcal{O}_P) + K_P^2 + \frac{3}{2}K_P\delta + \frac{1}{2}\delta^2 - \frac{1}{8}\sum(r_i - 2)(r_i - 4).\end{aligned}$$

□

Proof of Proposition 3:

a) From formulas (2) and (3),

$$\begin{aligned}t &= K_S^2 - 2K_W(K_W + L) - 2L(K_W + L) \\ &= K_S^2 + 2\chi(\mathcal{O}_S) - 2\chi(\mathcal{O}_W) - 2h^0(W, \mathcal{O}_W(2K_W + L)) - 4\chi(\mathcal{O}_S) + 8\chi(\mathcal{O}_W).\end{aligned}$$

b) (This is also a consequence of the holomorphic fixed point formula.) From (2),

$$\begin{aligned}4\chi(\mathcal{O}_S) - 8\chi(\mathcal{O}_W) &= 2L(K_W + L) = \left(B' + \sum_1^t A_i\right)(K_W + L) \\ &= B'(K_W + L) - t = \frac{1}{2}\pi^*(B')\pi^*(K_W + L) - t = R''K_S - t.\end{aligned}$$

Since S is of general type, $K_S R'' \geq 0$, thus

$$t \geq 8\chi(\mathcal{O}_W) - 4\chi(\mathcal{O}_S).$$

c) This is immediate from a) and b). □

Proof of Proposition 4:

a) This inequality is given by the following three claims.

Claim 1:
$$1 - p_a(B') = 3\chi(\mathcal{O}_W) - 3\chi(\mathcal{O}_S) - K_S^2 - K_W^2 + 3h^0(W, \mathcal{O}_W(2K_W + L)).$$

Proof. Formulas (2) and (3) give

$$\begin{aligned} L^2 - K_W^2 = {} & [2\chi(\mathcal{O}_S) - 4\chi(\mathcal{O}_W) - LK_W] \\ & - [h^0(W, \mathcal{O}_W(2K_W + L)) - \chi(\mathcal{O}_S) + \chi(\mathcal{O}_W) - K_W L], \end{aligned}$$

thus

$$L^2 = K_W^2 + 3\chi(\mathcal{O}_S) - 5\chi\big(\mathcal{O}_W\big) - h^0\big(W, \mathcal{O}_W(2K_W + L)\big). \tag{4}$$

Now we perform a straightforward calculation using the adjunction formula, (2), Proposition 3, a) and (4):

$$\begin{aligned} 2p_a(B') - 2 &= K_W B' + B'^2 = K_W 2L + (2L)^2 + 2t \\ &= 2L(K_W + L) + 2t + 2L^2 = 2[2\chi(\mathcal{O}_S) - 4\chi(\mathcal{O}_W)] \\ &\quad + 2[K_S^2 + 6\chi(\mathcal{O}_W) - 2\chi(\mathcal{O}_S) - 2h^0(W, \mathcal{O}_W(2K_W + L))] \\ &\quad + 2[K_W^2 + 3\chi(\mathcal{O}_S) - 5\chi(\mathcal{O}_W) - h^0(W, \mathcal{O}_W(2K_W + L))] \\ &= 2K_S^2 + 2K_W^2 + 6\chi(\mathcal{O}_S) - 6\chi(\mathcal{O}_W) - 6h^0(W, \mathcal{O}_W(2K_W + L)). \end{aligned}$$ $\square$

Denote by τ the number of rational curves of B'.

Claim 2:
$$1 - p_a(B') \leq \tau.$$

Proof. Write

$$B' = \sum_1^{\tau} B_i' + \sum_{\tau+1}^{h} B_i'$$

as a decomposition of B' in (smooth) connected components such that B_i', $i \leq \tau$, are the rational ones. The adjunction formula gives

$$2p_a(B') - 2 = \sum_1^h \big(K_W B_i' + B_i'^2\big) = \sum_1^{\tau}(2g(B_i') - 2) + \sum_{\tau+1}^{h}(2g(B_i') - 2) \geq -2\tau.$$ $\square$

Claim 3:
$$\tau \leq 8\left(\chi(\mathcal{O}_S) - \frac{1}{9}K_S^2\right).$$

Proof. Since B' does not contain (-2)-curves and it is contained in the branch locus of the cover $\pi : V \to W$, then each rational curve in B' corresponds to a rational curve in S. Now the result follows from Proposition 5 below. $\square$

Therefore $1 - p_a(B') \leq 8\big(\chi(\mathcal{O}_S) - \frac{1}{9}K_S^2\big)$ and using Claim 1 we obtain the desired inequality.

b) Proposition 3, a) says that

$$K_V^2 = K_S^2 - t = 2\chi(\mathcal{O}_S) - 6\chi(\mathcal{O}_W) + 2h^0\big(W, \mathcal{O}_W(2K_W + L)\big).$$

The second Betti number b_2 of a surface X satisfies

$$b_2(X) = 12\chi(\mathcal{O}_X) - K_X^2 + 4q(X) - 2.$$

Therefore

$$b_2(V) = 10\chi(\mathcal{O}_V) + 6\chi(\mathcal{O}_W) + 4q(V) - 2 - 2h^0(W, \mathcal{O}_W(2K_W + L)).$$

Since $b_2(V) \geq b_2(W)$, one has the result. $\square$

Proposition 5 ([13, Proposition 2.1.1])

Let X be a minimal surface of non-negative Kodaira dimension. Then the number of disjoint smooth rational curves in X is bounded by

$$8\left(\chi(\mathcal{O}_X) - \frac{1}{9}K_X^2\right).$$

2.3 Surfaces with an involution and $q = 1$

Let S be a surface of general type with $q = 1$. Then the Albanese variety of S is an elliptic curve E and the Albanese map is a connected fibration (see *e.g.* [2] or [1]).

Suppose that S has an involution i. Then i preserves the Albanese fibration (because $q(S) = 1$) and so we have a commutative diagram

$$\begin{array}{ccccc} V & \xrightarrow{h} & S & \longrightarrow & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\ W & \longrightarrow & S/i & \longrightarrow & \Delta \end{array} \tag{5}$$

where Δ is a curve of genus ≤ 1. Denote by

$$f_A : W \to \Delta$$

the fibration induced by the Albanese fibration of S.

Recall that

$$\rho : W \to P$$

is the projection of W onto its minimal model P and

$$\overline{B} := \rho(B),$$

where $B := B' + \sum_1^t A_i \subset W$ is the branch locus of π. Let

$$\overline{B'} := \rho(B') \quad \text{and} \quad \overline{A_i} = \rho(A_i).$$

When $\overline{B}$ has only negligible singularities the map ρ contracts only exceptional curves contained in fibres of f_A. In fact otherwise there exists a (-1)-curve $J \subset W$ such that $JB = 2$ and $\pi^*(J)$ is transverse to the fibres of the (genus 1 base) Albanese fibration of S. This is impossible because $\pi^*(J)$ is a rational curve. Moreover ρ contracts no curve meeting $\sum A_i$, thus the singularities of $\overline{B}$ are exactly the singularities of $\overline{B'}$, *i.e.* $\overline{B'} \bigcap \sum \overline{A_i} = \emptyset$. We denote the image of f_A on P by $\overline{f_A}$.

If $\Delta \cong \mathbb{P}^1$ then the double cover $E \to \Delta$ is ramified over 4 points p_j of Δ, thus the branch locus $B' + \sum_1^t A_i$ is contained in 4 fibres

$$F_A^j := f_A^*(p_j),\ j = 1, ..., 4,$$

of the fibration f_A. Hence by Zariski's Lemma (see *e.g.* [1]) the irreducible components B'_i of B' satisfy $B_i'^2 \leq 0$. If $\overline{B}$ has only negligible singularities then also $\overline{B'}^2 \leq 0$. Since $\pi^*(F_A^j)$ has even multiplicity, each component of F_A^j which is not a component of the branch locus $B' + \sum_1^t A_i$ must be of even multiplicity.

