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An exact Ramsey principle for block sequences

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Abstract

We prove an exact, i.e., formulated without Δ -expansions, Ramsey principle for infinite block sequences in vector spaces over countable fields, where the two sides of the dichotomic principle are represented by respectively winning strategies in Gowers' block sequence game and winning strategies in the infinite asymptotic game. This allows us to recover Gowers' dichotomy theorem for block sequences in normed vector spaces by a simple application of the basic determinacy theorem for infinite asymptotic games.

1. Introduction

The results presented here represent a new approach to the fundamental result of W.T. Gowers [7], whose uses in Banach space theory seem far from exhausted (for applications see, e.g., [7, 6]). Gowers' result is a Ramsey theoretic statement for Banach spaces that cleverly combines Ramsey theory and game theory to compensate for the fact that a true Ramsey theoretic result fails to hold in general. The proof of Gowers' theorem, however, involves approximation arguments, which at times are a bit delicate, as can be seen from the existing proofs [7, 3, 2, 8, 1], and also seemed to require tricks not previously used in infinite-dimensional Ramsey theory. Perhaps more importantly, the notion of *weakly Ramsey sets* extracted from the proof incorporates approximations, which makes it hard to induct over and extend beyond the class of analytic sets. For example, it was unknown whether Σ_2^1 sets are weakly Ramsey

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assuming Martin's axiom, though it was shown to hold under a strengthening of MA not equiconsistent with ZF by J. Bagaria and J. López-Abad [3].

The novelty of our approach lies in the replacement of both sides of the dichotomy with game theoretical statements, which completely eschew approximations and allow for a very simple inductive proof. The new tools are the *infinite asymptotic game* and the definition of *strategically Ramsey sets* in vector spaces over countable fields. Using these, one easily shows that under MA, Σ_2^1 sets are strategically Ramsey, and a version of the basic determinacy result for infinite asymptotic games [10] connects the notions of weakly Ramsey and strategically Ramsey sets.

2. Notation

Let \mathfrak{F} be a countable field and let E be a countable-dimensional \mathfrak{F} -vector space with basis (e_n) . We equip E with the discrete topology, whereby any subset is open, and equip its countable power E^{∞} with the product topology. Since E is a countable discrete set, E^{∞} is a Polish space. Notice that a basis for the topology on E^{∞} is given by sets of the form

$$N(x_0, \dots, x_k) = \{ (y_n) \in E^{\infty} \mid y_0 = x_0 \& \dots \& y_k = x_k \},\$$

where $x_0, \ldots, x_k \in E$. Let x, y, z, v be variables for *non-zero* elements of E. If $x = \sum a_n e_n \in E$, let supp $x = \{n \mid a_n \neq 0\}$ and set for $x, y \in E$,

$$x < y \Leftrightarrow \forall n \in \text{supp } x \; \forall m \in \text{supp } y \; n < m.$$

Similarly, if k is a natural number, we set

$$k < x \Leftrightarrow \forall n \in \text{supp } x \ k < n.$$

Analogous notation is used for finite subsets of \mathbb{N} . A finite or infinite sequence $(x_0, x_1, x_2, x_3, \ldots)$ of vectors is said to be a block sequence if for all $n, x_n < x_{n+1}$.

Notice that, by elementary linear algebra, for all infinite-dimensional subspaces $X \subseteq E$ there is a subspace $Y \subseteq X$ spanned by an infinite block sequence, called a *block* subspace. Henceforth, we use variables X, Y, Z, V, W to denote infinite-dimensional block subspaces of E. Also, denote infinite block sequences by variables $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and finite block sequences by variables $\vec{x}, \vec{y}, \vec{z}$.

3. Gowers' game and the infinite asymptotic game

Suppose $X \subseteq E$. We define *Gowers' game* G_X played below X between two players I and II as follows: I and II alternate (with I beginning) in choosing respectively infinite-dimensional subspaces $Y_0, Y_1, Y_2, \ldots \subseteq X$ and vectors $x_0 < x_1 < x_2 < \ldots$ according to the constraint $x_i \in Y_i$:

$$I Y_0 Y_1 Y_2 Y_3 \dots \\
 II x_0 x_1 x_2 x_3 \dots$$

Also, the *infinite asymptotic game* F_X played below X is defined as follows: I and II alternate (with I beginning) in choosing respectively natural numbers $n_0 < n_1 < n_2 < \ldots$ and vectors $x_0 < x_1 < x_2 < \ldots \in X$ according to the constraint $n_i < x_i$:

In both games we say that the sequence $(x_n)_{n \in \mathbb{N}}$ is the *outcome* of the game. Moreover, if \vec{x} is a finite block sequence, we define Gowers' game $G_X(\vec{x})$ and the infinite asymptotic game $F_X(\vec{x})$ as above except that the outcome is now $\vec{x}(x_0, x_1, x_2, \ldots)$.

If X and Y are subspaces, where Y is spanned by an infinite block sequence $\mathbf{y} = (y_0, y_1, y_2, \ldots)$, we write $Y \subseteq^* X$ if there is n such that $y_m \in X$ for all $m \ge n$. A simple diagonalisation argument shows that if $X_0 \supseteq X_1 \supseteq X_2 \supseteq \ldots$ is a decreasing sequence of block subspaces, then there is some $Y \subseteq X_0$ such that $Y \subseteq^* X_n$ for all n.

