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Controllability of partially prescribed matrices*

Glória Cravo

Departamento de Matemática e Engenharias, Universidade da Madeira
9000-390 Funchal, Madeira, Portugal

E-mail: gcravo@uma.pt

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Abstract

Let F be an infinite field and let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$ and $C_{2,1} \in F^{p_2 \times p_1}$. In this paper we show that appart from an exception, there always exist $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$ and $C_{2,3} \in F^{p_2 \times p_3}$ such that the pair

$$(A_1, A_2) = \left(\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}, \begin{bmatrix} C_{1,3} \\ C_{2,3} \end{bmatrix} \right)$$

is completely controllable. In other words, we study the possibility of the linear system $\dot{\chi}(t) = A_1\chi(t) + A_2\zeta(t)$ being completely controllable, when $C_{1,2}, C_{1,3}$ and $C_{2,1}$ are prescribed and the other blocks are unknown.

We also describe the possible characteristic polynomials of a partitioned matrix of the form

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix} \in F^{n \times n},$$

where $C_{1,1}, C_{2,2}, C_{3,3}$ are square submatrices (not necessarily with the same size), when $C_{1,2}, C_{1,3}$ and $C_{2,1}$ are fixed and the other blocks vary.

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1. Introduction

Control Theory is an important branch of mathematics that has several applications to technology, engineering, economics and sociology.

Very often we use applications of Control Theory, as the air-conditioning system, the oven, the iron, the hairdryer, the vehicle speed, and so on.

Currently a problem from Control Theory can be formalized as follows: "Given a mathematical description of a system how to manipulate the input variables in order to achieve a satisfactory performance, according to initial specifications?"

A very important problem in Control Theory is the following:

Problem. *Given a system*

$$\dot{\chi}(t) = A\chi(t) + B\zeta(t), \tag{1}$$

where $\chi(t) \in \mathbb{R}^n$ denotes the state of a certain physical system to be controllable by the input $\zeta(t) \in \mathbb{R}^m$, and $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, how to select the input $\zeta(t)$ in such way that $\chi(t)$ is driven to a certain desirable state?

In other words, the aim of this problem is to establish conditions under which the system (1) is completely controllable, i.e., the pair (A, B) is completely controllable. This problem is usually known as the *Pole Assignment Problem.*

Now consider F a field and let $A \in F^{n\times n}, B \in F^{n\times m}$. The characterization of (1) being completely controllable, when some entries of $[A\ B]$ are prescribed and the others are unknown has been often studied for many authors. In particular, when several entries of $[A\ B]$ are prescribed as 0, the problem is completely solved, see [12, 14, 15, 22]. When the prescribed entries are not necessarily equal to 0, there are only partial solutions, see [1, 6, 27, 28]. In this paper we characterize the possibility of (1) being completely controllable, where A and B are partitioned matrices of the forms:

$$A = \begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} \in F^{n\times n},\ B = \begin{bmatrix} C_{1,3} \\ C_{2,3} \end{bmatrix} \in F^{n\times m},$$

with $C_{1,1}$, $C_{2,2}$ square submatrices (not necessarily with the same size), when F is an infinite field, $C_{1,2}$, $C_{1,3}$ and $C_{2,1}$ are prescribed and the other blocks are unknown.

The approach used allows to solve another question in Matrix Completion Problems. In general, these problems consist in studying the possibility to "complete" a matrix, when some of its entries are prescribed (i.e., are fixed), such that the resulting matrix satisfies certain properties. In this context "to complete" means to attribute values to the remaining entries. In other words, given a matrix and a part of the given matrix (as a submatrix or some entries) the aim of these problems is to describe conditions under which we can fill the unknown entries, such that the resultant matrix satisfies the required properties. An important problem that motivates our work is the following, a particular case of the Matrix Completion Problems, proposed by G.N. de Oliveira in 1975.

Problem[18]. *Let F be a field and let n, p, q be positive integers such that*

$n = p + q$. Let $f(x) \in F[x]$ be a monic polynomial of degree n. Let

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}, \tag{2}$$

be a partitioned matrix, where $A_{1,1} \in F^{p\times p}$, $A_{2,2} \in F^{q\times q}$. Suppose that some of the blocks $A_{i,j}$,$i, j \in \{1, 2\}$, are known. Under which conditions does there exist a matrix of the form (2) *with characteristic polynomial $f(x)$?*

Note that this problem gives rise to essentially seven distinct problems, according to the prescription of some blocks of A:
(P_1) $A_{1,1}$ prescribed;
(P_2) $A_{1,2}$ prescribed;
(P_3) $A_{1,1}$ and $A_{1,2}$ prescribed;
(P_4) $A_{1,1}$ and $A_{2,2}$ prescribed;
(P_5) $A_{1,2}$ and $A_{2,1}$ prescribed;
(P_6) $A_{1,1}, A_{1,2}$ and $A_{2,2}$ prescribed;
(P_7) $A_{1,1}, A_{1,2}$ and $A_{2,1}$ prescribed.