3. List of possibilities

From now on S is a smooth minimal surface of general type with $p_g = q = 1$ having an involution i such that the bicanonical map ϕ_2 of S is not composed with i. Notice that then $2 \leq K_S^2 \leq 9$, by the Debarre's inequality for an irregular surface $\left(K_S^2 \geq 2p_g\right)$ and by the Miyaoka-Yau inequality $\left(K_S^2 \leq 9\chi(\mathcal{O}_S)\right)$.

Recall from Section 2.2 that

$$h^0(W, \mathcal{O}_W(2K_W + L)) \neq 0,$$

where W is the minimal resolution of S/i and $L \equiv \frac{1}{2}B$ is the line bundle which determines the double cover $V \to W$.

Let P be a minimal model of W and δ, $\overline{B} \equiv 2\delta$ and the numbers r_i be as defined in Section 2.2. Recall that t denotes the number of nodes of S/i. Notice that $p_g(P) \leq p_g(S) = 1$ and $q(P) \leq q(S) = 1$.

In the next sections the following result is useful:

Proposition 6 ([21])

Let S be a minimal smooth surface of general type and $C \subset S$ be a disjoint union of smooth elliptic curves. Then

$$-C^2 \leq 36\chi(\mathcal{O}_S) - 4K_S^2.$$

Proof. This follows from the inequality of [21, Corollary 7.8], using $KC + C^2 = 2p_a(C) - 2 = 0$. □

3.1 The case $\mathrm{Kod}(S/i) = 0$

Here we give a list of possibilities for the case $\mathrm{Kod}(S/i) = 0$.

Theorem 7

Let S and P be as above. If $\mathrm{Kod}(P) = 0$, only the following cases can occur:

a) *P is an Enriques surface and*

- $\{r_i \neq 2\} = \{4\}$, $\overline{B}^2 = 0$, $t - 2 = K_S^2 \in \{2, \ldots, 7\}$, *or*
- $\{r_i \neq 2\} = \{4, 4\}$, $\overline{B}^2 = 8$, $t = K_S^2 \in \{4, \ldots, 8\}$, *or*

· $\{r_i \neq 2\} = \{6\}$, $\overline{B}^2 = 16$, $t = K_S^2 \in \{4, \ldots, 8\}$;

b) *P is a bielliptic surface and*

· $\{r_i \neq 2\} = \emptyset$, $\overline{B}^2 = 8$, $t = 0$, $K_S^2 = 4$, *or*

· $\{r_i \neq 2\} = \{4\}$, $\overline{B}^2 = 16$, $t + 6 = K_S^2 = 6$ *or* 7, *or*

· $\{r_i \neq 2\} = \{4, 4\}$, $\overline{B}^2 = 24$, $t = 0$, $K_S^2 = 8$, *or*

· $\{r_i \neq 2\} = \{6\}$, $\overline{B}^2 = 32$, $t = 0$, $K_S^2 = 8$.

Furthermore, there are examples for

· a) *with* $K_S^2 = 8$;

· b) *with* $K_S^2 = 4$, 6, 7 *or* 8.

Proof. It is easy to see that P cannot be a $K3$ surface: in this case we get from Proposition 4, b) that

$$K_W^2 \geq 2h^0(W, \mathcal{O}_W(2K_W + L)) - 2,$$

which implies $h^0(W, \mathcal{O}_W(2K_W + L)) = 1$ and $K_W^2 = 0$. This contradicts the fact $\sum(r_i - 2) = 4 \neq 0$, given by Proposition 2, a).

So, from the classification of surfaces (see *e.g.* [2] or [1]), $p_g(P) = q(P) = 0$ or $p_g(P) = 0$, $q(P) = 1$ (notice that $p_g(P), q(P) \leq 1$), *i.e.* P is an Enriques surface or a bielliptic surface.

a) Suppose P is an Enriques surface: Proposition 4, a) implies that $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$, with equality holding only if $K_W^2 = 0$. In this case the branch locus $\overline{B}$ is smooth, *i.e.* $\sum(r_i - 2) = 0$, which contradicts Proposition 2, a). Therefore $h^0(W, \mathcal{O}_W(2K_W + L)) = 1$ or 2.

Now the only possibilities allowed by Propositions 2 and 3, a), b) are:

1) $\sum(r_i - 2) = 2$, $\overline{B}^2 = 0$, $t = K_S^2 + 2 \geq 4$;

2) $\sum(r_i - 2) = 4$, $\overline{B}^2 = 8$ or 16, $t = K_S^2 \geq 4$.

Moreover, if a nodal curve $A_i \subset B$ is not contracted to a point, then it is mapped onto a nodal curve of the Enriques surface P. Indeed, from the adjunction formula, $K_W A_i = 0$, which means that A_i does not intersect any (-1)-curve of W.

An Enriques surface has at most 8 disjoint (-2)-curves. In case 1), the non-negligible singularities of $\overline{B}$ are a 4-uple or $(3, 3)$-point, hence $t \leq 9$. In case 2), $t = 9$ only if $\overline{B}$ has a $(3, 3)$-point, which implies that S has an elliptic curve with negative self-intersection. Since in this case $K_S^2 = 9$, this is impossible from Proposition 6, therefore $t \leq 8$.

b) Suppose P is a bielliptic surface: from Proposition 4, a), one has $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 4$, with equality holding only if $K_W^2 = 0$. In this case we get from Proposition 2, a) that

$$\sum(r_i - 2) = 2h^0(W, \mathcal{O}_W(2K_W + L)) - 2 = 6 \neq 0,$$

which contradicts $K_W^2 = 0$. Hence $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$.

As in a), if a (-2)-curve $A_i \subset B$ is not contracted to a point, then it is mapped onto a (-2)-curve of P. But a bielliptic surface has no (-2)-curves (from Proposition 5), thus the nodal curves of B are contracted to singularities of $\overline{B}$.

Using Propositions 2 and 3, a) one obtains the following possibilities:

1) $\sum(r_i - 2) = 0$, $\overline{B}^2 = 8$, $K_S^2 = t + 4$;

2) $\sum(r_i - 2) = 2$, $\overline{B}^2 = 16$, $K_S^2 = t + 6$;

3) $\sum(r_i - 2) = 4$, $\overline{B}^2 = 24$, $K_S^2 = t + 8$;

4) $\sum(r_i - 2) = 4$, $\overline{B}^2 = 32$, $K_S^2 = t + 8$.

In case 1), $t = 0$, because $\overline{B}$ has only negligible singularities. In case 2), $\overline{B}$ can have a $(3,3)$-point, thus $t = 0$ or 1. In case 3), $t = 1$ only if $\overline{B}$ has a $(3,3)$-point, but then $K_S^2 = 9$ and S has an elliptic curve, which is impossible from Proposition 6. Finally, in case 4), the only non-negligible singularity of $\overline{B}$ is a point of multiplicity 6 (from Proposition 2, b)), thus $t = 0$.

The examples are constructed in Sections 4.3, 4.5, 4.6, 4.7 and 4.9. □

3.2 The case $\mathrm{Kod}(S/i) = 1$

Now we give a list of possibilities for the case $\mathrm{Kod}(S/i) = 1$.