The aim of the games above is for each of the players to ensure that the outcome **x** lies in some predetermined set depending on the player. By the asymptotic nature of the game, it is easily seen that if $\mathbb{A} \subseteq E^{\infty}$ and $Y \subseteq^* X$, then if II has a strategy in G_X to play in \mathbb{A} , i.e., to ensure that the outcome is in \mathbb{A} , then II will have a strategy in G_Y to play in \mathbb{A} too. Similarly, if I has a strategy in F_X to play in \mathbb{A} , then I also has a strategy in F_Y to play in \mathbb{A} .

DEFINITION 1 We say that a set $\mathbb{A} \subseteq E^{\infty}$ is *strategically Ramsey* if for all $V \subseteq E$ and all \vec{z} , there is $W \subseteq V$ such that either

- (a) II has a strategy in $G_W(\vec{z})$ to play in \mathbb{A} , or
- (b) I has a strategy in $F_W(\vec{z})$ to play in $\sim \mathbb{A}$.

Analytic sets are strategically Ramsey

Lemma 2

Open sets $\mathbb{U} \subseteq E^{\infty}$ are strategically Ramsey.

Proof. Noticing that for all open \mathbb{U} , $\mathbb{U}_{\vec{z}}^{V} = \{(x_i) \in V^{\infty} \mid \vec{z} \mid (x_i) \in \mathbb{U}\}$ is also an open subset of V^{∞} , we can suppose V = E and $\vec{z} = \emptyset$. We say that

- (1) (\vec{x}, X) is good if II has a strategy in $G_X(\vec{x})$ to play in \mathbb{U} ,
- (2) (\vec{x}, X) is bad if $\forall Y \subseteq X$, (\vec{x}, Y) is not good,
- (3) (\vec{x}, X) is worse if it is bad and $\exists n \forall y \in X \ (n < y \to (\vec{x} \cdot y, X) \text{ is bad}).$

We notice that the properties good, bad and worse are \subseteq^* -hereditary, i.e., if (\vec{x}, X) is good/bad/worse and $Y \subseteq^* X$, then (\vec{x}, Y) is good/bad/worse.

Sublemma 3

If (\vec{x}, X) is bad, then there is some $Z \subseteq X$ such that (\vec{x}, Z) is worse.

Proof. Notice that, as good and bad are \subseteq^* -hereditary, by diagonalising over all \vec{y} , we can find some $Y \subseteq X$ such that for all $\vec{y}, (\vec{y}, Y)$ is either good or bad. Suppose towards

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a contradiction that there is no $Z \subseteq Y$ such that (\vec{x}, Z) is worse. Then, as (\vec{x}, Z) is bad for all $Z \subseteq Y$, we have

$$\forall Z \subseteq Y \ \exists y \in Z \ (\vec{x} \ y, Z) \text{ is not bad.}$$

But then

$$\forall Z \subseteq Y \exists y \in Z \ (\vec{x} \cdot y, Y) \text{ is good.}$$

for if $Z \subseteq Y$ and $(\vec{x} \cdot y, Y)$ is not good, then $(\vec{x} \cdot y, Y)$ is bad and so $(\vec{x} \cdot y, Z)$ is bad too. In other words, for all $Z \subseteq Y$ there is some $y \in Z$ such that II has a strategy in $G_Y(\vec{x} \cdot y)$ to play in \mathbb{U} and therefore II also has a strategy in $G_Y(\vec{x})$ to play in \mathbb{U} , contradicting that (\vec{x}, X) was bad. \Box

Again, using the preceding sublemma and diagonalising, we can find some $X \subseteq E$ such that for all \vec{y} , either (\vec{y}, X) is good or worse. Now, if (\emptyset, X) is good, II has a strategy in G_X to play in \mathbb{U} , so suppose instead that (\emptyset, X) is worse. We claim that I has a strategy in F_X to produce block sequences (x_0, x_1, x_2, \ldots) so that for all m, $(x_0, x_1, \ldots, x_m, X)$ is worse. To see this, suppose that at some point of the game, \vec{x} has been played so that (\vec{x}, X) is worse. Then there is some n such that for all $y \in X$, if n < y, then $(\vec{x} \uparrow y, X)$ is bad and hence even worse. Thus, we can let I play n. But if I follows this strategy, then, in particular, for no m can II have a strategy in $G_X(x_0, \ldots, x_m)$ to play in \mathbb{U} and thus as \mathbb{U} is open, $(x_0, x_1, x_2, \ldots) \in \sim \mathbb{U}$. Therefore, I has a strategy in F_X to play in $\sim \mathbb{U}$.

Lemma 4

Suppose $\mathbb{A}_n \subseteq E^{\infty}$ and $\mathbb{B} = \bigcup_n \mathbb{A}_n$. Let \vec{x} and $X \subseteq E$ be given. Then there is $Z \subseteq X$ such that either

(a) II has a strategy in G_Z to play (z_i) such that

 $\exists n \ \forall V \subseteq Z \ I \ has no \ strategy \ in \ F_V(\vec{x}(z_0, \ldots, z_n)) \ to \ play \ in \sim \mathbb{A}_n,$

or

(b) I has a strategy in $F_Z(\vec{x})$ to play in $\sim \mathbb{B}$.

Proof. We say that (\vec{y}, n) accepts Y if I has a strategy in $F_Y(\vec{y})$ to play in $\sim \mathbb{A}_n$. Also, (\vec{y}, n) rejects Y if $\forall Z \subseteq Y$, (\vec{y}, n) does not accept Z. Notice that acceptance and rejection are \subseteq^* -hereditary, so there is $Y \subseteq X$ such that for all \vec{y} and n, either (\vec{y}, n) accepts or rejects Y. Set

$$\mathbb{D} = \{ (z_i) \mid \exists n \; (\vec{x} \; (z_0, \dots, z_n), n) \text{ rejects } Y \}$$

and notice that \mathbb{D} is open. It follows, by Lemma 2, that there is $Z \subseteq Y$ such that either II has a strategy in G_Z to play in \mathbb{D} or I has a strategy in F_Z to play in $\sim \mathbb{D}$.

