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Characterizations of localized $\mathrm{BMO}(\mathbb{R}^n)$ via commutators of localized Riesz transforms and fractional integrals associated to Schrödinger operators

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Abstract

Let $\mathcal{L} \equiv -\Delta + V$ be the Schrödinger operator in $\mathbb{R}^n$, where V is a nonnegative function satisfying the reverse Hölder inequality. Let ρ be an admissible function modeled on the known auxiliary function determined by V. In this paper, the authors establish several characterizations of the space $\mathrm{BMO}_\rho(\mathbb{R}^n)$ in terms of commutators of several different localized operators associated to ρ, respectively; these localized operators include localized Riesz transforms and their adjoint operators, the localized fractional integral and its adjoint operator, the localized fractional maximal operator and the localized Hardy-Littlewood-type maximal operator. These results are new even for the space $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ introduced by J. Dziubański, G. Garrigós *et al.*

1. Introduction

The space $\mathrm{BMO}(\mathbb{R}^n)$ of functions with bounded mean oscillation was introduced by John and Nirenberg in [12] and plays a crucial role in harmonic analysis and partial differential equations; see, for example, [15, 9]. It is known that the space $\mathrm{BMO}(\mathbb{R}^n)$ is essentially related to the Laplacian $\Delta \equiv \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ and $\mathrm{BMO}(\mathbb{R}^n)$ has a remarkable characterization via commutators of the Riesz transforms $\nabla(-\Delta)^{-1/2}$, where ∇ is the

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gradient operator. More generally, if T is a Calderón-Zygmund singular integral operator with smooth kernel, Coifman, Rochberg and Weiss [2] proved that $b \in \mathrm{BMO}(\mathbb{R}^n)$ is sufficient to guarantee the commutator $[b,T](f) \equiv bT(f) - T(bf)$ to be bounded on $L^p(\mathbb{R}^n)$ with all $p \in (1,\infty)$, and they also established a partial converse that if $[b, \nabla(-\Delta)^{-1/2}]$ are bounded on $L^p(\mathbb{R}^n)$ for certain $p \in (1,\infty)$, then $b \in \mathrm{BMO}(\mathbb{R}^n)$. The full converse of this result was obtained by Janson [11]. On the other hand, let $\beta \in (0,n)$, $p, q \in (1,\infty)$ and $1/q = 1/p - \beta/n$. Chanillo [1] proved that for a fractional integral I_β of order β, $b \in \mathrm{BMO}(\mathbb{R}^n)$ if and only if $[b, I_\beta]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

Let $\mathcal{L} \equiv -\Delta + V$ be the Schrödinger operator on $\mathbb{R}^n$, where the potential V is a nonnegative locally integrable function. Fefferman [7] and Shen [14] established some basic results, including some estimates of the fundamental solutions and the boundedness on Lebesgue spaces of Riesz transforms, for $\mathcal{L}$ on $\mathbb{R}^n$ with $n \ge 3$ and the nonnegative potential V satisfying the reverse Hölder inequality. Especially, the works of Shen [14] lay the foundation for developing harmonic analysis related to $\mathcal{L}$ on $\mathbb{R}^n$. On the other hand, denote by $\mathcal{B}_q(\mathbb{R}^n)$ the class of functions satisfying the reverse Hölder inequality of order q. For $V \in \mathcal{B}_{n/2}(\mathbb{R}^n)$ with $n \ge 3$, Dziubański and Zienkiewicz [6] introduced the Hardy space $H^1_{\mathcal{L}}(\mathbb{R}^n)$ associated with $\mathcal{L}$ and, in particular, characterized $H^1_{\mathcal{L}}(\mathbb{R}^n)$ in terms of the Riesz transforms $\nabla\mathcal{L}^{-1/2}$; Dziubański, Garrigós et al in [5] further introduced the BMO-type space $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ associated with $\mathcal{L}$ and proved that the dual space of $H^1_{\mathcal{L}}(\mathbb{R}^n)$ is $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$. It is now known that $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ in [5] is a special case of BMO-type spaces associated with operators studied by Duong and Yan in [3, 4]; see, in particular, [4, Proposition 6.11]. Let ρ be an admissible function introduced in [18], which is modeled on the known auxiliary function determined by V. In [16], the localized Riesz transforms $\{\widetilde{R}_j\}_{j=1}^n$ associated with ρ and their adjoint operators $\{\widetilde{R}_j^*\}_{j=1}^n$ were introduced; their boundedness on the localized BMO-type space $\mathrm{BMO}_\rho(\mathbb{R}^n)$, as well as the equivalent characterization of the localized Hardy spaces $H^1_\rho(\mathbb{R}^n)$ in terms of $\{\widetilde{R}_j\}_{j=1}^n$ and $\{\widetilde{R}_j^*\}_{j=1}^n$, was established. Recall that if $V \in \mathcal{B}_{n/2}(\mathbb{R}^n)$ with $n \ge 3$ and ρ is the auxiliary function determined by the potential V (see, for example, [14] or (1.1) below), then the spaces $\mathrm{BMO}_\rho(\mathbb{R}^n)$ and $H^1_\rho(\mathbb{R}^n)$ are just, respectively, the spaces $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ and $H^1_{\mathcal{L}}(\mathbb{R}^n)$.

On the other hand, Guo, Li and Peng [10] proved that for $n \ge 3$, $V \in \mathcal{B}_q(\mathbb{R}^n)$ with $q \in (n/2, n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$, the commutators of Riesz transforms $\nabla\mathcal{L}^{-1/2}$ are bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, p_0]$ with $1/p_0 = 1/q - 1/n$; and commutators of the adjoint Riesz transforms are bounded on $L^p(\mathbb{R}^n)$ for $p \in [p_0', \infty)$. Unlike the classical case, they gave a function f which is not in $\mathrm{BMO}(\mathbb{R}^n)$, while its commutators with the adjoint Riesz transforms associated with $\mathcal{L}$ are bounded on $L^2(\mathbb{R}^n)$. It is well known that $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$. Thus, this example mentioned above further implies that $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ can not be characterized by commutators of Riesz transforms associated with $\mathcal{L}$.

Let ρ be an admissible function. Different from the paper [10], in this paper, we characterize the space $\mathrm{BMO}_\rho(\mathbb{R}^n)$ via several classes of commutators of different localized operators associated to ρ, respectively; these localized operators include localized Riesz transforms $\{\widetilde{R}_j\}_{j=1}^n$ and their adjoint operators $\{\widetilde{R}_j^*\}_{j=1}^n$, the localized fractional integral and its adjoint operator, the localized fractional maximal operator and the localized Hardy-Littlewood-type maximal operator. In particular, if ρ is as in (1.1) below, then these results give new characterizations of $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ in terms of these localized commutators as above.

To be precise, we begin with recalling some necessary notions and notation.