Theorem 8

Let S and P be as above. If $\mathrm{Kod}(P) = 1$, only the following cases can occur:

a) $\chi(\mathcal{O}_P) = 2$, $q(P) = 0$ *and*

- $\{r_i\} = \emptyset$, $K_P\overline{B} = 4$, $\overline{B}^2 = -32$, $t - 8 = K_S^2 \in \{4, \ldots, 8\}$;

b) $\chi(\mathcal{O}_P) = 1$, $q(P) = 0$ *and*

- $\{r_i \neq 2\} = \emptyset$, $K_P\overline{B} = 2$, $\overline{B}^2 = -12$, $t - 2 = K_S^2 \in \{2, 3, 4\}$, *or*
- $\{r_i \neq 2\} = \emptyset$, $K_P\overline{B} = 4$, $\overline{B}^2 = -16$, $t = K_S^2 \in \{4, \ldots, 8\}$, *or*
- $\{r_i \neq 2\} = \{4\}$, $K_P\overline{B} = 2$, $\overline{B}^2 = -4$, $t = K_S^2 \in \{4, \ldots, 8\}$;

c) $\chi(\mathcal{O}_P) = 1$, $q(P) = 1$ *and*

- $\{r_i \neq 2\} = \emptyset$, $K_P\overline{B} = 2$, $\overline{B}^2 = -12$, $t - 2 = K_S^2 \in \{2, \ldots, 6\}$, *or*
- $\{r_i\} = \emptyset$, $K_P\overline{B} = 4$, $\overline{B}^2 = -16$, $t = K_S^2 \in \{4, \ldots, 8\}$.

d) $\chi(\mathcal{O}_P) = 0$, $q(P) = 1$ *and*

- $\{r_i \neq 2\} = \emptyset$, $K_P\overline{B} = 2$, $\overline{B}^2 = 4$, $t = 0$, $K_S^2 = 6$, *or*
- $\{r_i \neq 2\} = \emptyset$, $K_P\overline{B} = 4$, $\overline{B}^2 = 0$, $t = 0$, $K_S^2 = 8$, *or*
- $\{r_i \neq 2\} = \{4\}$, $K_P\overline{B} = 2$, $\overline{B}^2 = 12$, $t = 0$, $K_S^2 = 8$.

Furthermore, there exist examples for

- a) *with* $K_S^2 = 8$;
- b) *with* $K_S^2 = 4$, 6 *or* 7;
- c) *with* $K_S^2 = 8$;
- d) *with* $K_S^2 = 6$ *or* 8.

Proof. Since $p_g(P), q(P) \leq 1$, we have the following cases:

a) $\chi(\mathcal{O}_P) = 2$, $q(P) = 0$.
From Proposition 4, b) it is immediate that $h^0(W, \mathcal{O}_W(2K_W + L)) = 1$ and $K_W^2 = 0$ (thus $\overline{B}$ is smooth). Proposition 2 gives $K_P\overline{B} = 4$ and $\overline{B}^2 = -32$. If $K_S^2 = 9$, then the number of nodal curves of B is $t = K_S^2 + 8 = 17$, from Proposition 3, a). This is impossible because Proposition 5 implies $t \leq 16$. Proposition 3, c) gives $K_S^2 \geq 4$.

b) $\chi(\mathcal{O}_P) = 1$, $q(P) = 0$.
Proposition 4, a) implies $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$, with equality only if $K_S^2 = 9$ and $K_W^2 = 0$ (hence $\sum(r_i - 2) = 0$ and $W = P$). In this case Proposition 2, a) implies $K_W B' = 6$ and then $B' \neq \emptyset$. Now $p_a(B') = 1$ (see Claim 1 in the proof of Proposition 4), thus B' has a rational or elliptic component. But Propositions 5 and 6 imply that a minimal surface of general type with $\chi = 1$ and $K^2 = 9$ contains no rational or elliptic curves. Therefore $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 2$.

Since $\mathrm{Kod}(P) = 1$, $K_P\overline{B} = 0$ implies that $\overline{B}$ is contained in the elliptic fibration of P and then S has an elliptic fibration, which is impossible because S is of general type.

So $K_P\overline{B} \neq 0$. Now Propositions 2 and 3, a) give the following possibilities:

1) $\sum(r_i - 2) = 0$, $K_P\overline{B} = 2$, $\overline{B}^2 = -12$, $t = K_S^2 + 2$;

2) $\sum(r_i - 2) = 0$, $K_P\overline{B} = 4$, $\overline{B}^2 = -16$, $t = K_S^2$;

3) $\sum(r_i - 2) = 2$, $K_P\overline{B} = 2$, $\overline{B}^2 = -4$, $t = K_S^2$.

In case 1), $t > 6$ implies $\overline{B'}^2 = \overline{B}^2 + 2t > 0$, a contradiction (see Section 2.3).

Similarly $t \leq 8$, in case 2). Proposition 3, c) gives $K_S^2 \geq 4$, in this case.

In case 3), the quadruple or $(3,3)$-point of $\overline{B}$ gives rise to an elliptic curve in S, thus $K_S^2 \neq 9$, from Proposition 6. Again Proposition 3, c) implies $K_S^2 \geq 4$.

c) $\chi(\mathcal{O}_P) = 1$, $q(P) = 1$.
This is analogous to the proof of b): just notice that Proposition 4, b) excludes case 3) and implies $K_W^2 = 0$ in case 2); in case 1) is no longer true that $t \leq 6$, instead use Proposition 5 to obtain $t \leq 8$ (thus $K_S^2 \leq 6$).

d) $\chi(\mathcal{O}_P) = 0$, $q(P) = 1$.
As in b), one shows that $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$ and $K_P\overline{B} \neq 0$. Propositions 2 and 3, a) give the following possibilities:

1) $\sum(r_i-2)=0$, $K_P\overline{B}=2$, $\overline{B}^2=4$, $t=K_S^2-6$;

2) $\sum(r_i-2)=0$, $K_P\overline{B}=4$, $\overline{B}^2=0$, $t=K_S^2-8$;

3) $\sum(r_i-2)=2$, $K_P\overline{B}=2$, $\overline{B}^2=12$, $t=K_S^2-8$.

As in the proof of b), the existence of a quadruple or $(3,3)$-point on $\overline{B}$ implies $K_S^2 \neq 9$, in case 3).

Consider now cases 1) and 2). From Proposition 5, P has no smooth rational curves. Since $p_g(P)=0$ and $q(P)=1$, the Albanese variety of P is an elliptic curve (see *e.g.* [2]). Therefore any singular rational curve D of $\overline{B}$ is necessarily contained in a fibre of the Albanese fibration of P and as such satisfies $D^2 \leq 0$. So a desingularization $\widehat{D}$ of D verifies $\widehat{D}^2 \leq -4$ and thus B has no (-2)-curves, *i.e.* $t=0$.

The examples are given in Sections 4.1, 4.2, 4.4, 4.6, 4.7 and 4.8. □

3.3 The case $\mathrm{Kod}(S/i)=2$

Finally we give a list of possibilities for the case $\mathrm{Kod}(S/i)=2$.

Theorem 9

Let S and P be as above. If $\mathrm{Kod}(P)=2$, then $\overline{B}$ has at most negligible singularities and only the following cases can occur:

a) $\chi(\mathcal{O}_P)=2$, $q(P)=0$ *and*

- $K_P\overline{B}=0$, $\overline{B}^2=-24$, $t=12$, $K_S^2=2K_P^2$, $K_P^2=2,3,4$, *or*
- $K_P\overline{B}=2$, $\overline{B}^2=-28$, $t-10+2K_P^2=K_S^2\in\{2K_P^2+2,\dots,2K_P^2+4\}$, $K_P^2=1,2$;

b) $\chi(\mathcal{O}_P)=1$, $q(P)=1$ *and*

- $K_P\overline{B}=0$, $\overline{B}^2=-8$, $t=4$, $K_S^2=2K_P^2$, $K_P^2=2,3,4$, *or*
- $K_P\overline{B}=2$, $\overline{B}^2=-12$, $K_P^2=2$, $t+2=K_S^2\in\{6,7,8\}$;

c) $\chi(\mathcal{O}_P)=1$, $q(P)=0$ *and*

- $K_P\overline{B}=0$, $\overline{B}^2=-8$, $t=4$, $K_S^2=2K_P^2$, $K_P^2=1,\dots,4$, *or*
- $K_P\overline{B}=2$, $\overline{B}^2=-12$, $t+2K_P^2-2=K_S^2\in\{2K_P^2+2,\dots,2K_P^2+4\}$, $K_P^2=1,2$, *or*
- $K_P\overline{B}=4$, $\overline{B}^2=-16$, $K_P^2=1$, $t+2=K_S^2\in\{6,7,8\}$.