In the first case, II has a strategy in G_Z to play (z_i) such that

 $\exists n \ \forall V \subseteq Y \ I \ has no \ strategy in \ F_V(\vec{x}(z_0, \ldots, z_n)) \ to \ play \ in \ \sim \mathbb{A}_n$

which immediately implies (a). So suppose instead that I has a strategy in F_Z to play in $\sim \mathbb{D}$, i.e., that I has a strategy in F_Z to play (z_i) such that

$$\forall n \ (\vec{x} \ (z_0, \ldots, z_n), n) \text{ accepts } Z.$$

Thus, I has a strategy σ in F_Z to play (z_i) such that for all n, I has a strategy $\sigma_{(z_0,\ldots,z_n)}$ in $F_Z(\vec{x} \ (z_0,\ldots,z_n))$ to play in $\sim \mathbb{A}_n$. By successively putting more and more strategies into play, I thus has a strategy in $F_Z(\vec{x})$ to play in $\bigcap_n \sim \mathbb{A}_n = \sim \mathbb{B}$, which gives us (b). Concretely, if at step n + 1, (z_0,\ldots,z_n) has been played, then I will respond with

$$\max\{\sigma(z_0,\ldots,z_n),\sigma_{(z_0)}(z_1,z_2,\ldots,z_n),\ldots,\sigma_{(z_0,\ldots,z_n)}(\emptyset)\}.$$

It follows that if (z_i) is the outcome of the game, then for all n, as II has responded to a stronger strategy than $\sigma_{(z_0,\ldots,z_n)}$ when playing $(z_{n+1}, z_{n+2}, \ldots)$, we see that $\vec{x}(z_0,\ldots,z_n)(z_{n+1}, z_{n+2},\ldots) \in \mathcal{A}_n$. Therefore, $\vec{x}(z_i) \in \bigcap_n \sim \mathcal{A}_n$.

Notice that both conclusions (a) and (b) in Lemma 4 are \subseteq^* -hereditary in Z.

Theorem 5

Analytic sets are strategically Ramsey.

Proof. Suppose $\mathbb{A} \subseteq E^{\infty}$ is analytic. Noticing that for all $V \subseteq E$ and $\vec{z}, \mathbb{A}_{\vec{z}}^{V} = \{(x_i) \in V^{\infty} \mid \vec{z}^{\hat{}}(x_i) \in \mathbb{A}\}$ is also an analytic subset of V^{∞} , we can suppose V = E and $\vec{z} = \emptyset$. Let $F \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{A}$ be a continuous surjection and set for every $s \in \mathbb{N}^{<\mathbb{N}}, \mathbb{A}_s = F[N_s]$, where $N_s = \{\alpha \in \mathbb{N}^{\mathbb{N}} \mid s \subseteq \alpha\}$. We note that $\mathbb{A}_s = \bigcup_{n \in \mathbb{N}} \mathbb{A}_{s n}$. Let $\mathbb{D}(s, \vec{x}, X)$ be the set

 $\{(z_i) \mid \exists n \; \forall W \subseteq X \text{ I has no strategy in } F_W(\vec{x}(z_0, \ldots, z_n)) \text{ to play in } \sim \mathbb{A}_{s^n}\}.$

By Lemma 4, there is $X \subseteq E$ such that for all \vec{x} and all $s \in \mathbb{N}^{<\mathbb{N}}$ either

- (a) II has a strategy in G_X to play in $\mathbb{D}(s, \vec{x}, X)$, or
- (b) I has a strategy in $F_X(\vec{x})$ to play in $\sim \mathbb{A}_s$.

Suppose that I has no strategy in F_X to play in $\sim \mathbb{A} = \sim \mathbb{A}_{\emptyset}$. We describe a strategy for II in G_X to play in \mathbb{A} .

First, as II has a strategy in G_X to play in $\mathbb{D}(\emptyset, \emptyset, X)$, he follows this strategy until (z_0, \ldots, z_{n_0}) has been played such that I does not have a strategy in $F_X(z_0, \ldots, z_{n_0})$ to play in $\sim \mathbb{A}_{n_0}$.

Thus, by the assumption on X, II must have a strategy in G_X to play in $\mathbb{D}((n_0), (z_0, \ldots, z_{n_0}), X)$. II follows this until further $(z_{n_0+1}, \ldots, z_{n_0+n_1+1})$ has been played such that I does not have a strategy in $F_X(z_0, \ldots, z_{n_0}, z_{n_0+1}, \ldots, z_{n_0+n_1+1})$ to play in $\sim \mathbb{A}_{(n_0, n_1)}$.

By the same reasoning as before, II must have a strategy in G_X to play in the set $\mathbb{D}((n_0, n_1), (z_0, \ldots, z_{n_0+n_1+1}), X)$. He follows this strategy until yet another $(z_{n_0+n_1+2}, \ldots, z_{n_0+n_1+n_2+2})$ has been played such that I does not have a strategy in $F_X(z_0, \ldots, z_{n_0+n_1+n_2+2})$ to play in $\sim \mathbb{A}_{(n_0, n_1, n_2)}$.