Definition 1.1 ([18]) A positive function ρ on $\mathbb{R}^n$ is called admissible if there exist positive constants $\widetilde{C}$ and k_0 such that for all $x,\, y \in \mathbb{R}^n$,

$$\rho(y) \le \widetilde{C}[\rho(x)]^{1/(1+k_0)}[\rho(x) + |x-y|]^{k_0/(1+k_0)}.$$

Obviously, constant functions are admissible. Moreover, let $s \in (-\infty, 1)$ and $\rho(y) \equiv (1+|y|)^s$ for all $y \in \mathbb{R}^n$. Then $\rho(y)$ also satisfies Definition 1.1 with $k_0 = s/(1-s)$ when $s \in [0,1)$ and $k_0 = -s$ when $s \in (-\infty, 0)$. Another non-trivial class of admissible functions is given by the well-known reverse Hölder class $\mathcal{B}_q(\mathbb{R}^n)$. Recall that a nonnegative potential V is said to be in $\mathcal{B}_q(\mathbb{R}^n)$ with $q \in (1,\, \infty]$ if there exists a positive constant C such that for all open balls B of $\mathbb{R}^n$,

$$\left(\frac{1}{|B|}\int_B [V(y)]^q\,dy\right)^{1/q} \le \frac{C}{|B|}\int_B V(y)\,dy$$

with the usual modification made when $q=\infty$. It is known that if $V \in \mathcal{B}_q(\mathbb{R}^n)$ for certain $q \in (1,\, \infty]$, then V is an $A_\infty(\mathbb{R}^n)$ weight in the sense of Muckenhoupt, and also $V \in \mathcal{B}_{q+\epsilon}(\mathbb{R}^n)$ for certain $\epsilon > 0$; see, for example, [15]. Thus $\mathcal{B}_q(\mathbb{R}^n) = \cup_{q_1 > q}\mathcal{B}_{q_1}(\mathbb{R}^n)$. For any $V \in \mathcal{B}_q(\mathbb{R}^n)$ with certain $q \in (1,\, \infty]$ and all $x \in \mathbb{R}^n$, set

$$\rho(x) \equiv [m(x,V)]^{-1} \equiv \sup\left\{r > 0 : \frac{r^2}{|B(x,\, r)|}\int_{B(x,\, r)} V(y)\,dy \le 1\right\}; \tag{1.1}$$

see, for example, [14]. It was proved in [14] that ρ in (1.1) is an admissible function if $n \ge 3$ and $V \in \mathcal{B}_{n/2}(\mathbb{R}^n)$. Moreover, as pointed out in [18], ρ is admissible if $n \ge 1$, $q > \max\{1,\, n/2\}$ and $V \in \mathcal{B}_q(\mathbb{R}^n)$.

We next recall the notion of the space $\mathrm{BMO}_\rho(\mathbb{R}^n)$ in [16] associated to a given admissible function ρ. Throughout this paper, $\mathcal{D}$ denotes the set of all open balls $B(x,\, r)$ such that $r \ge \rho(x)$.

Definition 1.2 Let ρ be an admissible function. A function $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be in the space $\mathrm{BMO}_\rho(\mathbb{R}^n)$ if there exists a nonnegative constant C such that for all open balls $B \notin \mathcal{D}$,

$$\frac{1}{|B|}\int_B |f(y) - f_B|\,dy \le C,$$

and for all open balls $B \in \mathcal{D}$,

$$\frac{1}{|B|}\int_B |f(y)|\,dy \le C, \tag{1.2}$$

where and in what follows, f_B denotes the mean of f over B for any ball B, that is, $f_B \equiv \frac{1}{|B|}\int_B f(x)\,dx$. Moreover, the minimal constant C as above is defined to be the norm of f in the space $\mathrm{BMO}_\rho(\mathbb{R}^n)$ and denoted by $\|f\|_{\mathrm{BMO}_\rho(\mathbb{R}^n)}$.

Remark 1.1 If ρ is in (1.1), the space $\mathrm{BMO}_\rho(\mathbb{R}^n)$ is just the space $\mathrm{BMO}_{\mathcal{L}}(\mathbb{R}^n)$ in [5]. When ρ is an admissible function as in Definition 1.1, it was proved in [16] that $\mathrm{BMO}_\rho(\mathbb{R}^n)$ also satisfies the John-Nirenberg inequality.

The organization of the paper is as follows.

In Section 2, letting ρ be an admissible function on $\mathbb{R}^n$ and $j \in \{1, \cdots, n\}$, we show that $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$ if and only if b satisfies (1.2) and $[b, \widetilde{R}_j]$ (or $[b, \widetilde{R}_j^*]$) is bounded on $L^q(\mathbb{R}^n)$ for all (or certain) $q \in (1, \infty)$. We remark that in fact, we prove that if $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[b, \widetilde{R}_j]$ (or $[b, \widetilde{R}_j^*]$) is bounded on $L^q(\mathbb{R}^n)$ for all $q \in (1, \infty)$. From this, it follows that the fact that $[b, \widetilde{R}_j]$ (or $[b, \widetilde{R}_j^*]$) is bounded on $L^q(\mathbb{R}^n)$ for all (or certain) $q \in (1, \infty)$ is not enough to guarantee that $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$, due to the fact that $\mathrm{BMO}_\rho(\mathbb{R}^n) \subsetneq \mathrm{BMO}(\mathbb{R}^n)$. To be precise, the boundedness of these commutators can not control the behavior of $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$ on big balls, namely, balls in $\mathcal{D}$, since these commutators are localized. In other words, by (6.5) in [14] and definitions of $\{\widetilde{R}_j\}_{j=1}^n$ and $\{\widetilde{R}_j^*\}_{j=1}^n$ in Section 2 below as well as classical Riesz transforms, we see that kernels of $\{\widetilde{R}_j\}_{j=1}^n$ and $\{\widetilde{R}_j^*\}_{j=1}^n$ decay faster over balls in $\mathcal{D}$ than the kernels of the classical Riesz transforms, but slower than the kernels of $\nabla \mathcal{L}^{-1/2}$. Also in this section, we introduce the localized fractional integral I_α^ρ and its adjoint operator $I_\alpha^{\rho,*}$ with $\alpha \in (0, n)$ and prove that $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$ if and only if b satisfies (1.2) and for all (or certain) $p,\ q \in (1, \infty)$ with $1/q = 1/p - \alpha/n$, $[b, I_\alpha^{\rho,*}]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ or $[b, I_\alpha^\rho]$ is bounded from $L^{q'}(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$.

In Section 3, we introduce localized versions of the fractional maximal operator $\mathcal{M}_\beta^{\mathrm{loc}}$ with $\beta \in (0, 1)$ and the Hardy-Littlewood-type maximal operator $\mathcal{M}^{\mathrm{loc},p}$ with $p \in [1, \infty)$, and establish equivalent characterizations for $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$ in terms of commutators of $\mathcal{M}_\beta^{\mathrm{loc}}$ and $\mathcal{M}^{\mathrm{loc},p}$, respectively. Especially, we prove that $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$ if and only if b satisfies (1.2) and $\mathcal{M}_{b,\beta}^{\mathrm{loc}}$, the commutator of $\mathcal{M}_\beta^{\mathrm{loc}}$, is bounded from $L^r(\mathbb{R}^n)$ to $L^s(\mathbb{R}^n)$ for all (or certain) $r \in (1, 1/\beta)$ and $s \in (1, \infty)$ with $1/s = 1/r - \beta$, and if and only if b satisfies (1.2) and $\mathcal{M}_b^{\mathrm{loc},p}$, the commutator of $\mathcal{M}^{\mathrm{loc},p}$, is bounded on $L^q(\mathbb{R}^n)$ for all (or certain) $q \in (p, \infty)$.