Moreover, there exist examples for

- a) *with* $K_S^2=4$, 6, 7 *or* 8;
- b) *with* $K_S^2=4$.

Proof.

Claim : If $K_P\overline{B} = 0$, then $\overline{B} = B$ is a disjoint union of nodal curves.

Proof. Since P is minimal of general type, K_P is nef and big and therefore every component of $\overline{B}$ is a nodal curve (*i.e.* a (-2)-curve) and the intersection form on the components of the reduced effective divisor $\overline{B}$ is negative definite by the Algebraic Index Theorem. The claim is true if each connected component of $\overline{B}$ is irreducible. Let C be a connected component of $\overline{B}$. Then, if C is not irreducible, there is one component θ of C such that $\theta(C-\theta) = 1$ and this implies that B has a (-3)-curve, contradicting $B \equiv 0 \pmod 2$. $\square$

Since $p_g(P), q(P) \leq 1$ and $p_g(P) \geq q(P)$, we have only the following three cases:

a) $\chi(\mathcal{O}_P) = 2$, $q(P) = 0$.
Propositions 2, a) and 4, b) give:

$$h^0(W, \mathcal{O}_W(2K_W + L)) = K_P^2 + K_P\delta + \frac{1}{2}\sum(r_i - 2) - 1,$$

$$h^0(W, \mathcal{O}_W(2K_W + L)) \leq \frac{1}{2}K_W^2 + 1 \leq \frac{1}{2}K_P^2 + 1.$$

From this we get

$$\frac{1}{2}K_P^2 + K_P\delta + \frac{1}{2}\sum(r_i - 2) \leq 2, \tag{6}$$

with equality only if $K_W^2 = K_P^2$. Since $K_P^2 > 0$, $K_P\delta = 0$ or 1.

If $K_P\delta = 1$, then $\sum(r_i - 2) = 0$ and $K_P^2 = 1$ or 2.

If $K_P\delta = 0$, then $\sum(r_i-2) = 0$, from the Claim above. As $K_S^2 \leq 9$, Proposition 3 implies $h^0(W, \mathcal{O}_W(2K_W + L)) \leq 3$.

Now the result follows from Propositions 2 and 3, a). Notice that Proposition 2 gives $\overline{B}^2 \geq -2(12+2K_P\delta)$. This implies $t \leq 12+2K_P\delta$, because, since $q(P) = 0$ and $\overline{B}$ has only negligible singularities, every component of $\overline{B}$ has non-positive self-intersection.

b) $\chi(\mathcal{O}_P) = 1$, $q(P) = 1$.
Propositions 2, a) and 4, b) give:

$$h^0(W, \mathcal{O}_W(2K_W + L)) = K_P^2 + K_P\delta + \frac{1}{2}\sum(r_i - 2),$$

$$h^0(W, \mathcal{O}_W(2K_W + L)) \leq \frac{1}{2}K_W^2 + 2 \leq \frac{1}{2}K_P^2 + 2,$$

thus equation (6) is still valid here, which implies $K_P\delta = 0$ or 1. As $p_g(P) = q(P) = 1$, then $K_P^2 \geq 2$.

If $K_P\delta = 1$, then $K_P^2 = 2$ and $\sum(r_i - 2) = 0$, from equation (6). Since then $h^0(W, \mathcal{O}_W(2K_W + L)) = 3$, Proposition 2, b) implies $\overline{B}^2 = -12$.
If $K_P\delta = 0$, then $\overline{B}$ is a disjoint union of (-2)-curves, from the Claim above. Hence $\sum(r_i - 2) = 0$, $h^0(W, \mathcal{O}_W(2K_W + L)) = K_P^2$ and $K_P^2 = 2, 3$ or 4. In this case Proposition 2, b) gives $\delta^2 = -2$. Therefore $\overline{B}^2 = -8$ and then $t = 4$, from the Claim above.

Now the result follows from Proposition 3, a) (notice that $K_P^2=2$ implies $t\neq 7$, by Theorem 5).

c) $\chi(\mathcal{O}_P)=1$, $q(P)=0$.

Propositions 2, a), 3, c) and 4, a) imply

$$h^0(W,\mathcal{O}_W(2K_W+L))=K_P(K_P+\delta)+\frac{1}{2}\sum(r_i-2)\leq 4 \qquad (7)$$

and

$$h^0(W,\mathcal{O}_W(2K_W+L))\leq 3+\frac{1}{3}K_W^2, \qquad (8)$$

with equality only if $K_S^2=9$. As $K_P^2\geq 1$ and $K_W^2\leq K_P^2$, this implies $K_P\delta\leq 2$.

• Suppose $K_P\delta=0$.

We have $\sum(r_i-2)=0$, by the Claim above. Hence

$$h^0(W,\mathcal{O}_W(2K_W+L))=K_P^2\leq 4,$$

by (7). Now from Proposition 2, b) and Proposition 3, b), we have $\overline{B}^2=(2\delta)^2=-8$ and $t\geq 4$. Thus $t=4$ and, using Proposition 3, a), we conclude that

$$K_S^2=2K_P^2,\ 1\leq K_P^2\leq 4.$$

• Suppose $K_P\delta=1$.

Then $h^0(W,\mathcal{O}_W(2K_W+L))=4$ only if $K_S^2=9$, from (7) and (8). In this case Propositions 3, b) and 2, a) imply $K_P^2=3$ and B contains an elliptic curve, which contradicts Proposition 6.

If $K_P^2=1/2\sum(r_i-2)=1$, then $K_W^2=0$ and $K_S^2=9$, by (8). The quadruple or $(3,3)$-point of $\overline{B}$ gives rise to an elliptic curve in S, which is impossible from Proposition 6.

Now using (7), Proposition 2, b) and Proposition 3, we obtain

$$K_P^2=2,\ \sum(r_i-2)=0,\ \delta^2=-3,\ t=K_S^2-2\geq 4$$

or

$$K_P^2=1,\ \sum(r_i-2)=0,\ \delta^2=-3,\ t=K_S^2\geq 4.$$

Hence $\overline{B}^2=-12$ and then $t\leq 6$, because, since $q(P)=0$ and $\overline{B}$ has only negligible singularities, every component of $\overline{B}$ has non-positive self-intersection.

• Suppose $K_P\delta=2$.

Then $K_P^2\leq 2$, from (7), and $h^0(W,\mathcal{O}_W(2K_W+L))\leq 3$, from (8). The only possibility allowed by (7), Proposition 2, b) and Proposition 3 is:

$$K_P^2=1,\ \delta^2=-4,\ t=K_S^2-2\geq 4.$$

It remains to be shown that $K_S^2\neq 9$. In this case, the curve $\overline{B}$ has at least 8 disjoint components contained in a fibration $\overline{f_A}$ of P (see Section 2.3). These are independent in $\mathrm{Pic}(P)$ from a general fibre of $\overline{f_A}$ and from K_P, so $\mathrm{Pic}(P)$ has 10 independent classes. This is a contradiction because the second Betti number of P is

$$b_2(P) = 12\chi(\mathcal{O}_P) - K_P^2 + 4q(P) - 2 = 9.$$

The examples can be found in Sections 4.2, 4.3, 4.4, 4.5, 4.6, 4.7 and 4.9. □

4. (Bi)double cover examples

The next sections contain constructions of minimal smooth surfaces of general type S with $p_g = q = 1$ which present examples for cases a), b), ... of Theorems 7, 8 and 9. Only the existence of case c) of Theorem 9 remains an open problem.