 $F_X(z_0, \ldots, z_{n_0+n_1+n_2+2})$ to play in $\sim \mathbb{A}_{(n_0,n_1,n_2)}$. Continuing in this way and letting $m_k = (\sum_{j \le k} n_j) + k$, the outcome of the game will be a sequence

$$\mathbf{z} = (z_0, z_1, z_2, \dots, z_{m_0}, \dots, z_{m_1}, \dots, z_{m_2}, \dots)$$

such that for the sequence $\alpha = (n_0, n_1, n_2, ...)$ and all k, I does not have a strategy in $F_X(z_0, ..., z_{m_k})$ to play in $\sim \mathbb{A}_{(n_0, n_1, ..., n_k)}$. It follows that for all k, there must be

an infinite block sequence \mathbf{z}_k end-extending (z_0, \ldots, z_{m_k}) such that $\mathbf{z}_k \in \mathbb{A}_{(n_0, n_1, \ldots, n_k)}$. So for some $\beta_k \in N_{(n_0, n_1, \ldots, n_k)}$, we have $F(\beta_k) = \mathbf{z}_k$. But, by continuity of F, we have $F(\beta_k) \xrightarrow[k \to \infty]{} F(\alpha)$, while $\mathbf{z}_k \xrightarrow[k \to \infty]{} \mathbf{z}$, so $F(\alpha) = \mathbf{z}$ and $\mathbf{z} \in \mathbb{A}$. Therefore, this describes a strategy for II in G_X to play in \mathbb{A} .

Infinite asymptotic games in normed vector spaces

Suppose now that \mathfrak{F} is a countable *sub*field of \mathbb{R} or \mathbb{C} and $\|\cdot\|$ is a norm on E taking values in \mathfrak{F} . For $X \subseteq E$, denote by \mathcal{B}_X the unit ball of X and by $\mathfrak{B}(X)$ the set of block sequences (x_i) of X with $\|x_i\| \leq 1$. A set $\mathbb{A} \subseteq E^{\infty}$ is said to be *large* if for all $X \subseteq E$, $\mathbb{A} \cap \mathfrak{B}(X) \neq \emptyset$. Also, if $\Delta = (\delta_i)$ is a sequence of strictly positive real numbers, denoted by $\Delta > 0$, and $\mathbb{A} \subseteq E^{\infty}$, we define

$$\mathbb{A}_{\Delta} = \{ (z_i) \in E^{\infty} \mid \exists (x_i) \in \mathbb{A} \; \forall i \; ||x_i - z_i|| < \delta_i \},$$
$$\operatorname{Int}_{\Delta}(\mathbb{A}) = \sim (\sim \mathbb{A})_{\Delta} = \{ (x_i) \mid \forall (z_i) \; (\forall i \; ||x_i - z_i|| < \delta_i \to (z_i) \in \mathbb{A}) \}.$$

To get a stronger statement in (b) of the definition of strategically Ramsey sets, we need to allow approximations. For this, we use a variant of a result from [10], though the proof given here is in the same spirit as that presented in [6].

Theorem 6

Suppose there is a strategy σ for I in F_X to play in the set $\mathbb{B} \subseteq E^{\infty}$. Then for any sequence $\Delta > 0$ there are intervals $I_0 < I_1 < I_2 < \ldots$ of \mathbb{N} such that for any block sequence $(x_i) \in \mathfrak{B}(X)$, if

$$\forall n \; \exists m \; I_0 < x_n < I_m < x_{n+1},$$

then $(x_i) \in \mathbb{B}_{\Delta}$.

Proof. Choose sets $\mathbb{D}_n \subseteq \mathcal{B}_X$ such that for each finite $d \subseteq \mathbb{N}$, the number of $x \in \mathbb{D}_n$ such that supp x = d is finite, and for every $x \in \mathcal{B}_X$ there is some $y \in \mathbb{D}_n$ with supp x = supp y and $||x - y|| < \delta_n$. This is possible since the unit ball in $[e_i]_{i \in d}$ is totally bounded for all finite $d \subseteq \mathbb{N}$.

For each position $p = (n_0, y_0, \ldots, n_i, y_i)$ in F_X played according to σ in which $y_j \in \mathbb{D}_j$ for all j, we write p < k if $n_j, y_j < k$ for all j. Notice that for all k there are only finitely many such p with p < k, so we can define

$$\alpha(k) = \max(k, \max\{\sigma(p) \mid p < k\})$$

and set $I_k = [k, \alpha(k)]$. The I_k are not necessarily successive, but their minimal elements tend to ∞ . So, modulo passing to a subsequence, it is enough to show that if $(x_i) \in \mathfrak{B}(X)$ and

$$\forall n \exists m \ I_0 < x_n < I_m < x_{n+1},$$

then $(x_i) \in \mathbb{B}_{\Delta}$.

Suppose such (x_i) is given. Find $y_i \in \mathbb{D}_i$ such that $||x_i - y_i|| < \delta_i$ and supp $x_i = \sup y_i$ for all i and let $0 = b_0 < b_1 < b_2 < \ldots$ be integers such that

$$I_{b_0} < y_0 < I_{b_1} < y_1 < I_{b_2} < y_2 < \dots$$

We claim that there are natural numbers $n_i \leq \max I_{b_i}$ such that each

$$p_i = (n_0, y_0, \dots, n_i, y_i)$$

is a position in F_X in which I has played according to σ . To see this, notice first that $n_0 = \alpha(\emptyset) \in I_{b_0}$, so $p_0 = (n_0, y_0)$ is played according to σ . Now, for the induction step, suppose that p_i is played according to σ , and notice that $p_i < \min I_{b_{i+1}} = b_{i+1}$. We set $n_{i+1} = \sigma(p_i) \leq \alpha(b_{i+1}) = \max I_{b+1}$, whereby p_{i+1} is played according to σ . This finishes the induction and proves the claim.