We now make some conventions. Throughout this paper, we always use C to denote a positive constant that is independent of the main parameters involved but whose value may differ from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. If $f \le Cg$, we then write $f \lesssim g$ or $g \gtrsim f$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$. Denote any ball B of $\mathbb{R}^n$ by $B \equiv B(x_B, r_B)$, where $x_B \in \mathbb{R}^n$ is its center and $r_B > 0$ its radius. For any ball $B \equiv B(x_B, r_B) \subset \mathbb{R}^n$ and $\lambda > 0$, $\lambda B \equiv B(x_B, \lambda r_B)$. Also, χ_E denotes the characteristic function of any set $E \subset \mathbb{R}^n$.

2. Localized commutators

Let ρ be a given admissible function. In this section, we establish characterizations of $\mathrm{BMO}_\rho(\mathbb{R}^n)$ via commutators of localized Riesz transforms and their adjoint operators in [17], as well as localized fractional integrals and their adjoint operators associated with a given admissible function ρ. We begin with definitions of localized Riesz transforms and their adjoint operators.

Let ρ be an admissible function as in Definition 1.1. For all $j \in \{1, \cdots, n\}$,

$f \in \cup_{p=1}^{\infty} L^p(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$\widetilde{R}_j(f)(x) \equiv \mathrm{p.\,v.}\, c_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} \eta\left(\frac{|x-y|}{\rho(x)}\right) f(y)\, dy, \tag{2.1}$$

where and in what follows, $c_n \equiv \Gamma((n+1)/2)/[\pi^{(n+1)/2}]$, $\eta \in C^1(\mathbb{R})$ supported in $(-1, 1)$ and $\eta(t) = 1$ if $|t| \le 1/2$. The adjoint operators of $\widetilde{R}_j$, $j \in \{1, \cdots, n\}$, have the forms

$$\widetilde{R}_j^*(f)(x) \equiv -\mathrm{p.\,v.}\, c_n \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} \eta\left(\frac{|x-y|}{\rho(y)}\right) f(y)\, dy,$$

where f and x are the same as in (2.1).

The following lemma was established in [17].

Lemma 2.1

Let ρ be an admissible function, and for all $j \in \{1, \cdots, n\}$, let $\widetilde{R}_j$ and $\widetilde{R}_j^*$ be defined as above. Then $\widetilde{R}_j$ and $\widetilde{R}_j^*$ are bounded on $L^p(\mathbb{R}^n)$ for $p \in (1, \infty)$ and bounded from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$.

Let $q \in [1, \infty)$, $x \in \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The classical maximal functions $\mathcal{M}^\sharp(f)$ and $\mathcal{M}_p(f)$ are, respectively, defined by

$$\mathcal{M}^\sharp(f)(x) \equiv \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y) - f_B|\, dy,$$

and

$$\mathcal{M}_p(f)(x) \equiv \left[\sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|^p\, dy\right]^{1/p}.$$

Notice that when $p = 1$, $\mathcal{M}_p$ is the classical Hardy-Littlewood maximal function and we denote $\mathcal{M}_p$ simply by $\mathcal{M}$.

Lemma 2.2

Let ρ be an admissible function, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $j \in \{1, \cdots, n\}$ and $p, r \in (1, \infty)$. Then there exists a positive constant C such that for all $f \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}^\sharp\left(\left[b, \widetilde{R}_j^*\right] f\right)(x) \le C \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \left[\mathcal{M}_p\left[\widetilde{R}_j^*(f)\right](x) + \mathcal{M}_{pr}(f)(x)\right].$$

Proof. By the homogeneity of BMO($\mathbb{R}^n$), we may assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. Let $x \in \mathbb{R}^n$, $B \equiv B(x_0, r_0)$ be any ball containing x, $f_1 \equiv f\chi_{2B}$ and $f_2 \equiv f\chi_{\mathbb{R}^n \setminus (2B)}$. We first prove that for all $j \in \{1, \cdots, n\}$ and all B as above,

$$\frac{1}{|B|} \int_B \left|\left[b, \widetilde{R}_j^*\right](f)(y) + \widetilde{R}_j^*\left[(b - b_B) f_2\right](x_0)\right| dy \lesssim \mathcal{M}_p\left[\widetilde{R}_j^*(f)\right](x) + \mathcal{M}_{pr}(f)(x). \tag{2.2}$$

From the linearity of $\widetilde{R}_j^*$, we deduce that for all $y \in \mathbb{R}^n$,

$$\begin{aligned}\left[b, \widetilde{R}_j^*\right](f)(y) &= (b(y) - b_B)\widetilde{R}_j^*(f)(y) - \widetilde{R}_j^*\left[(b - b_B) f_1\right](y) - \widetilde{R}_j^*\left[(b - b_B) f_2\right](y)\\ &\equiv \mathrm{I}_1(y) + \mathrm{I}_2(y) + \mathrm{I}_3(y).\end{aligned}$$

By the Hölder inequality and the John-Nirenberg inequality (see Remark 1.1), we have that

$$\begin{aligned}\frac{1}{|B|}\int_B |\mathrm{I}_1(y)|\,dy &\le \left[\frac{1}{|B|}\int_B |b(y)-b_B|^{p'}\,dy\right]^{1/p'}\left[\frac{1}{|B|}\int_B \left|\widetilde{R}_j^*(f)(y)\right|^p\,dy\right]^{1/p}\\ &\lesssim \mathcal{M}_p\left[\widetilde{R}_j^*(f)\right](x). \qquad (2.3)\end{aligned}$$

On the other hand, by Lemma 2.1, we obtain that for all $j\in\{1,\cdots,n\}$, $\widetilde{R}_j^*$ are bounded on $L^r(\mathbb{R}^n)$ for all $r\in(1,\infty)$. It then follows from this and the Hölder inequality that

$$\begin{aligned}\frac{1}{|B|}\int_B |\mathrm{I}_2(y)|\,dy &\lesssim \left[\frac{1}{|B|}\int_{2B} |[b(y)-b_B]f(y)|^r\,dy\right]^{1/r}\\ &\lesssim \left[\frac{1}{|B|}\int_{2B} |b(y)-b_B|^{p'r}\,dy\right]^{1/(p'r)}\left[\frac{1}{|B|}\int_{2B} |f(y)|^{pr}\,dy\right]^{1/(pr)}\\ &\lesssim \mathcal{M}_{pr}(f)(x). \qquad (2.4)\end{aligned}$$

By Definition 1.1, for any given positive constant a, there exists a positive constant $\widetilde{C}_a\in[1,\infty)$ such that for all $y\in\mathbb{R}^n$ and $x\in B(y,\,a\rho(y))$,

$$\frac{1}{\widetilde{C}_a}\rho(y)\le\rho(x)\le\widetilde{C}_a\rho(y). \qquad (2.5)$$