Each example is obtained as the smooth minimal model of a bidouble cover of a ruled surface (irregular only in Sections 4.8 and 4.9).

The verifications that the surfaces S and the corresponding quotient surfaces are as claimed are not written here. These can be found in the author's Ph.D. thesis, available at http://home.utad.pt/˜crito .

A bidouble cover is a finite flat Galois morphism with Galois group $\mathbb{Z}_2^2$. Following [7] or [14], to define a bidouble cover cover $\psi : V \to X$, with V, X smooth surfaces, it suffices to present:

· smooth divisors $D_1, D_2, D_3 \subset X$ with pairwise transverse intersections and no common intersection;

· line bundles L_1, L_2, L_3 such that $2L_g \equiv D_j + D_k$ for each permutation (g, j, k) of $(1, 2, 3)$.

If $\mathrm{Pic}(X)$ has no 2-torsion, the L_i's are uniquely determined by the D_i's.

Let $N := 2K_X + \sum_1^3 L_i$. One has:

$$\begin{aligned}
p_g(V) &= p_g(X) + \sum_1^3 h^0\big(X, \mathcal{O}_X(K_X + L_i)\big),\\
\chi(\mathcal{O}_V) &= 4\chi(\mathcal{O}_X) + \frac{1}{2}\sum_1^3 L_i(K_X + L_i),\\
2K_V &\equiv \psi^*(N)
\end{aligned}$$

and

$$H^0\big(V, \mathcal{O}_V(2K_V)\big) \simeq H^0\big(X, \mathcal{O}_X(N)\big) \oplus \bigoplus_{i=1}^3 H^0\big(X, \mathcal{O}_X(N - L_i)\big).$$

The bicanonical map of V is composed with the involution i_g, associated to L_g, if and only if

$$h^0\big(X, \mathcal{O}_X(2K_X + L_g + L_j)\big) = h^0\big(X, \mathcal{O}_X(2K_X + L_g + L_k)\big) = 0.$$

For more information on bidouble covers see [7] or [14].

Denote by i_1, i_2, i_3 the involutions of V corresponding to L_1, L_2, L_3, respectively. In each example we give the invariants of the quotient surfaces $W_j := V/i_j$, $j = 1, 2, 3$.

We use the following:

Notation 10

Let $p_0, \ldots, p_j, \ldots, p_{j+s} \in \mathbb{P}^2$ be distinct points and define T_i as the line through p_0 and p_i, $i = 1, \ldots, j$. We say that a plane curve is of type

$$d(m, (n,n)_T^j, r^s)$$

if it is of degree d and if it has: an m-uple point at p_0, an (n,n)-point at $p_1, \ldots, p_j$, an r-uple point at $p_{j+1}, \ldots, p_{j+s}$ and no other non-negligible singularities. The index $_T$ is used if T_i is tangent to the (n,n)-point at p_i.

An obvious generalization is used if there are other singularities.

Let $p_1', \ldots, p_j'$ be the infinitely near points to $p_1, \ldots, p_j$, respectively. We denote by

$$\mu : X \to \mathbb{P}^2$$

the blow-up with centers

$$p_0, p_1, p_1', \ldots, p_j, p_j', p_{j+1}, \ldots, p_{j+s}$$

and by

$$E_0, E_1, E_1', \ldots, E_j, E_j', E_{j+1}, \ldots, E_{j+s}$$

the corresponding exceptional divisors (with self-intersection -1).

The notation $\widetilde{\cdot}$ stands for the total transform $\mu^*(\cdot)$ of a curve.

The letter T is reserved for a general line of $\mathbb{P}^2$.

The genus of a general Albanese fibre of S is denoted by g.

4.1 $K^2 = 8$, $g = 3$, S/i_1 ruled, S/i_2 rational, $\mathrm{Kod}(S/i_1) = 1$

Let Q be a reduced curve of type $4(0, (2,2)_T^2)$, *i.e.* Q is the union of two conics tangent to the lines T_1 and T_2 at p_1, p_2. Let C be another non-degenerate conic tangent to T_1, T_2 at p_1, p_2 and let $T_3, \ldots, T_6 \neq T_1, T_2$ be distinct lines through $p_0 \in T_1 \bigcap T_2$.

Set

$$\begin{aligned} D_1 &:= \widetilde{T_1} + \cdots + \widetilde{T_6} - \textstyle\sum_1^2 (E_i + E_i') - 6E_0, \\ D_2 &:= \widetilde{Q} - \textstyle\sum_1^2 (E_i + 3E_i'), \\ D_3 &:= \widetilde{C} - \textstyle\sum_1^2 (E_i + E_i') \end{aligned}$$

and let $V \to X$ be the bidouble cover determined by D_1, D_2, D_3. One has $p_g(V) = \chi(\mathcal{O}_V) = 1$ and the bicanonical map of V is composed with the involution i_2 and is not composed with the involutions i_1 and i_3. The quotients $W_j := V/i_j$, $j = 1, 2, 3$, satisfy:

- W_1 is ruled, $q(W_1) = 1$;

· W_2 is rational;

· $\mathrm{Kod}(W_3) = p_g(W_3) = 1$, $q(W_3) = 0$.

Moreover $K_S^2 = 8$, where S is the minimal model of V. The pencil of conics tangent to T_1, T_2 at p_1, p_2 induces the Albanese fibration of S, which is of genus 3.

This gives an example for case a) of Theorem 8.

The surface S is a Du Val double plane of type III described in [16].

4.2 $K^2 = 6$, $g = 4$, $\mathrm{Kod}(S/i_1) = 2$, S/i_2 **rational,** $\mathrm{Kod}(S/i_3) = 1$

From Proposition 12 in Appendix A.1, there is a pencil l, with no base component, of curves of type $7(3,(2,2)_T^5)$. Let Q be a general element of this pencil and C be a reduced curve of type $4(2,(1,1)_T^5)$.

Set

$$D_1 := \widetilde{T_1} + \cdots + \widetilde{T_4} - \textstyle\sum_1^4 (E_i + E_i') + (E_5 - E_5') - 4E_0,$$
$$D_2 := \widetilde{T_5} + \widetilde{Q} - \textstyle\sum_1^4 (E_i + 3E_i') - 3E_5 - 3E_5' - 4E_0,$$
$$D_3 := \widetilde{C} - \textstyle\sum_1^5 (E_i + E_i') - 2E_0$$

and let $V \to X$ be the bidouble cover determined by D_1, D_2, D_3. One has $p_g(V) = \chi(\mathcal{O}_V) = 1$ and the bicanonical map of V is composed with the involution i_2 and is not composed with the involutions i_1 and i_3. The quotients $W_j := V/i_j$ satisfy:

· $\mathrm{Kod}(W_1) = 2$, $p_g(W_1) = 1$, $q(W_1) = 0$;

· W_2 is rational;

· $\mathrm{Kod}(W_3) = 1$, $p_g(W_3) = 0$, $q(W_3) = 1$.

Moreover $K_S^2 = 6$, where S is the minimal model of V. The pencil l induces the Albanese fibration of S, which is of genus 4.

This is an example for Theorems 8, d) and 9, a).

One can verify that S is a Du Val double plane obtained imposing a 4-uple point to the branch locus of a Du Val's ancestor of type $\mathcal{D}_5$ (*cf.* [19]).