Thus, $(n_0, y_0, n_1, y_1, \ldots)$ is a run of the game in which I has followed the strategy σ and so $(y_i) \in \mathbb{B}$, whereby $(x_i) \in \mathbb{B}_{\Delta}$.

Theorem 7

Suppose $\mathbb{A} \subseteq E^{\infty}$ is strategically Ramsey and for some $\Delta > 0$, $\operatorname{Int}_{\Delta}(\mathbb{A})$ is large. Then there is $X \subseteq E$ such that II has a strategy in G_X to play in \mathbb{A} .

Proof. Suppose for a contradiction that for some $X \subseteq E$, I has a strategy in F_X to play in $\sim \mathbb{A} = E^{\infty} \setminus \mathbb{A}$. Then, using Theorem 6, we can find some $Y \subseteq X$ such that $\mathfrak{B}(Y) \subseteq (\sim \mathbb{A})_{\Delta}$, contradicting that $\operatorname{Int}_{\Delta}(\mathbb{A})$ is large. So since \mathbb{A} is strategically Ramsey there is instead $X \subseteq E$ such that II has a strategy in G_X to play in \mathbb{A} . \Box

Theorem 7 is a slight variation of Theorem 4.1 of Gowers' paper [7] in that his result concerns analytic sets of block sequences in Banach spaces and not normed vector spaces over countable fields. Moreover, in his result the Δ -expansions occur on the opposite side of the dichotomy. Though for applications it is mostly immaterial whether one works with all of \mathbb{R} or only a countable subfield, for completeness, we indicate here how to deduct his statement from Theorems 5 and 7. So suppose Bis a real Banach space with a Schauder basis (e_n) . Then we can define the Gowers game G_B^* as before, except that we are now allowing both players to play vectors and block subspaces with coefficients in \mathbb{R} . We similarly modify the definitions of $\mathfrak{B}(B)$, \mathcal{B}_B , \mathbb{A}_Δ etc. to allow for real scalars. Now suppose \mathbb{A} is an analytic subset of $\mathfrak{B}(B)$ in the topology induced from the countable power B^{∞} of the Banach space B, such that any \mathbb{R} -block sequence has a further \mathbb{R} -block sequence belonging to \mathbb{A} . Let also $\Delta > 0$. We will show that there is an \mathbb{R} -block subspace $X \subseteq B$ such that II has a strategy in G_X^* to play in \mathbb{A}_{Δ} . To see this, let \mathfrak{F} be a countable subfield of \mathbb{R} such that any finite \mathfrak{F} -linear combination of (e_n) has norm belonging to \mathfrak{F} and denote by E the \mathfrak{F} -linear span of (e_n) . Then the intersection $\mathbb{B} = \mathbb{A}_{\Delta/3} \cap \mathfrak{B}(E)$ is analytic in $\mathfrak{B}(E)$, since Δ -expansions preserve analyticity and the topology of $\mathfrak{B}(E)$ is finer than that induced from B^{∞} . Moreover, as E is dense in B, any \mathfrak{F} -block sequence has a further \mathfrak{F} -block sequence belonging to \mathbb{B} , i.e., \mathbb{B} is large and so $\operatorname{Int}_{\Lambda/3}(\mathbb{B}_{\Lambda/3})$ is large too. It follows, by Theorems 5 and 7, that for some \mathfrak{F} -block subspace $X \subseteq E$, II has a strategy in G_X to play in $\mathbb{B}_{\Delta/3}$. Standard perturbation arguments then show that II also has a strategy in G_X^* to play in $(\mathbb{B}_{\Delta/3})_{\Delta/3}$, which is a subset of \mathbb{A}_{Δ} .

Suppose $X \subseteq E$. We define the unraveled Gowers game H_X played below X between two players I and II as follows: I and II alternate (with I beginning) in choosing infinite dimensional subspaces $Y_0, Y_1, Y_2, \ldots \subseteq X$, respectively vectors

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 $x_0 < x_1 < x_2 < \dots$ and digits $\epsilon_i \in \{0, 1\}$, according to the constraint $x_i \in Y_i$.

$$\mathbf{I} \qquad Y_0 \qquad Y_1 \qquad Y_2 \qquad Y_3 \qquad \dots \\ \mathbf{II} \qquad x_0, \epsilon_0 \qquad x_1, \epsilon_1 \qquad x_2, \epsilon_2 \qquad x_3, \epsilon_3 \qquad \dots$$

We say that the pair of sequences $((x_n)_{n \in \mathbb{N}}, (\epsilon_n)_{n \in \mathbb{N}})$ is the *outcome* of the game.

The following result is exceedingly useful in applications. For example, it is an integral part of the proofs of the three main dichotomies of [6] and the case of G_{δ} sets also lies at the heart of the alternative proofs of Gowers' Theorem in [2, 8]. Its utility lies in the fact that if one tries to play into some analytic set, which could be, e.g., the isomorphism class of a particular space B, then often not only can this be done, but one can at the same time continuously produce witnesses for belonging to this analytic set, e.g., the operator realising the isomorphism with B.