We denote the kernel of $\widetilde{R}_j^*$ still by $\widetilde{R}_j^*$. Let $y\in B$. From (2.5), we deduce that $\operatorname{supp}(\widetilde{R}_j^*(y,\cdot))\subset B(y,\widetilde{C}_1\rho(y))$. Because $x_0,\ y\in B$, this together with another application of (2.5) yields that there exists positive constant C_1 such that $\operatorname{supp}(\widetilde{R}_j^*(y,\cdot))\subset B(x_0,C_1\rho(x_0))$ if $B\notin\mathcal{D}$ and $\operatorname{supp}(\widetilde{R}_j^*(y,\cdot))\subset B(x_0,C_1r_0)$ if $B\in\mathcal{D}$. From this, the Hölder inequality and the John-Nirenberg inequality, it follows that

$$\begin{aligned}&|\mathrm{I}_3(y)-\mathrm{I}_3(x_0)|\\ &\quad\le\int_{[C_1(\rho(x_0)/r_0+1)B]\setminus(2B)}|\widetilde{R}_j^*(y,z)-\widetilde{R}_j^*(x_0,z)||b(z)-b_B||f(z)|\,dz\\ &\quad\lesssim\int_{[C_1(\rho(x_0)/r_0+1)B]\setminus(2B)}\frac{|y-x_0||b(z)-b_B||f(z)|}{|x_0-z|^{n+1}}\,dz\\ &\quad\lesssim\sum_{k=1}^{\infty}\left[\frac{r_0}{(2^kr_0)^{n+1}}\int_{(2^{k+1}B)\setminus(2^kB)}|b(z)-b_{2^{k+1}B}|\,|f(z)|\,dz+\frac{|b_B-b_{2^{k+1}B}|}{2^k}\mathcal{M}(f)(x)\right]\\ &\quad\lesssim\sum_{k=1}^{\infty}\left[\frac{r_0}{(2^kr_0)^{n+1}}\left\|[b-b_{2^{k+1}B}]\,\chi_{2^{k+1}B}\right\|_{L^{(pr)'}(\mathbb{R}^n)}\left\|f\chi_{2^{k+1}B}\right\|_{L^{(pr)}(\mathbb{R}^n)}+\frac{k}{2^k}\mathcal{M}(f)(x)\right]\\ &\quad\lesssim\mathcal{M}_{pr}(f)(x).\end{aligned}$$

This together with estimates of I_1 and I_2 yields (2.2), which implies that

$$\frac{1}{|B|}\int_B\left|\left[b,\widetilde{R}_j^*\right](f)(y)-\left(\left[b,\widetilde{R}_j^*\right](f)\right)_B\right|\,dy\lesssim\mathcal{M}_p\left[\widetilde{R}_j^*(f)\right](x)+\mathcal{M}_{pr}(f)(x),$$

and hence, completes the proof of Lemma 2.2. □

Theorem 2.1

Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, ρ be an admissible function and $j \in \{1, \cdots, n\}$. Then the following assertions are equivalent:

(i) $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$.

(ii) *b satisfies* (1.2) *and $[b, \widetilde{R}_j^*]$ is bounded on $L^q(\mathbb{R}^n)$ for all $q \in (1, \infty)$.*

(iii) *b satisfies* (1.2) *and $[b, \widetilde{R}_j^*]$ is bounded on $L^q(\mathbb{R}^n)$ for certain $q \in (1, \infty)$.*

(iv) *b satisfies* (1.2) *and $[b, \widetilde{R}_j]$ is bounded on $L^q(\mathbb{R}^n)$ for all $q \in (1, \infty)$.*

(v) *b satisfies* (1.2) *and $[b, \widetilde{R}_j]$ is bounded on $L^q(\mathbb{R}^n)$ for certain $q \in (1, \infty)$.*

Proof. We now show that (i) implies (ii). We first claim that if $b \in \mathrm{BMO}(\mathbb{R}^n)$, then $[b, \widetilde{R}_j^*]$ is bounded on $L^q(\mathbb{R}^n)$ for all $q \in (1, \infty)$. Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. For each $N \in \mathbb{N}$, define $b_N \equiv \min\{N, |b|\} sgn(b)$. Then $b_N \in L^\infty(\mathbb{R}^n)$ and $\|b_N\|_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Furthermore, by Lemma 2.1, we see that for all $f \in C_c^\infty(\mathbb{R}^n)$, $[b_N, \widetilde{R}_j^*](f) \in L^q(\mathbb{R}^n)$ and $\|[b_N, \widetilde{R}_j^*](f)\|_{L^q(\mathbb{R}^n)} \lesssim N\|f\|_{L^q(\mathbb{R}^n)}$. On the other hand, recall that for all $f \in L^q(\mathbb{R}^n)$, $\mathcal{M}^\sharp(f) \in L^q(\mathbb{R}^n)$ and

$$\|f\|_{L^q(\mathbb{R}^n)} \lesssim \|\mathcal{M}^\sharp(f)\|_{L^q(\mathbb{R}^n)}; \tag{2.6}$$

see, for example, [15, p. 148]. Let $p,\ r \in (1, \infty)$ with $pr < q$. By these facts and Lemma 2.2 together with the boundedness of $\mathcal{M}_p$ and $\mathcal{M}_{pr}$ on $L^q(\mathbb{R}^n)$ for $q > pr$ and Lemma 2.1, we have that for all $f \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\left\|\left[b_N, \widetilde{R}_j^*\right](f)\right\|_{L^q(\mathbb{R}^n)} &\lesssim \left\|\mathcal{M}^\sharp\left(\left[b_N, \widetilde{R}_j^*\right](f)\right)\right\|_{L^q(\mathbb{R}^n)} \\ &\lesssim \|b_N\|_{\mathrm{BMO}(\mathbb{R}^n)} \left\|\mathcal{M}_p\left[\widetilde{R}_j^*(f)\right] + \mathcal{M}_{pr}(f)\right\|_{L^q(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^q(\mathbb{R}^n)}.\end{aligned}$$

A standard argument together with the Fatou lemma (see, for example, [9, p. 564]) then leads to that for all $f \in L^q(\mathbb{R}^n), [b, \widetilde{R}_j^*](f) \in L^q(\mathbb{R}^n)$ and

$$\|[b, \widetilde{R}_j^*](f)\|_{L^q(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^q(\mathbb{R}^n)}.$$

This implies the claim. From this, Definition 1.2 and the fact that $\mathrm{BMO}_\rho(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$, we deduce that (ii) holds.