4.3 $K^2 = 4$, $g = 3$, $\mathrm{Kod}(S/i_1) = 2$, S/i_2 **rational,** $\mathrm{Kod}(S/i_3) = 0$

From Proposition 12, there is a pencil l, with no base component, of curves of type $6(2,(2,2)_T^4)$, through points $p_0, \ldots, p_4$ (*i.e.* of plane curves of degree 6 with a double point at p_0 and a tacnode at p_i with tangent line through p_0, p_i, $i = 1, \ldots, 4$). Let Q be a general element of this pencil, C be a reduced curve of type $4(2,(1,1)_T^4)$ and set

$$D_1 := \widetilde{T_1} + \cdots + \widetilde{T_4} - \textstyle\sum_1^4 (E_i + E_i') - 4E_0,$$
$$D_2 := \widetilde{Q} - \textstyle\sum_1^4 (E_i + 3E_i') - 2E_0,$$
$$D_3 := \widetilde{C} - \textstyle\sum_1^4 (E_i + E_i') - 2E_0.$$

Let $V \to X$ be the bidouble cover determined by D_1, D_2, D_3 and S be the minimal model of V. One has $p_g(S) = \chi(\mathcal{O}_S) = 1$, $K_S^2 = 4$ and the bicanonical map of V is composed with the involution i_2 and is not composed with the involutions i_1 and i_3, associated to the bidouble cover. The quotients $W_j := V/i_j$ satisfy:

· $\mathrm{Kod}(W_1) = 2$, $p_g(W_1) = 1$, $q(W_1) = 0$;

· W_2 is rational;

· $\mathrm{Kod}(W_3) = 0$, $p_g(W_3) = 0$, $q(W_3) = 1$.

The pencil l induces the (genus 3) Albanese fibration of S.

This gives an example for Theorems 7, b) and 9, a).

One can verify that S is a Du Val double plane obtained imposing two 4-uple points to the branch locus of a Du Val's ancestor of type $\mathcal{D}_4$ (*cf.* [19]).

4.4 $K^2 = 4$, $g = 2$, S/i_1 ruled, $\mathrm{Kod}(S/i_2) = 1$, $\mathrm{Kod}(S/i_3) = 2$

We recall Notation 10.

This section contains the construction of a surface of general type S with $p_g = q = 1$, $K^2 = 4$, $g = 2$ and bicanonical map ϕ_2 of degree 2.

By Proposition 12 in Appendix A.1, there is a pencil l, with no base component, of curves of type $6(2,(2,2)_T^4)$. Let Q_1 be a general element of this pencil, Q_2 be a smooth curve of type $3(1,(1,1)_T^4)$ and $Q := Q_1 + Q_2$. Let T_5 be a line through p_0 transverse to Q and set

$$\begin{aligned} D_1 &:= \widetilde{T_1} + \widetilde{Q} - 4E_1 - 4E_1' - \textstyle\sum_2^4(3E_i + 3E_i') - 4E_0,\\ D_2 &:= \widetilde{T_2} + \cdots + \widetilde{T_5} - \textstyle\sum_2^4(E_i + E_i') - 4E_0,\\ D_3 &:= \textstyle\sum_2^4(E_i - E_i'). \end{aligned}$$

Let $\psi : V \to X$ be the bidouble cover determined by D_1, D_2, D_3. The bicanonical map of V is composed with the involution i_1 and is not composed with the involutions i_2 and i_3. The quotients $W_j := V/i_j$ satisfy:

· W_1 is ruled, $q(W_1) = 1$;

· $\mathrm{Kod}(W_2) = 1$, $p_g(W_2) = q(W_2) = 0$;

· $\mathrm{Kod}(W_3) = 2$, $p_g(W_3) = 1$, $q(W_3) = 0$.

The surface S is the minimal model of V. The Albanese fibration of S is induced by the pullback of the pencil of lines through p_0. It is of genus 2.

This gives an example for Theorems 8, b) and 9, a).

Let N be as above. One has $\deg(\phi_2) = 2$ because the system $|\psi^*(N)|$ is strictly contained in the bicanonical system of V, ϕ_2 is composed with i_1 and the map $X \dashrightarrow \mathbb{P}^2$ induced by $|N|$ is birational (this can be verified using the Magma function *IsInvertible*).

4.5 $K^2 = 8$, $g = 3$,
$\mathrm{Kod}(S/i_1) = 2$, $\mathrm{Kod}(S/i_2) = 0$, $\mathrm{Kod}(S/i_3) = 0$

A smooth projective surface S of general type is said to be a *standard isotrivial fibration* if there exists a finite group G which acts faithfully on two smooth projective curves C and F so that S is isomorphic to the minimal desingularization of $T := (C \times F)/G$. The paper [17] contains examples of such surfaces with $K^2 = 8$.

This section contains the construction of the first surface of general type with $p_g = q = 1$, $K^2 = 8$ and $g = 3$ which is not a standard isotrivial fibration.

Let G be a curve of type $6(2,(2,2)_T^4)$ and C be a curve of type $8(4,(2,2)_T^4,(3,3))$ such that $G + C$ is reduced and the $(3,3)$-point of C is tangent to G. The existence of these curves is shown in Appendix A.2.

Set

$$\begin{aligned}
D_1 &:= \widetilde{T_1} + \widetilde{T_2} - \textstyle\sum_1^2 2E_i' + (E_5 - E_5') - 2E_0,\\
D_2 &:= \widetilde{G} - \textstyle\sum_1^4 (2E_i + 2E_i') - (E_5 + E_5') - 2E_0,\\
D_3 &:= \widetilde{T_3} + \widetilde{T_4} + \widetilde{C} - \textstyle\sum_1^2 (2E_i + 2E_i') - \sum_3^4 (2E_i + 4E_i') - (3E_5 + 3E_5') - 6E_0
\end{aligned}$$

and let $V \to X$ be the bidouble cover determined by D_1, D_2, D_3. The bicanonical map of V is not composed with any of the involutions i_1, i_2, i_3, associated to the bidouble cover. The quotients $W_j := V/i_j$ satisfy:

- $\mathrm{Kod}(W_1) = 2$, $p_g(W_1) = 1$, $q(W_1) = 0$;
- $\mathrm{Kod}(W_2) = 0$, $p_g(W_2) = 0$, $q(W_2) = 1$;
- $\mathrm{Kod}(W_3) = 0$, $p_g(W_3) = 0$, $q(W_3) = 0$.

Let S be the minimal model of V. One has $p_g(S) = q(S) = 1$ and $K_S^2 = 8$. The Albanese fibration of S is induced by a pencil of curves of type $14(6,(4,4)_T^4,(4,4))$, which contains an element equal to $G+C$ (see Appendix A.2). From [18, Theorem 3.2], the existence of such reducible fibre implies that S is not a standard isotrivial fibration, so this is not one of Polizzi's examples.

This is an example for Theorems 7 a), b) and 9 a).

4.6 $K^2 = 7$, $g = 3$,
$\mathrm{Kod}(S/i_1) = 2$, $\mathrm{Kod}(S/i_2) = 1$, $\mathrm{Kod}(S/i_3) = 0$

This section contains the construction of a bidouble cover $V \to X$, with X rational, such that the minimal model S of V is a surface of general type with $K^2 = 7$, $p_g = q = 1$ and $g = 3$ having birational bicanonical map.

From Appendix A.2, there exist a curve C of type $7(3,(2,2)_T^4,3)$ (*i.e.* C is a plane curve of degree 7 with triple points at p_0, p_5 and a tacnode at p_i tangent to the line T_i through p_0, p_i, $i = 1, \ldots, 4$) and a curve G of type $6(2,(2,2)_T^4,1)$, both through points $p_0, \ldots, p_5$, such that $C + G$ is reduced.

Set

$$D_1 := \widetilde{T_1} + \widetilde{T_2} + \widetilde{T_3} - \textstyle\sum_1^3 2E_i' + E_5 - 3E_0,$$
$$D_2 := \widetilde{T_4} + \widetilde{G} - \textstyle\sum_1^3 (2E_i + 2E_i') - (2E_4 + 4E_4') - E_5 - 3E_0,$$
$$D_3 := \widetilde{C} - \textstyle\sum_1^4 (2E_i + 2E_i') - 3E_5 - 3E_0$$

and let $\psi : V \to X$ be the bidouble cover determined by D_1, D_2, D_3. The bicanonical map of V is not composed with any of the involutions i_1, i_2, i_3, associated to the bidouble cover. The quotients $W_j := V/i_j$, $j = 1, 2, 3$, satisfy:

- $\mathrm{Kod}(W_1) = 2$, $p_g(W_1) = 1$, $q(W_1) = 0$;
- $\mathrm{Kod}(W_2) = 1$, $p_g(W_2) = 0$, $q(W_2) = 0$;
- $\mathrm{Kod}(W_3) = 0$, $p_g(W_3) = 0$, $q(W_3) = 1$.