Theorem 8

Let $\mathbb{B} \subseteq E^{\infty} \times 2^{\infty}$ be analytic such that $\mathbb{A} = \operatorname{proj}_{E^{\infty}}(\mathbb{B})$ is large. Then for every $\Delta > 0$ there is $X \subseteq E$ such that II has a strategy in H_X to play in

$$\mathbb{B}_{\Delta} = \{ ((y_n), (\epsilon_n)) \mid \exists (x_n) \forall n \mid ||y_n - x_n|| < \delta_n \& ((x_n), (\epsilon_n)) \in \mathbb{B} \}.$$

Proof. We can suppose that $\frac{1}{4} > \delta_0 > \delta_1 > \dots$ Also, for simplicity of notation, let us suppose temporarily that 2 is the set $\{\frac{1}{2}, 1\}$, so $\mathbb{B} \subseteq E^{\infty} \times \{\frac{1}{2}, 1\}^{\infty}$. Define $\mathbb{D} \subseteq E^{\infty}$ as follows:

$$\mathbb{D} = \{ (x_i) \in E^{\infty} \mid ((x_{2i})_{i=0}^{\infty}, (||x_{2i+1}||)_{i=0}^{\infty}) \in \mathbb{B} \}.$$

We claim that \mathbb{D} is large. For suppose $X \subseteq E$ is spanned by a block sequence (z_n) , let $Z = [z_{2n}]$ and find some $((y_n), (\epsilon_n)) \in \mathbb{B}$ such that $(y_n) \in \mathbb{A} \cap \mathfrak{B}(Z)$. Now for all n, find some $v_n \in [z_{2n+1}]$ such that $y_n < v_n < y_{n+1}$ and $||v_n|| = \epsilon_n$. (This is where we use that the norm takes values in \mathfrak{F} and hence that we can normalise). Then $(y_0, v_0, y_1, v_1, \ldots) \in \mathbb{D} \cap \mathfrak{B}(X)$, verifying the largeness of \mathbb{D} . Since $\mathbb{D} \subseteq \operatorname{Int}_{\Delta}(\mathbb{D}_{\Delta})$ and \mathbb{D}_{Δ} is analytic, by Theorem 7 there is some $X \subseteq E$ such that II has a strategy in G_X to play in \mathbb{D}_{Δ} . Since for $(x_i) \in \mathbb{D}$, $||x_{2i+1}||$ is either 1 or $\frac{1}{2}$ and moreover $\delta_{2i+1} < \frac{1}{4}$, this easily implies that II has a strategy in G_X to play in \mathbb{D}_{Δ} such that moreover $||x_{2i+1}||$ is either 1 or $\frac{1}{2}$ for all *i*. Using this, II evidently has a strategy in H_X to play in \mathbb{B}_{Δ} . \square

6. Strategically Ramsey sets under set theoretical hypotheses

Theorem 9

or

The class of strategically Ramsey sets is closed under countable unions.

Proof. Let \mathbb{A}_n be strategically Ramsey for every n and set $\mathbb{B} = \bigcup_n \mathbb{A}_n$. Let \vec{x} and $X \subseteq E$ be given. Since each \mathbb{A}_n is strategically Ramsey, by diagonalising, there is some $Y \subseteq X$ such that for all \vec{y} and n, either II has a strategy in $F_Y(\vec{y})$ to play in \mathbb{A}_n or I has a strategy in $G_Y(\vec{y})$ to play in $\sim \mathbb{A}_n$. Also, by Lemma 4 there is $Z \subseteq Y$ such that either

(a) II has a strategy in G_Z to play (z_i) such that

 $\exists n \ \forall V \subseteq Z$ I has no strategy in $F_V(\vec{x}(z_0, \ldots, z_n))$ to play in $\sim \mathbb{A}_n$,

(b) I has a strategy in $F_Z(\vec{x})$ to play in $\sim \mathbb{B}$.

Note that (a) implies that II has a strategy in G_Z to play (z_i) such that

 $\exists n \text{ II has a strategy in } G_Z(\vec{x} (z_0, \ldots, z_n)) \text{ to play in } \mathbb{A}_n.$

And, in this case, II first follows the strategy to play some (z_0, \ldots, z_n) such that II has a strategy in $G_Z(\vec{x} (z_0, \ldots, z_n))$ to play in \mathbb{A}_n and thereafter continues with this other strategy. This, combined, is a strategy for II in $G_Z(\vec{x})$ to play in $\mathbb{B} = \bigcup_m \mathbb{A}_m$.

Theorem 10 (MA $_{\omega_1}$)

A union of \aleph_1 many strategically Ramsey sets is again strategically Ramsey.

Proof. By Theorem 9, it is enough to consider well-ordered increasing unions of length ω_1 . So suppose $\mathbb{A}_{\xi} \subseteq \mathbb{A}_{\zeta} \subseteq E^{\infty}$ are strategically Ramsey for all $\xi < \zeta < \omega_1$ and $\mathbb{B} = \bigcup_{\zeta < \omega_1} \mathbb{A}_{\zeta}$. Fix \vec{x} and $X \subseteq E$. Since every \mathbb{A}_{ξ} is strategically Ramsey, we can define a decreasing sequence $\ldots \subseteq^* X_{\xi} \subseteq^* \ldots \subseteq^* X_2 \subseteq^* X_1 \subseteq^* X_0 \subseteq X$ of length ω_1 such that for all $\xi < \omega_1$ either

- (a) II has a strategy in $G_{X_{\xi}}(\vec{x})$ to play in \mathbb{A}_{ξ} , or
- (b) I has a strategy in $F_{X_{\xi}}(\vec{x})$ to play in $\sim \mathbb{A}_{\xi}$.