Since the implications from (ii) to (iii) and from (iv) to (v) are obvious, and the implications from (ii) to (iv) and from (iii) to (v) follow from a standard duality argument, Theorem 2.1 is reduced to showing that (v) implies (i). To this end, let $\{R_j\}_{j=1}^n$ be the kenels of the classical Riesz transforms. Observe that for all $j \in \{1, \cdots, n\}$, $1/R_j \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Therefore, there exist $z_0 \in \mathbb{R}^n \setminus \{0\}$ and $\delta \in (0, \infty)$ such that $1/R_j(z)$ is expressed as an absolutely convergent Fourier series in $B(z_0, \delta)$ (see, for example, [9, Theorem 3.2.16]). That is, there exist $\{\nu_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and numbers $\{a_k\}_{k\in\mathbb{N}}$ with $\sum_{k=1}^\infty |a_k| < \infty$ such that for all $z \in B(z_0, \delta)$, $1/R_j(z) = \sum_{k=1}^\infty a_k e^{i\nu_k\cdot z}$. Let $z_1 \equiv \delta^{-1}z_0$. If $|z - z_1| < 1$, then we have that $|\delta z - z_0| < \delta$ and

$$\frac{1}{R_j(z)} = \frac{\delta^{-n}}{R_j(\delta z)} = \delta^{-n}\sum_{k=1}^\infty a_k e^{i\nu_k\cdot(\delta z)}. \tag{2.7}$$

Let $B \equiv B(\widetilde{x}, r) \notin \mathcal{D}$ being any ball and $C_2 \equiv [4(1+|z_1|)\widetilde{C}_1]$, where $\widetilde{C}_1$ is as in (2.5) with $a = 1$. Then we have that $r < \rho(\widetilde{x})$. To show (i), we first consider the case that $r < \rho(\widetilde{x})/C_2$. Let $\widetilde{y} \equiv \widetilde{x} - 2rz_1$ and $\widetilde{B} \equiv B(\widetilde{y}, r)$. Then we obtain that for all $x \in B$ and $y \in \widetilde{B}$,

$$\left|\frac{x-y}{2r} - z_1\right| \le \frac{|x-\widetilde{x}|}{2r} + \frac{|y-\widetilde{y}|}{2r} < 1 \tag{2.8}$$

and $|\widetilde{x} - x| < r < \rho(\widetilde{x})$, which together with (2.5) in turn implies that $\rho(\widetilde{x}) \le \widetilde{C}_1 \rho(x)$. Therefore, we see that

$$|x-y| \le |x-\widetilde{x}| + |\widetilde{x}-\widetilde{y}| + |\widetilde{y}-y| < \frac{\rho(\widetilde{x})}{2\widetilde{C}_1} \le \frac{\rho(x)}{2}. \tag{2.9}$$

Observe that for all $x,\ y \in \mathbb{R}^n$ with $|x-y| < \rho(x)/2$ and

$$j \in \{1, \cdots, n\},\ \widetilde{R}_j(x,y) = R_j(x-y).$$

From this, (2.7), (2.8), the Hölder inequality and the boundedness of $[b, \widetilde{R}_j]$ on $L^q(\mathbb{R}^n)$, we then deduce that

$$\begin{aligned}
&\int_B |b(x) - b_{\widetilde{B}}|\,dx \\
&= \int_{\mathbb{R}^n} [b(x) - b_{\widetilde{B}}] sgn(b - b_{\widetilde{B}}) \chi_B(x)\,dx \\
&= \frac{1}{|\widetilde{B}|} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [b(x) - b(y)] sgn(b(x) - b_{\widetilde{B}}) \chi_B(x)\chi_{\widetilde{B}}(y) \frac{(2r)^n \widetilde{R}_j(x,y)}{R_j(\frac{x-y}{2r})}\,dy\,dx \\
&\lesssim \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [b(x) - b(y)] sgn(b(x) - b_{\widetilde{B}}) \chi_B(x)\chi_{\widetilde{B}}(y) \widetilde{R}_j(x,y) \sum_{k=1}^{\infty} a_k e^{i(\delta\nu_k)/2r\cdot(x-y)}\,dy\,dx \\
&\lesssim \sum_{k=1}^{\infty} |a_k| \int_{\mathbb{R}^n} \left|\left[b, \widetilde{R}_j\right]\left(\chi_{\widetilde{B}} e^{-i(\delta\nu_k)/2r}\right)(x)\right| \chi_B(x)\,dx \\
&\lesssim \sum_{k=1}^{\infty} |a_k| \left\|\left[b, \widetilde{R}_j\right]\left(\chi_{\widetilde{B}} e^{-i(\delta\nu_k)/2r}\right)\right\|_{L^q(\mathbb{R}^n)} \|\chi_B\|_{L^{q'}(\mathbb{R}^n)} \lesssim |B|,
\end{aligned}$$

which implies that

$$\int_B |b(x) - b_B|\,dx \lesssim |B|. \tag{2.10}$$

Now consider the case that $r \in [\rho(\widetilde{x})/C_2, \rho(\widetilde{x}))$. In this case, we have that $C_2 B \in \mathcal{D}$. From this fact together with $C_2 > 1$ and (1.2), we deduce that

$$\frac{1}{|B|}\int_B |b(y) - b_B|\,dy \lesssim \frac{1}{|B|}\int_B |b(y)|\,dy \lesssim \frac{1}{|C_2 B|}\int_{C_2 B} |b(y)|\,dy \lesssim 1.$$

This and the estimate for the case that $r < \rho(\widetilde{x})/C_2$ imply that (2.10) holds for all balls $B \notin \mathcal{D}$, which together with (1.2) further yields (i), and hence, completes the proof of Theorem 2.1. $\square$

We now consider the fractional integral I_α^ρ. Let $\alpha \in (0, n)$, η be as in (2.1) and ρ an admissible function. For all $f \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define I_α^ρ, its adjoint operator

$I_\alpha^{\rho,*}$ and the classical fractional integral I_α, respectively, by

$$I_\alpha^\rho(f)(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \eta\left(\frac{|x-y|}{\rho(x)}\right) dy,$$
$$I_\alpha^{\rho,*}(f)(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \eta\left(\frac{|x-y|}{\rho(y)}\right) dy$$

and

$$I_\alpha(f)(x) \equiv \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Moreover, for all $A \in (0,\infty)$ and $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$, let

$$I_{\alpha,b}^{\rho,A}(f)(x) \equiv \int_{\mathbb{R}^n} \frac{|b(x)-b(y)||f(y)|}{|x-y|^{n-\alpha}} \eta\left(\frac{|x-y|}{A\rho(y)}\right) dy.$$

If $A = 1$, we denote $I_{\alpha,b}^{\rho,A}$ simply by $I_{\alpha,b}^\rho$. It is easy to see that

$$|[b, I_\alpha^{\rho,*}](f)| \le I_{\alpha,b}^\rho(f). \tag{2.11}$$

Let $p \in [1,\infty)$ and $\beta \in (0, 1/p)$. The fractional maximal function $\mathcal{M}_p^\beta(f)$ is defined by setting, for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}_p^\beta(f)(x) \equiv \left[\sup_{B \ni x} \frac{1}{|B|^{1-p\beta}} \int_B |f(y)|^p\, dy\right]^{1/p}.$$

If $p = 1$, we denote $\mathcal{M}_p^\beta$ simply by $\mathcal{M}^\beta$.