One has $p_g(S) = \chi(S) = 1$ and $K_S^2 = 7$, where S is the minimal model of V. The Albanese fibration of S is induced by the pencil of curves of type $6(2, (2, 2)_T^4)$. It is of genus 3.

This is an example for Theorems 7, b), 8, b) and 9, a).

It remains to verify that the bicanonical map of S is birational. Let N be as above. The system $|\psi^*(N)|$ is strictly contained in the bicanonical system of V. The bicanonical map of V is not composed with any of the involutions i_1, i_2, i_3, hence it is birational if the map τ given by $|N| = N_1 + |N_2|$ is birational. This is in fact the case, see Appendix A.2, where Magma is used to show that the image of τ is of degree $7 = N_2^2$.

4.7 $K^2 = 6,\ g = 3,$ $\mathrm{Kod}(S/i_1) = 2,\ \mathrm{Kod}(S/i_2) = 1,\ \mathrm{Kod}(S/i_3) = 0$

One can obtain a construction analogous to the one in Section 4.6, but with $K_S^2 = 6$ instead: replace the triple point of C by a $(2, 2)$-point, tangent to G. Such a curve exists, see Appendix A.2. With this change the branch locus in W_3 has a 4-uple point instead of a $(3, 3)$-point.

4.8 $K^2 = 8,\ g = 3,$ $\mathrm{Kod}(S/i_1) = 1,\ S/i_2$ **ruled,** $\mathrm{Kod}(S/i_3) = 1$

Here we give the construction of a surface of general type S, with $K^2 = 8$, $p_g = q = 1$ and $g = 3$, as a bidouble cover of a ruled surface Z with $q(Z) = 1$.

Let $F_1, \ldots, F_4$ be disjoint fibres of the Hirzebruch surface $\mathbb{F}_0$ and $Z \to \mathbb{F}_0$ be the double cover with branch locus $F_1 + \cdots + F_4$. Clearly Z is a ruled surface with irregularity 1. Denote by γ the rational fibration of Z.

Let $G, G_1, \ldots, G_6$ be distinct smooth elliptic sections of γ and $\Gamma_1, \ldots, \Gamma_4$ be distinct fibres of γ such that $\Gamma_1 + \Gamma_2 \equiv 2\Gamma_3 \equiv 2\Gamma_4$.

Set

$$D_1 := \Gamma_1 + \Gamma_2,$$
$$D_2 := G_1 + \cdots + G_4,$$
$$D_3 := G_5 + G_6$$

and

$$L_1 := 3G + \Gamma_3 - \Gamma_4,$$
$$L_2 := G + \Gamma_4,$$
$$L_3 := 2G + \Gamma_3.$$

The bidouble cover $V \to Z$ is determined by the curves D_i and by the divisors L_i. The surface S is the minimal model of V.

The bicanonical map of V is not composed with any of the involutions i_1, i_2, i_3, associated to the bidouble cover. The quotients $W_j := S/i_j$, $j = 1, 2, 3$, satisfy:

- $\mathrm{Kod}(W_1) = 1$, $p_g(W_1) = 0$, $q(W_1) = 1$;
- W_2 is ruled, $q(W_2) = 1$;
- $\mathrm{Kod}(W_3) = p_g(W_3) = q(W_3) = 1$.

This is an example for cases c) and d) of Theorem 8.

4.9 $K^2 = 4$, $g = 3$, S/i_1 ruled, $\mathrm{Kod}(S/i_2) = 0$, $\mathrm{Kod}(S/i_3) = 2$

This section contains the construction of a bidouble cover $V \to Z$, with Z ruled and $q(Z) = 1$, such that the minimal model S of V is a surface of general type with $K^2 = 4$, $p_g = q = 1$, $g = 3$ and that the bicanonical map ϕ_2 of S is not composed with any of the involutions i_1, i_2, i_3 associated to the bidouble cover.

We use Notation 10.

Let Q_1 be a general curve of type $5(1, (2, 2)^3_T)$ (there is a pencil of such curves, see Appendix A.1) and Q_2 be a general curve of type $3(1, (1, 1)^3_T)$, both through points $p_0, \dots, p_3$.

Let

$$Q_1' := \widetilde{Q_1} - \sum_1^3 (2E_i + 2E_i') - E_0 \equiv 5\widetilde{T} - \sum_1^3 (2E_i + 2E_i') - E_0,$$

$$Q_2' := \widetilde{Q_2} - \sum_1^3 (E_i + E_i') - E_0 \equiv 3\widetilde{T} - \sum_1^3 (E_i + E_i') - E_0$$

and consider the double cover $\psi : Z \to X$ with branch locus

$$\widetilde{T_1} + \cdots + \widetilde{T_4} - \sum_1^3 2E_i' - 4E_0,$$

where T_4 is a general line through p_0.

Let

$$\Gamma := \frac{1}{2}\psi^*(\widetilde{T_4} - E_0),\ \Gamma_i := \frac{1}{2}\psi^*(\widetilde{T_i} - E_0),$$
$$C_0 := \psi^*(E_0),$$
$$e_i := \frac{1}{2}\psi^*(E_i - E_i'),$$
$$e_i' := \psi^*(E_i'),\quad i = 1, 2, 3,$$

and set

$$D_1 := \psi^*(Q_1') \equiv 4C_0 + 10\Gamma - \textstyle\sum_1^3(4e_i + 4e_i'),$$
$$D_2 := \psi^*(Q_2') \equiv 2C_0 + 6\Gamma - \textstyle\sum_1^3(2e_i + 2e_i'),$$
$$D_3 := 0,$$

$$L_1 := C_0 + 3\Gamma - \textstyle\sum_1^3(e_i + e_i'),$$
$$L_2 := 2C_0 + 5\Gamma - \textstyle\sum_1^3(2e_i + 2e_i'),$$
$$L_3 := 3C_0 + 8\Gamma - \textstyle\sum_1^3(3e_i + 3e_i').$$

The bidouble cover $V \to Z$ is determined by the curves D_i and by the divisors L_i. The surface S is the minimal model of V.

The quotients $W_j := V/i_j$, $j = 1, 2, 3$, satisfy:

- W_1 is ruled, $q(W_1) = 1$;
- $\mathrm{Kod}(W_2) = 0$, $p_g(W_2) = 0$, $q(W_2) = 1$;
- $\mathrm{Kod}(W_3) = 2$, $p_g(W_3) = 1$, $q(W_3) = 1$; the branch locus of the cover $V \to W_3$ is an union of four (-2)-curves.

This is an example for Theorems 7, b) and 9, b).

A. Appendix: Construction of plane curves

A.1 Useful pencils

Here we show the existence of some pencils of plane curves that are useful on some of the constructions of Section 4. Recall Notation 10.

Lemma 11

Let $C \subset \mathbb{P}^2$ be a smooth conic and $p_0 \notin C$, $p_1, \ldots, p_4 \in C$ be distinct points. Consider the points $p_5, p_6 \in C$ such that the lines through p_0, p_5 and p_0, p_6 are tangent to C.

There exists a smooth curve Q of type $3(1, (1,1)_T^4, 1^2)$, through $p_0, \ldots, p_6$.

Proof. Let C_x, $x \in \mathbb{P}^1$, be a parametrization of the pencil of conics through $p_1, \ldots, p_4$. Let p_x^1, p_x^2 be the points of C_x (not distinct if C_x is singular) such that the lines through p_0, p_x^1 and p_0, p_x^2 are tangent to C_x.