Concerning problem (P_1), G.N. de Oliveira presented the complete answer in [17]. The complete answer to problem (P_2) was established by G.N. de Oliveira in [18]. H.K. Wimmer in [30] gave the complete answer to problem (P_3). Problem (P_4) has only some partial answers obtained by G.N. de Oliveira in [20, 21] and by F.C. Silva in [25]. Concerning problem (P_5), as in the previous case, there exist some partial results, established by G.N. de Oliveira in [19], F.C. Silva in [24] and M.G. Marques and F.C. Silva in [13]. In [26] F.C. Silva presented a partial solution for problem (P_6). Concerning problem (P_7) we do not know any reference with nontrivial results.

It is remarkable the fact that after more than 30 years, many of these questions are still unsolved.

Motivated by this problem, a natural question that arises is the following. Let F be an arbitrary field. Let $n, k, p_1, \ldots, p_k$ be positive integers such that $n = p_1+\cdots+p_k$. Let

$$C = \begin{bmatrix} C_{1,1} & \cdots & C_{1,k} \\ \vdots & & \vdots \\ C_{k,1} & \cdots & C_{k,k} \end{bmatrix} \in F^{n\times n}, \tag{3}$$

where the blocks $C_{i,j} \in F^{p_i\times p_j}$, $i, j \in \{1, \ldots, k\}$ and $C_{1,1}, \ldots, C_{k,k}$ are square submatrices.

Problem. *Suppose that some of the blocks $C_{i,j}$ are prescribed. Under which conditions does there exist a matrix of the form* (3) *with prescribed eigenvalues or characteristic polynomial?*

Obviously the prescription of the characteristic polynomial is more general since it covers the situation of the eigenvalues of the matrix being outside of the field F. Clearly, if all the eigenvalues of (3) are in F, the description of the possible characteristic polynomials of (3) simply consists in studying the eigenvalues of (3).

In [3] we showed that given a matrix of the form (3) partitioned into $k \times k$ blocks of the same size $p \times p$, with entries in an arbitrary field F, it is always possible to prescribe $2k-3$ blocks of the matrix and the eigenvalues in F, except if, either all the principal blocks are prescribed, or all the blocks of one row or column are prescribed. In these exceptional cases, we identified necessary and sufficient conditions under which it is possible to prescribe $2k-3$ blocks of the matrix and the eigenvalues in F. We also noticed that there are additional necessary conditions if more than $2k-3$ blocks are fixed.

Later, in [2] we described the possible characteristic polynomials of a matrix of the form (3) partitioned into $k \times k$ blocks of the same size $p \times p$, with entries in an arbitrary field F. Our answer shows that it is always possible to prescribe $k-1$ blocks of the matrix and the characteristic polynomial, except if all the nonprincipal blocks of a row or column are prescribed equal to 0 and the characteristic polynomial has not any divisor of degree p.

When the blocks are not necessarily of the same size, the description of the eigenvalues of a matrix of the form (3), when some of its blocks are prescribed and the others are unknown, becomes more difficult. In [4] we studied the possible eigenvalues of a matrix of the form (3) with entries in an arbitrary field F, where $C_{i,j} \in F^{p_i \times p_j}$ and $C_{1,1}, \ldots, C_{k,k}$ are square submatrices, when a diagonal of blocks is prescribed. Notice that when the prescribed positions correspond to "large" submatrices, then there are necessary interlacing inequalities for the invariant factors [23, 29].

In this paper we study a particular case, when $k = 3$ and F is infinite. In fact, we describe the possible characteristic polynomials of

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} \\ C_{2,1} & C_{2,2} & C_{2,3} \\ C_{3,1} & C_{3,2} & C_{3,3} \end{bmatrix}, \tag{4}$$

where $C_{i,j} \in F^{p_i \times p_j}$, $i, j \in \{1,2,3\}$ and $C_{1,1}$, $C_{2,2}$, $C_{3,3}$ are square submatrices, when $C_{1,2}, C_{1,3}$ and $C_{2,1}$ are prescribed and the remaining blocks are unknown.

2. Background

Let F be a field.

Let $D = F$ or $D = F[x]$ and let m, n be positive integers. We denote by $D^{m \times n}$ the set of all matrices in D of type $m \times n$.