Lemma 2.3

Let ρ be an admissible function, $b \in \mathrm{BMO}(\mathbb{R}^n)$, $A \in (0,\infty)$, $\alpha \in (0,n)$ and $p, r \in (1,\infty)$ with $pr < n/\alpha$. Then there exists a positive constant C such that for all $f \in C_c^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{M}^\sharp\left(I_{\alpha,b}^{\rho,A}(f)\right)(x) \le C\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \left[\mathcal{M}_p(I_\alpha(|f|))(x) + \mathcal{M}_{pr}^{\alpha/n}(f)(x)\right].$$

Proof. By the homogeneity of $\mathrm{BMO}(\mathbb{R}^n)$, we may assume that $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = 1$. Let $B \equiv B(x_0, r_0)$ being any ball containing x, $f_1 \equiv f\chi_{2B}$ and $f_2 \equiv f\chi_{\mathbb{R}^n \setminus 2B}$. For all $x \in \mathbb{R}^n$, let

$$\mathrm{J}_0(x) \equiv \int_{\mathbb{R}^n} \frac{|b(z)-b_B||f_2(z)|}{|x-z|^{n-\alpha}} \eta\left(\frac{|x-z|}{A\rho(z)}\right) dz.$$

Then, for all $y \in B$, we have

$$\begin{aligned}\left|I_{\alpha,b}^{\rho,A}(f)(y) - \mathrm{J}_0(x_0)\right| &\le |b(y)-b_B| I_\alpha(|f|)(y) + I_\alpha\left[|(b-b_B)f_1|\right](y) + |\mathrm{J}_0(y) - \mathrm{J}_0(x_0)| \\ &\equiv \mathrm{J}_1(y) + \mathrm{J}_2(y) + |\mathrm{J}_0(y) - \mathrm{J}_0(x_0)|.\end{aligned}$$

By the Hölder inequality and the boundedness of I_α from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ with $1/p - 1/q = \alpha/n$, we obtain

$$\frac{1}{|B|}\int_B [\mathrm{J}_1(y) + \mathrm{J}_2(y)]\, dy \lesssim \mathcal{M}_p\left[I_\alpha(|f|)\right](x) + \mathcal{M}_{pr}^{\alpha/n}(f)(x). \tag{2.12}$$

Using an argument as in the proof of Lemma 2.2 via the Hölder inequality and the John-Nirenberg inequality, we have that for all $y \in B$,

$$\begin{aligned}|\mathrm{J}_0(y) - \mathrm{J}_0(x_0)| &\lesssim \int_{[C_3(\rho(x_0)/r_0+1)B]\setminus(2B)} \frac{|y-x_0||b(z)-b_B||f(z)|}{|x_0-z|^{n-\alpha+1}}\,dz \\ &\lesssim \sum_{k=1}^{\infty}\left[\frac{r_0}{(2^k r_0)^{n-\alpha+1}}\int_{(2^{k+1}B)\setminus(2^kB)}|b(z)-b_{2^{k+1}B}||f(z)|\,dz\right. \\ &\quad \left. + \frac{|b_B - b_{2^{k+1}B}|}{2^k}\mathcal{M}^{\alpha/n}(f)(x)\right] \lesssim \mathcal{M}^{\alpha/n}_{pr}(f)(x),\end{aligned}$$

where $C_3 \in [1,\infty)$. This further yields that

$$\frac{1}{|B|}\int_B |\mathrm{J}_0(y) - \mathrm{J}_0(x_0)|\,dy \lesssim \mathcal{M}^{\alpha/n}_{pr}(f)(x).$$

Combining this and (2.12), we have that

$$\frac{1}{|B|}\int_B \left|I^{\rho,\,A}_{\alpha,\,b}(f)(y) - \mathrm{J}_0(x_0)\right|\,dy \lesssim \mathcal{M}_p\left[I_\alpha(|f|)\right](x) + \mathcal{M}^{\alpha/n}_{pr}(f)(x),$$

which implies that

$$\frac{1}{|B|}\int_B \left|I^{\rho,\,A}_{\alpha,\,b}(f)(y) - \left[I^{\rho,\,A}_{\alpha,\,b}(f)\right]_B\right|\,dy \lesssim \mathcal{M}_p\left[I_\alpha(|f|)\right](x) + \mathcal{M}^{\alpha/n}_{pr}(f)(x).$$

This finishes the proof of Lemma 2.3. □

Theorem 2.2

Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, ρ be an admissible function and $\alpha \in (0,n)$. Then the following assertions are equivalent:

(i) $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$.

(ii) *b satisfies (1.2) and $[b, I^{\rho,\,*}_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all p, $q \in (1,\infty)$ with $1/q = 1/p - \alpha/n$.*

(iii) *b satisfies (1.2) and $[b, I^{\rho,\,*}_\alpha]$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for certain p, $q \in (1,\infty)$ with $1/q = 1/p - \alpha/n$.*

(iv) *b satisfies (1.2) and $[b, I^{\rho}_\alpha]$ is bounded from $L^{q'}(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ for all p, $q \in (1,\infty)$ with $1/q = 1/p - \alpha/n$.*

(v) *b satisfies (1.2) and $[b, I^{\rho}_\alpha]$ is bounded from $L^{q'}(\mathbb{R}^n)$ to $L^{p'}(\mathbb{R}^n)$ for certain p, $q \in (1,\infty)$ with $1/q = 1/p - \alpha/n$.*

Proof. As in the proof of Theorem 2.1, we only need to prove that (i) implies (ii) and that (v) implies (i). To show that (i) implies (ii), let $b \in \mathrm{BMO}(\mathbb{R}^n)$. We now show that for all $f \in C^\infty_c(\mathbb{R}^n)$, $[b, I^{\rho,\,*}_\alpha] \in L^q(\mathbb{R}^n)$ and

$$\|[b, I^{\rho,\,*}_\alpha](f)\|_{L^q(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}. \tag{2.13}$$

For each $N \in \mathbb{N}$, let b_N be as in Theorem 2.1. Then $b_N \in L^\infty(\mathbb{R}^n)$ and $\|b_N\|_{\mathrm{BMO}(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$. Furthermore, by the boundedness of I_α from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all p,

$q \in (1, \infty)$ with $1/q = 1/p - \alpha/n$, we see that for all $f \in C_c^\infty(\mathbb{R}^n)$, $I^\rho_{\alpha,\, b_N}(f) \in L^q(\mathbb{R}^n)$ and

$$\left\| I^\rho_{\alpha,\, b_N}(f) \right\|_{L^q(\mathbb{R}^n)} \lesssim \|b_N\|_{L^\infty(\mathbb{R}^n)} \|I_\alpha(|f|)\|_{L^q(\mathbb{R}^n)} \lesssim N \|f\|_{L^p(\mathbb{R}^n)}.$$