The correspondence

$$\{p_x^1, p_x^2\} \leftrightarrow x$$

gives a plane algebraic curve Q which is a double cover of $\mathbb{P}^1$. This cover is ramified over four points, corresponding to the three degenerate conics which contain the points $p_1, \ldots, p_4$ plus the conic which contains p_0. Therefore, by the Hurwitz formula, Q is a cubic.

The conic through $p_0, \ldots, p_4$ is not tangent to the line T_i (through p_0, p_i) at p_0, thus also Q is not tangent to T_i at p_0, $i = 1, \ldots, 4$. Since each conic C_x can be tangent to T_i only at p_i, $i = 1, \ldots, 4$, then Q intersects T_i only at p_0 and p_i, $i = 1, \ldots, 4$. This means that Q is tangent to T_i at p_i, $i = 1, \ldots, 4$, and then Q is smooth. □

Proposition 12

In the notation of Notation 10, there exist pencils, without base components, of plane curves of type:

[3] **a)** $5(1, (2,2)_T^3)$;

b) $6(2, (2,2)_T^4)$;

c) $7(3, (2,2)_T^5)$;

d) $8(4, (2,2)_T^6)$.

Proof.

a) This is proved in [3]. Notice that we are imposing 19 conditions to a linear system of dimension 20.

b) Let $\mathbb{A}(\mathbb{C})$ be an affine plane and $a, b, c, d \in \mathbb{C}\backslash\{0\}$ be numbers such that $a \neq c$ and $bc \neq \pm ad$. Consider the points of $\mathbb{A}$:

$$p_0 := (0,0),\ p_1 := (a,b),\ p_2 := (c,d),\ p_3 := (c,-d), p_4 := (a,-b)$$

and let T_i be the line through p_0 and p_i, $i = 1, \ldots, 4$. Let C_1 be the conic through $p_1, \ldots, p_4$ tangent to T_1, T_4 and C_2 be the conic through $p_1, \ldots, p_4$ tangent to T_2, T_3.

The curves

$$2C_1 + T_2 + T_3 \quad \text{and} \quad 2C_2 + T_1 + T_4$$

generate a pencil whose general member is a curve of type $6(2, (2,2)_T^4)$.

c) Let $C \subset \mathbb{P}^2$ be a non-degenerate conic and $p_0 \notin C$, $p_1, \ldots, p_5 \in C$ be distinct points such that the lines T_1, T_5, defined by p_0, p_1 and p_0, p_5, are tangent to C. From Lemma 11, there exists a curve Q of type $3(1, (1,1)_T^4, 1)$, through $p_0, \ldots, p_5$, respectively.

The curves

$$2C + T_2 + T_3 + T_4 \quad \text{and} \quad 2Q + T_5$$

generate a pencil whose general member is a curve of type $7(3, (2,2)_T^5)$.

d) This is analogous to the previous case, but now the pencil is generated by

$$2C + T_2 + \cdots + T_5 \quad \text{and} \quad 2Q + T_1 + T_6.$$

□

A.2 Constructions using Magma

In this appendix we construct some plane curves using the Computational Algebra System Magma ([4]). We use the Magma procedure *LinSys*, defined in [19]. This procedure calculates the linear system L of plane curves of degree d, in an affine plane $\mathbb{A}$, having singular points p_i of order $(m1_i, m2_i)$ with tangent direction given by the slope td_i.

Consider, in a affine plane $\mathbb{A}$, the points

$$p_0 := (0,0),\ p_1 := (2,2),\ p_2 := (-2,2),\ p_3 := (3,1),\ p_4 := (-3,1).$$

From Appendix A.1, there exists a pencil of curves of type $6(2,(2,2)^4_T)$, with singularities at $p_0,\ldots,p_4$, respectively. Let G be the element of this pencil which contains the point $p_5 := (0,5)$. Using the above Magma procedure, it is easy to verify the following (the respective code lines are available at http://home.utad.pt/~crito/thesis.html):

· the curve G is reduced and the tangent line to G at p_5 is horizontal;
· there exists a reduced curve C of type $8(4,(2,2)^4_T,(3,3))$, singular at $p_0,\ldots,p_5$, such that the $(3,3)$-point is tangent to G. Moreover, $G+C$ is a reduced element of a pencil of curves of type $14(6,(4,4)^4_T,(4,4))$;
· there exists a reduced curve of type $7(3,(2,2)^4_T,(2,2))$, singular at $p_0,\ldots,p_5$, such that the $(2,2)$-point is tangent to G.

Now we will see that p_5 can be chosen such that

· there exist reduced curves C_1 of type $7(3,(2,2)^4_T,3)$ and C_2 of type $6(2,(2,2)^4_T,1)$, both through $p_0,\ldots,p_5$, such that C_1+C_2 is reduced and the singularity of C_1+C_2 at p_5 is ordinary.

```
> A<x,y>:=AffineSpace(Rationals(),2);
> p:=[A![2,2],A![-2,2],A![3,1],A![-3,1],Origin(A)];
> d:=7;m1:=[2,2,2,2,3];m2:=[2,2,2,2];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m1,m2,td,~L);
> #Sections(L);BaseComponent(L);
6
Scheme over Rational Field defined by
1
```

Now we impose a triple point to the elements of L. This is done by asking for the vanishing of minors of a matrix of derivatives.

```
> R<x,y,n>:=PolynomialRing(Rationals(),3);
> h:=hom<PolynomialRing(L)->R|[x,y]>;
> H:=h(Sections(L));
> M:=[[H[i],D(H[i],1),D(H[i],2),D2(H[i],1,1),D2(H[i],1,2),\
> D2(H[i],2,2)]:i in [1..#H]];
> Mt:=Matrix(M);min:=Minors(Mt,#H);
```

```
> A:=AffineSpace(R);
> S:=Scheme(A,min cat [x-3,1+n*(y-x)*(y+x)*(3*y-x)*(3*y+x)]);
> //The condition 1+n*(..)=0 implies that
> //the solution is not in p.
> Dimension(S);
0
> PointsOverSplittingField(S);
```

We choose one of the solutions and we compute the curves C_1 and C_2:

```
> R<r1>:=PolynomialRing(Rationals());
> K<r1>:=NumberField(r1^2 - 1761803/139426560*r1 + \
> 1387488001/33730073395200);
> A<x,y>:=AffineSpace(K,2);
> y1:=-33462374400/102856069*r1 + 419793163/102856069;
> p:=[A![2,2],A![-2,2],A![3,1],A![-3,1],A![3,y1],Origin(A)];
> d:=7;m1:=[2,2,2,2,3,3];m2:=[2,2,2,2];
> td:=[p[i][2]/p[i][1]:i in [1..#m2]];
> LinSys(A,d,p,m1,m2,td,~L);#Sections(L);
1
> C1:=Curve(A,Sections(L)[1]);
> d:=6;m1:=[2,2,2,2,1,2];m2:=[2,2,2,2];
> LinSys(A,d,p,m1,m2,td,~L);#Sections(L);
1
> C2:=Curve(A,Sections(L)[1]);
```

The verification that the singularities are no worst than stated is left to the reader (use the Magma functions *ProjectiveClosure*, *SingularPoints*, *HasSingularPointsOverExtension* and *ResolutionGraph*).

The calculations for Section 4.6 (verification that ϕ_2 is birational) are as follows:

```
> d:=7;m1:=[2,2,2,2,1,3];m2:=[2,2,2,2];
> LinSys(A,d,p,m1,m2,td,~L);
> #Sections(L);BaseComponent(L);
5 Scheme over K defined by
1
> P4:=ProjectiveSpace(K,4);
> tau:=map<A->P4|Sections(L)>;
> Degree(tau(Scheme(A,Sections(L)[3])));
7
```

thus an hyperplane section of the image of τ is of degree 7.

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