The symbol $|$ is used in the following way: if $f(x)$, $g(x) \in F[x]$, then $f(x)|g(x)$ means "$f(x)$ divides $g(x)$".

Given $a_1, \ldots, a_n \in F$, we denote by $\operatorname{diag}(a_1, \ldots, a_n)$ the following matrix:

$$\begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix}.$$

Now we present some definitions and results that are necessary for the rest of the paper. In general these results can be found in many books on Linear Algebra, for example see [5, 7, 8, 10, 11, 16].

Let R be the set of all monic polynomials and the zero polynomial.

Definition 1 Let $A(x) \in F[x]^{m\times n}$. The greatest common divisor choosen in R, of the determinants of the submatrices of size $k \times k$ of $A(x)$, $k \in \{1, \dots, \min\{m, n\}\}$ is denoted by $d_k(x)$. If $k \leq \mathrm{rank}A(x)$, we say that $d_k(x)$ is the k-th determinantal divisor of $A(x)$. Make convention that $d_0(x) = 1$.

It is known that if $A(x) \in F[x]^{m\times n}$ and $\mathrm{rank}A(x) = r$, then:

(i) $d_k(x) \neq 0$ if and only if $k \leq r$;
(ii) $d_{k-1}(x)|d_k(x), k \in \{1, \dots, r\}$.

Definition 2 The k-th invariant factor of $A(x)$ is the element

$$i_k(x) = \frac{d_k(x)}{d_{k-1}(x)}, k \in \{1, \dots, \mathrm{rank}A(x)\},$$

with the convention that $i_0(x) = 1$.

Note that according to the previous definitions, the determinantal divisors and the invariant factors of the matrix $A(x)$ are monic polynomials.

It is known that $i_{k-1}(x)|i_k(x), k \in \{1, \dots, \mathrm{rank}A(x)\}$.

It is also known that:

(i) $A(x)$, $B(x) \in F[x]^{m\times n}$ are equivalent if and only if they have the same invariant factors.
(ii) $A(x)$, $B(x) \in F[x]^{m\times n}$ are equivalent if and only if they have the same determinantal divisors.

Let $A \in F^{m\times m}$. The polynomial matrix $xI_m - A$ is called the *characteristic matrix* of A and its determinant is called the *characteristic polynomial* of A.

The invariant factors of $xI_m - A$ are called the *invariant polynomials* of A.

Note that the matrix $xI_m - A$ has rank m, since its determinant is different from zero. Consequently A has m invariant polynomials,

$$f_1(x) \mid \cdots \mid f_m(x).$$

It is also known that the characteristic polynomial of a matrix $A \in F^{m\times m}$ it is equal to the product of its invariant polynomials.

The invariant polynomials of A which are equal to 1, are called the *trivial invariant polynomials* of A. The remaining invariant polynomials of A are called the *nontrivial invariant polynomials* of A and are denoted by $i(A)$.

Remark 1 $A, B \in F^{m\times m}$ are similar matrices in F if and only if they have the same invariant polynomials.

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, $n \geq 1$, be a monic polynomial of $F[x]$. The matrix

$$C(f) = \left[\begin{array}{c|c} 0 & I_{n-1} \\ \hline -a_0 & -a_1 \cdots - a_{n-1} \end{array}\right]$$

is known as the *companion matrix* of $f(x)$.

It is not hard to see that the only nontrivial polynomial of $C(f)$ is $f(x)$ and consequently, its characteristic polynomial is $f(x)$.

If $A \in F^{p\times p}, B \in F^{p\times q}$, $C \in F^{q\times p}$, we denote by $i[A\ B]$ the number of nontrivial invariant factors of the matrix pencil $[xI_p - A| - B]$ and by

$$i\begin{bmatrix} A \\ C \end{bmatrix}$$

the number of nontrivial invariant factors of the matrix pencil

$$\begin{bmatrix} xI_p - A \\ -C \end{bmatrix}.$$

It is known from systems theory [9] that a pair (A, B), where $A \in F^{p\times p}$, $B \in F^{p\times q}$ is *completely controllable* if and only if all the invariant factors of the matrix pencil

$$[xI_p - A| - B]$$

are equal to 1, if and only if the controllability matrix

$$\mathcal{C}(A, B) = \begin{bmatrix} B & AB & \cdots & A^{p-1}B \end{bmatrix} \in F^{p\times pq}$$

has rank equal to p, if and only if

$$\min_{\lambda\in\overline{F}} \operatorname{rank} [\lambda I_p - A| - B] = p,$$

where $\overline{F}$ is an algebraic closure of F.