Using (2.11), (2.6), Lemma 2.3 with $A = 1$, the boundedness of $\mathcal{M}_s$ on $L^q(\mathbb{R}^n)$ and the facts that I_α and $\mathcal{M}^{\alpha/n}_{sr}$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all $1 < sr < p < n/\alpha$ and $1/q = 1/p - \alpha/n$ (see [1, Lemma 2]), we have that for all $f \in C_c^\infty(\mathbb{R}^n)$,

$$\begin{aligned} \|[b_N, I^{\rho,\,*}_\alpha](f)\|_{L^q(\mathbb{R}^n)} &\le \left\| I^\rho_{\alpha,\, b_N}(f) \right\|_{L^q(\mathbb{R}^n)} \lesssim \left\| \mathcal{M}^\sharp \left[I^\rho_{\alpha,\, b_N}(f) \right] \right\|_{L^q(\mathbb{R}^n)} \\ &\lesssim \|b_N\|_{\mathrm{BMO}(\mathbb{R}^n)} \left\| \mathcal{M}_s(I_\alpha(|f|)) + \mathcal{M}^{\alpha/n}_{sr}(f) \right\|_{L^q(\mathbb{R}^n)} \\ &\lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{2.14}$$

Then using an argument similar to that used in the proof of Theorem 2.1, we see that (2.13) holds for all $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $f \in L^p(\mathbb{R}^n)$ with $1/q = 1/p - \alpha/n$. This together with $\mathrm{BMO}_\rho(\mathbb{R}^n) \subset \mathrm{BMO}(\mathbb{R}^n)$ implies (ii).

Now assume that (v) holds. Let $I_\alpha(x) \equiv 1/|x|^{n-\alpha}$ for all $x \in \mathbb{R}^n \setminus \{0\}$ and $I^\rho_\alpha(x, y)$ for all $x,\, y \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{x = y\}$ be the kernels of I_α and I^ρ_α, respectively. As in the proof of (2.7), we then see that $1/I_\alpha \in C^\infty(\mathbb{R}^n \setminus \{0\})$, and there exist $z_1 \in \mathbb{R}^n \setminus \{0\}$, $\delta \in (0, \infty)$, $\{\nu_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n$ and numbers $\{a_k\}_{k\in\mathbb{N}}$ with $\sum_{k=1}^\infty |a_k| < \infty$ such that for all $z \in \mathbb{R}^n$ with $|z - z_1| < 1$,

$$\frac{1}{I_\alpha(z)} = \frac{\delta^{\alpha-n}}{I_\alpha(\delta z)} = \delta^{\alpha-n} \sum_{k=1}^\infty a_k e^{i\nu_k\cdot(\delta z)}. \tag{2.15}$$

Let $B \equiv B(\widetilde{x}, r) \notin \mathcal{D}$ being any ball and C_2 be as in Theorem 2.1. We first consider the case that $r < \rho(\widetilde{x})/C_2$. Let $\widetilde{y} \equiv \widetilde{x} - 2rz_1$ and $\widetilde{B} \equiv B(\widetilde{y}, r)$. Observe that for all $x,\, y \in \mathbb{R}^n$ with $|x - y| < \rho(x)/2$, $I^\rho_\alpha(x, y) = I_\alpha(x - y)$. From this, (2.9), (2.8) and (2.15), we deduce that

$$\begin{aligned}
&\int_B |b(x) - b_{\widetilde{B}}|\,dx \\
&\quad = \int_B [b(x) - b_{\widetilde{B}}] sgn(b - b_{\widetilde{B}})\,dx \\
&\quad = \frac{1}{|\widetilde{B}|}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [b(x) - b(y)] sgn(b(x) - b_{\widetilde{B}})\chi_B(x)\chi_{\widetilde{B}}(y)\frac{(2r)^{n-\alpha} I_\alpha^\rho(x,y)}{I_\alpha(\frac{x-y}{2r})}\,dy\,dx \\
&\quad \lesssim r^{-\alpha}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} [b(x) - b(y)] sgn(b(x) - b_{\widetilde{B}})\chi_B(x)\chi_{\widetilde{B}}(y) I_\alpha^\rho(x,y) \\
&\qquad \times \sum_{k=1}^\infty a_k e^{i(\delta\nu_k)/2r\cdot(x-y)}\,dy\,dx \\
&\quad \lesssim r^{-\alpha}\sum_{k=1}^\infty |a_k| \int_{\mathbb{R}^n} \left|[b, I_\alpha^\rho]\left(\chi_{\widetilde{B}} e^{-i(\delta\nu_k)/2r}\right)(x)\right| \chi_B(x)\,dx \\
&\quad \lesssim r^{-\alpha}\sum_{k=1}^\infty |a_k| \left\|[b, I_\alpha^\rho]\left(\chi_{\widetilde{B}} e^{-i(\delta\nu_k)/2r}\right)\right\|_{L^q(\mathbb{R}^n)} \|\chi_B\|_{L^{q'}(\mathbb{R}^n)} \\
&\quad \lesssim r^{-\alpha}\sum_{k=1}^\infty |a_k| \|\chi_{\widetilde{B}}\|_{L^p(\mathbb{R}^n)} \|\chi_B\|_{L^{q'}(\mathbb{R}^n)} \lesssim |B|.
\end{aligned}$$

This yields that

$$\int_B |b(x) - b_B|\,dx \lesssim |B|. \tag{2.16}$$

Now we consider the case that $r \in [\rho(\widetilde{x})/C_2, \rho(\widetilde{x}))$. In this case, we have that $C_2 B \in \mathcal{D}$, which together with (1.2) further implies that

$$\frac{1}{|B|}\int_B |b(y) - b_B|\,dy \lesssim \frac{1}{|C_2 B|}\int_{C_2 B} |b(y)|\,dy \lesssim 1.$$

Combining this and the estimate for the case that $r < \rho(\widetilde{x})/C_2$ yields that (2.16) holds for all balls $B \notin \mathcal{D}$, which together with (1.2) further yields (i), and hence, completes the proof of Theorem 2.2. □

3. Localized maximal commutators

Let ρ be a given admissible function. In this section, we consider characterizations of $\mathrm{BMO}_\rho(\mathbb{R}^n)$ in terms of commutators of the localized fractional maximal operator and the localized Hardy-Littlewood-type maximal operator.

Let ρ be an admissible function and $\mathcal{D}$ the set of balls as in Section 1. For any $\beta \in (0,1)$, $x \in \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the localized fractional maximal operator $\mathcal{M}_\beta^{\mathrm{loc}}$ is defined by

$$\mathcal{M}_\beta^{\mathrm{loc}}(f)(x) \equiv \sup_{B\ni x,\, B\notin\mathcal{D}} \frac{1}{|B|^{1-\beta}}\int_B |f(y)|\,dy.$$

Moreover, for any function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define the localized maximal commutator $\mathcal{M}^{\mathrm{loc}}_{b,\beta}$ of $\mathcal{M}^{\mathrm{loc}}_{\beta}$ by

$$\mathcal{M}^{\mathrm{loc}}_{b,\beta}(f)(x) \equiv \sup_{B \ni x,\, B \notin \mathcal{D}} \frac{1}{|B|^{1-\beta}} \int_B |b(x) - b(y)||f(y)|\,dy.$$

Let $\eta \in C^1(\mathbb{R}^n)$ being as in Section 2 and $C_4 \in (4\widetilde{C}_1, \infty)$, where $\widetilde{C}_1$ is as in (2.5) with $a = 1$ there. By the definition of η and (2.5), we see that for all $x \in \mathbb{R}^n$,

$$\begin{aligned} \mathcal{M}^{\mathrm{loc}}_{b,\beta}(f)(x) &= \sup_{B \ni x,\, B \notin \mathcal{D}} \frac{1}{|B|^{1-\beta}} \int_B \left| [b(x) - b(y)] f(y) \eta\left(\frac{|x-y|}{C_4 \rho(y)}\right)\right| dy \qquad (3.1)\\ &\lesssim \int_{\mathbb{R}^n} \frac{|b(x) - b(y)||f(y)|}{|x-y|^{n(1-\beta)}} \eta\left(\frac{|x-y|}{C_4\rho(y)}\right) dy = I^{\rho, C_4}_{n\beta, b}(f)(x). \end{aligned}$$

Theorem 3.1

Let $\beta \in (0,1)$, $r \in (1, 1/\beta)$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then the following assertions are equivalent:

(i) $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$.