If for some ξ , II has a strategy in $G_{X_{\xi}}(\vec{x})$ to play in \mathbb{A}_{ξ} , then II also has a strategy in $G_{X_{\xi}}(\vec{x})$ to play in $\mathbb{B} = \bigcup_{\zeta < \omega_1} \mathbb{A}_{\zeta}$ and we are done. So suppose instead that for every ξ , I has a strategy in $F_{X_{\xi}}(\vec{x})$ to play in $\sim \mathbb{A}_{\xi}$. By [5, Lemma 5], under MA_{ω_1} there is a $Y \subseteq X$ such that $Y \subseteq^* X_{\xi}$ for all ξ . Thus, for every ξ , I has a strategy σ_{ξ} in $F_Y(\vec{x})$ to play in $\sim \mathbb{A}_{\xi}$.

Notice that σ_{ξ} is formally a function from the countable set D of finite block sequences \vec{y} of Y to the set of natural numbers and hence a member of \mathbb{N}^D . By $\operatorname{MA}_{\omega_1}$, the family $\{\sigma\}_{\xi < \omega_1}$ cannot be \leq^* unbounded in \mathbb{N}^D and hence for some $\sigma \in \mathbb{N}^D$ we have $\sigma_{\xi} \leq^* \sigma$ for all ξ , i.e., for all ξ there is a finite set $p_{\xi} \subseteq D$ such that

$$\forall \vec{y} \in D \setminus p_{\xi} \quad \sigma_{\xi}(\vec{y}) \le \sigma(\vec{y}).$$

By reason of cardinality, there is some $p \subseteq D$ such that for an unbounded set $S \subseteq \omega_1$ we have $p_{\xi} = p$ for all $\xi \in S$. Now let n_0 be large enough such that $n_0 \not\leq y_0$ for all $\vec{y} = (y_0, \ldots, y_m) \in p$. We modify σ so that $\sigma(\emptyset) = n_0$ and otherwise leave it unaltered. Then σ is a strategy for I in $F_Y(\vec{x})$ to play in $\sim \mathbb{B} = \bigcap_{\xi < \omega_1} \sim \mathbb{A}_{\xi} = \bigcap_{\xi \in S} \sim \mathbb{A}_{\xi}$. To see this, suppose that (z_i) is the outcome of a game in which I has followed σ . Then as $n_0 < z_0$, we must have $(z_0, \ldots, z_m) \notin p$ for all m, and hence for all $\xi \in S$ and $m, \sigma(z_0, \ldots, z_m) = \sigma_{\xi}(z_0, \ldots, z_m)$. If follows that for every $\xi \in S$, I has followed the strategy σ_{ξ} and hence $(z_i) \notin \mathbb{A}_{\xi}$.

Since Σ_2^1 sets are unions of \aleph_1 many Borel sets, we have the following strengthening of a result of Bagaria and López-Abad [3]. They essentially proved the conclusion of Theorem 7 for Σ_2^1 sets, but only under a hypothesis relatively consistent with the existence of a large cardinal. On the other hand, our hypothesis, namely MA_{ω_1} , is equiconsistent with ZF, which permits the use of absoluteness arguments.

Corollary 11 (MA_{ω_1})

 Σ_2^1 sets are strategically Ramsey.

We do not know if the axiom of projective determinacy suffices to prove that all projective sets are strategically Ramsey, though we very much suspect so. Again, Bagaria and López-Abad [4] proved that under PD, projective sets are weakly Ramsey.

7. Relational games

In this section we consider relational versions of Gowers' game and the infinite asymptotic game in which both players contribute to the outcome. Unfortunately, we can in this case only prove the Ramsey principle for open and closed sets. Simpler relational games were first considered by A. M. Pełczar [9], where a specific instance of Theorem 12 below was used to prove that any space saturated with subsymmetric sequences must contain a minimal subspace. Related uses of Theorem 12 can be found in [6].

Suppose $X \subseteq E$. We define the game A_X played below X between two players I and II as follows: I and II alternate in choosing block subspaces $Z_0, Z_1, Z_2, \ldots \subseteq X$ and vectors $x_0 < x_1 < x_2 < \ldots \in X$, respectively integers $n_0 < n_1 < n_2 < \ldots$ and vectors $y_0 < y_1 < y_2 < \ldots \in X$ according to the constraints $n_i < x_i$ and $y_i \in Z_i$:

$$I n_0 < x_0, Z_0 n_1 < x_1, Z_1 n_2 < x_2, Z_2 \dots$$

$$II n_0 y_0 \in Z_0, n_1 y_1 \in Z_1, n_2 \dots$$

We say that the sequence $(x_0, y_0, x_1, y_1, ...)$ is the *outcome* of the game.

If \vec{x} is a finite block sequence of *even* length, the game $A_X(\vec{x})$ is defined as above except that the outcome is now $\vec{x}(x_0, y_0, x_1, y_1, \ldots)$.

On the other hand, if \vec{x} is a finite block sequence of *odd* length, $A_X(\vec{x})$ is defined in a similar way as before except that I begins

and the *outcome* is now $\vec{x}(y_0, x_0, y_1, x_1, ...)$ rather than $\vec{x}(x_0, y_0, x_1, y_1, ...)$.

We define the game B_X in a similar way to A_X except that we now have I playing integers and II playing block subspaces:

with $x_i \in Z_i \subseteq X$ and $n_i < y_i \in X$. Again, the *outcome* is $(x_0, y_0, x_1, y_1, \ldots)$.