(ii) *b satisfies (1.2) and $\mathcal{M}^{\mathrm{loc}}_{b,\beta}$ is bounded from $L^r(\mathbb{R}^n)$ to $L^s(\mathbb{R}^n)$ for all $s \in (r, \infty)$ with $1/s = 1/r - \beta$.*

(iii) *b satisfies (1.2) and $\mathcal{M}^{\mathrm{loc}}_{b,\beta}$ is bounded from $L^r(\mathbb{R}^n)$ to $L^s(\mathbb{R}^n)$ for certain $s \in (r, \infty)$ with $1/s = 1/r - \beta$.*

Proof. We first show that (i) implies (ii). Let $b \in \mathrm{BMO}(\mathbb{R}^n)$. Using the argument similar to the proof for the implication from (i) to (ii) in Theorem 2.2, we see that for all $f \in C^\infty_c(\mathbb{R}^n)$, $I^{\rho, C_4}_{n\beta, b}(f) \in L^s(\mathbb{R}^n)$ and

$$\left\| I^{\rho, C_4}_{n\beta, b}(f) \right\|_{L^s(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^r(\mathbb{R}^n)};$$

see (2.14). This together with (3.1) further yields that for all $f \in C^\infty_c(\mathbb{R}^n)$, $\mathcal{M}^{\mathrm{loc}}_{b,\beta}(f) \in L^s(\mathbb{R}^n)$ and

$$\left\| \mathcal{M}^{\mathrm{loc}}_{b,\beta}(f) \right\|_{L^s(\mathbb{R}^n)} \lesssim \|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \|f\|_{L^r(\mathbb{R}^n)}. \qquad (3.2)$$

Moreover, a standard argument, via the density of $C^\infty_c(\mathbb{R}^n)$ in $L^r(\mathbb{R}^n)$ and the fact that $\mathcal{M}^{\mathrm{loc}}_{b,\beta}(f)$ is sublinear, implies that (3.2) holds for all $f \in L^r(\mathbb{R}^n)$. On the other hand, the fact that $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$ implies (1.2). These two facts further yield (ii).

Because (ii) obviously implies (iii), it remains to prove that (iii) implies (i). Let $B \notin \mathcal{D}$. By the definition of $\mathcal{M}^{\mathrm{loc}}_{b,\beta}$, the Hölder inequality and the boundedness of $\mathcal{M}^{\mathrm{loc}}_{b,\beta}$ from $L^r(\mathbb{R}^n)$ to $L^s(\mathbb{R}^n)$, we see that

$$\begin{aligned} \frac{1}{|B|}\int_B |b(x) - b_B|\,dx &\le \frac{1}{|B|^2} \int_B \int_B |b(x) - b(y)|\,dy\,dx \\ &\le \frac{1}{|B|^{1+\beta}} \int_B \mathcal{M}^{\mathrm{loc}}_{b,\beta}(\chi_B)(x)\,dx \lesssim 1. \end{aligned}$$

This together with (1.2) implies that $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$, which completes the proof of Theorem 3.1. $\square$

Let $p \in [1, \infty)$, $x \in \mathbb{R}^n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The localized Hardy-Littlewood-type maximal operator $\mathcal{M}^{\mathrm{loc},p}$ is defined by

$$\mathcal{M}^{\mathrm{loc},p}(f)(x) \equiv \left[\sup_{B \ni x,\, B \notin \mathcal{D}} \frac{1}{|B|} \int_B |f(y)|^p \, dy\right]^{1/p}.$$

For any function $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define the localized maximal commutator $\mathcal{M}_b^{\mathrm{loc},p}$ of $\mathcal{M}^{\mathrm{loc},p}$ by

$$\mathcal{M}_b^{\mathrm{loc},p}(f)(x) \equiv \left[\sup_{B \ni x,\, B \notin \mathcal{D}} \frac{1}{|B|} \int_B |b(x) - b(y)||f(y)|^p \, dy\right]^{1/p}.$$

Theorem 3.2

Let $p \in [1, \infty)$ and $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The following assertions are equivalent:

(i) $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$.

(ii) *b satisfies* (1.2) *and $\mathcal{M}_b^{\mathrm{loc},p}$ is bounded on $L^q(\mathbb{R}^n)$ for all $q \in (p, \infty)$.*

(iii) *b satisfies* (1.2) *and $\mathcal{M}_b^{\mathrm{loc},p}$ is bounded on $L^q(\mathbb{R}^n)$ for certain $q \in (p, \infty)$.*

Proof. We show the implication from (i) to (ii). For all $x \in \mathbb{R}^n$, let

$$\mathcal{M}_b(f)(x) \equiv \sup_{B \ni x} \frac{1}{|B|} \int_B |b(x) - b(y)||f(y)| \, dy.$$

By the boundedness of $\mathcal{M}_b$ on $L^r(\mathbb{R}^n)$ with $r \in (1, \infty)$ (see [13, Theorem 3]), we see that for all $f \in L^q(\mathbb{R}^n)$ with $q > p$,

$$\left\|\mathcal{M}_b^{\mathrm{loc},p}(f)\right\|_{L^q(\mathbb{R}^n)} \le \left\|[\mathcal{M}_b(|f|^p)]^{1/p}\right\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{R}^n)}.$$

On the other hand, (1.2) follows from the fact that $b \in \mathrm{BMO}_\rho(\mathbb{R}^n)$. These facts yield (ii).

Since the implication from (ii) to (iii) is trivial, it remains to prove that (iii) implies (i). Let $B \notin \mathcal{D}$. By the definition of $\mathcal{M}_b^{\mathrm{loc},p}$, the Hölder inequality and the boundedness of $\mathcal{M}_b^{\mathrm{loc},p}$ on $L^q(\mathbb{R}^n)$, we see that

$$\begin{aligned}\frac{1}{|B|} \int_B |b(x) - b_B| \, dx &\le \frac{1}{|B|^2} \int_B \int_B |b(x) - b(y)| \, dy \, dx \\ &\le \frac{1}{|B|} \int_B \mathcal{M}_b^{\mathrm{loc},p}(\chi_B)(x) \, dx \lesssim 1.\end{aligned}$$

This together with (1.2) implies (i), and hence, finishes the proof of Theorem 3.2. $\square$

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