If \vec{x} is a finite block sequence of *even* length, the game $B_X(\vec{x})$ is defined as above except that the outcome is now $\vec{x}(x_0, y_0, x_1, y_1, \ldots)$.

On the other hand, if \vec{x} is a finite block sequence of *odd* length, $B_X(\vec{x})$ is defined by letting I begin

and the *outcome* is now $\vec{x}(y_0, x_0, y_1, x_1, \ldots)$.

Thus, in both games A_X and B_X , one should remember that I is the *first* to play a vector. And in A_X , I plays block subspaces and II plays tail subspaces, while in B_X , II takes the role of playing block subspaces and I plays tail subspaces.

Suppose $\mathbb{A} \subseteq E^{\infty}$, $Y \subseteq^* X$ and \vec{x} are given. Then one easily sees that if II has a strategy in $A_X(\vec{x})$ to play in \mathbb{A} , then II also has a strategy in $A_Y(\vec{x})$ to play in \mathbb{A} . Similarly, if I has a strategy in $B_X(\vec{x})$ to play in \mathbb{A} , then I also has a strategy in $B_Y(\vec{x})$ to play in \mathbb{A} . Also, if II has a strategy in $A_X(\vec{x})$ to play in \mathbb{A} , then II also has a strategy in $B_X(\vec{x})$ to play in \mathbb{A} .

Theorem 12

Suppose $\mathbb{A} \subseteq E^{\infty}$ is open or closed. Then there is $X \subseteq E$ such that either

- (1) II has a strategy in A_X to play in \mathbb{A} , or
- (2) I has a strategy in B_X to play in $\sim \mathbb{A}$.

Proof. Suppose first that \mathbb{A} is open. We say that

- (a) (\vec{x}, X) is good if II has a strategy in $A_X(\vec{x})$ to play in A.
- (b) (\vec{x}, X) is *bad* if $\forall Y \subseteq X$, (\vec{x}, Y) is not good.
- (c) (\vec{x}, X) is *worse* if it is bad and either
 - (1) $|\vec{x}|$ is odd and $\exists n \ \forall y \in X \ (n < y \to (\vec{x} \ y, X) \text{ is bad}), \text{ or }$
 - (2) $|\vec{x}|$ is even and $\forall Y \subseteq X \exists x \in Y (\vec{x} \cdot x, X)$ is bad).

One checks as always that good, bad and worse are all \subseteq^* -hereditary.

Lemma 13

If (\vec{x}, X) is bad, then there is some $Z \subseteq X$ such that (\vec{x}, Z) is worse.

Proof. By diagonalisation, we can find some $Y \subseteq X$ such that for all \vec{y} , (\vec{y}, Y) is either good or bad.

Assume first that $|\vec{x}|$ is even. Since (\vec{x}, Y) is bad, we have $\forall V \subseteq X$ II has no strategy in $A_V(\vec{x})$ to play in \mathbb{A} . So $\forall V \subseteq X \exists x \in V$ such that II has no strategy in $A_V(\vec{x} \cdot x)$ to play in \mathbb{A} , and hence such that $(\vec{x} \cdot x, V)$ is not good. Thus,

$$\forall V \subseteq X \; \exists x \in V \; (\vec{x} \; \hat{x}, Y) \text{ is bad},$$

and so already (\vec{x}, Y) is worse.

Now suppose instead that $|\vec{x}|$ is odd and, towards a contradiction, that there is no $Z \subseteq Y$ such that (\vec{x}, Z) is worse. Then, as (\vec{x}, Y) is bad, $\forall Z \subseteq Y \exists y \in Z \ (\vec{x} \land y, Z)$ is not bad and thus also $\forall Z \subseteq Y \exists y \in Z \ (\vec{x} \land y, Y)$ is good. So

 $\forall Z \subseteq Y \exists y \in Z$ II has a strategy in $A_Y(\vec{x} \cdot y)$ to play in \mathbb{A} ,

and hence II also has a strategy in $A_Y(\vec{x})$ to play in \mathbb{A} , contradicting that (\vec{x}, Y) is bad.

Diagonalising, we now find $X \subseteq E$ such that for all \vec{x} , either (\vec{x}, X) is good or worse. Assume that II has no strategy in A_X to play in \mathbb{A} , whereby (\emptyset, X) is worse. Then, by unraveling the definition of worse and using that bad and worse

coincide below X, one sees that I has a strategy in B_X to produce block sequences $(z_0, z_1, z_2, ...)$ so that for all $m, (z_0, z_1, ..., z_m, X)$ is worse. In particular, for no m does II have a strategy in $A_X(z_0, ..., z_m)$ to play in \mathbb{A} , and so, as \mathbb{A} is open, we must have $(z_0, z_1, z_2, ...) \in \mathbb{A}$. So I has a strategy in B_X to play in $\sim \mathbb{A}$, which finishes the proof for open sets.

Now if instead \mathbb{A} is closed, set

$$\mathbb{B} = \{x \, \hat{\mathbf{x}} \mid x \in E \& \mathbf{x} \notin \mathbb{A}\} = E \times \sim \mathbb{A},\$$

which is open. So find some $X \subseteq E$ such that either

- (1) II has a strategy in A_X to play in \mathbb{B} , or
- (2) I has a strategy in B_X to play in $\sim \mathbb{B}$.

Now if II has a strategy in A_X to play in \mathbb{B} , then I has a strategy in B_X to play in $\sim \mathbb{A}$. And if I has a strategy in B_X to play in $\sim \mathbb{B}$, then II has a strategy in A_X to play in \mathbb{A} , which is what needed proof.

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