Let $A_1, A_2 \in F^{p\times p}$, $B_1, B_2 \in F^{p\times q}$. The matrices $[A_1\ B_1]$ and $[A_2\ B_2]$ are said to be *block-similar* if there exist nonsingular matrices $P \in F^{p\times p}$, $Q \in F^{q\times q}$ and a matrix $R \in F^{q\times p}$, such that

$$\begin{bmatrix} A_2 & B_2 \end{bmatrix} = P^{-1}\begin{bmatrix} A_1 & B_1 \end{bmatrix}\begin{bmatrix} P & 0 \\ R & Q \end{bmatrix}.$$

It is known that $[A_1\ B_1]$ and $[A_2\ B_2]$ are block-similar if and only if the matrix pencils $[xI_p - A_1| - B_1]$ and $[xI_p - A_2| - B_2]$ are strictly equivalent.

It is also known that the matrices $[A_1\ B_1]$ and $[A_2\ B_2]$ are block-similar if and only if the matrix pencils $[xI_p - A_1| - B_1]$ and $[xI_p - A_2| - B_2]$ have the same invariant factors and the same column minimal indices.

3. Main Results

Let F be a field and let $n, k, p_1, \ldots, p_k$ be positive integers such that $n = p_1 + \cdots + p_k$. Let $(r_1, s_1), \ldots, (r_k, s_k) \in \{1, \ldots, k\}\times\{1, \ldots, k\}$ and assume that $r_i < k, i \in \{1, \ldots, k\}$. Let $A_{r_i,s_i} \in F^{p_{r_i}\times p_{s_i}}, i \in \{1, \ldots, k\}$. Our main goal is to solve the following problem.

Problem. *Under which conditions does there exist a completely controllable pair of the form*

$$(A_1, A_2) = \left(\begin{bmatrix} C_{1,1} & \cdots & C_{1,k-1} \\ \vdots & & \vdots \\ C_{k-1,1} & \cdots & C_{k-1,k-1} \end{bmatrix}, \begin{bmatrix} C_{1,k} \\ \vdots \\ C_{k-1,k} \end{bmatrix}\right) \tag{5}$$

with $C_{r_i,s_i} = A_{r_i,s_i}, i \in \{1, \ldots, k\}$?

In [2] we established conditions under which there exists a completely controllable pair of the form (5), when $k-1$ blocks of the same size are prescribed and the others are unknown.

Proposition 3

Let F be an arbitrary field. Let $A_{r_i,s_i} \in F^{p_{r_i} \times p_{s_i}}$, $i \in \{1,\dots,k\}$. If one of the following conditions holds, then there exists no completely controllable pair of the form (5) such that $C_{r_i,s_i} = A_{r_i,s_i}, i \in \{1,\dots,k\}$. The conditions are the following:

(i_3) *There exists $r \in \{1,\dots,k-1\}$ such that all the positions (r,j), with $j \in \{1,\dots,k\}\backslash\{r\}$, are prescribed equal to 0.*

(ii_3) *All the positions (i,k), with $i \in \{1,\dots,k-1\}$, are prescribed equal to 0.*

Proof. Let (A_1, A_2) be a pair of the form (5) such that $C_{r_i,s_i} = A_{r_i,s_i}, i \in \{1,\dots,k\}$, and assume that one of the conditions (i_3), (ii_3) occurs.

Case 1. Suppose that (i_3) holds. We may assume, without loss of generality, that $r = 1$. Let $\lambda \in \overline{F}$ be an eigenvalue of $C_{1,1}$. Hence,

$$\operatorname{rank}\left[\lambda I_{n-p_1} - A_1 | - A_2\right] < n - p_1. \tag{6}$$

Consequently, (A_1, A_2) is not completely controllable.

Case 2. Suppose that (ii_3) holds. Now let $\lambda \in \overline{F}$ be an eigenvalue of A_1. Again, as in the previous case, (6) holds. Therefore (A_1, A_2) is not completely controllable. □

In the previous result we identified exceptional conditions for this problem, nevertheless its solution is still an open problem. In order to give some insight into this question, we start by studying the case $k = 3$. In this paper we identify conditions under which the pair of the form (5) is completely controllable, when $C_{1,2}, C_{1,3}, C_{2,1}$ are prescribed and the other blocks are unkown.

Our main result is the following.

Theorem 4

Let F be an infinite field. Let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$ and $C_{2,1} \in F^{p_2 \times p_1}$. Then, there exist $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$ such that the pair

$$(A_1, A_2) = \left(\begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}, \begin{bmatrix} C_{1,3} \\ C_{2,3} \end{bmatrix} \right) \tag{7}$$

is completely controllable, except if the following condition (E) holds:

(E) $C_{1,2} = 0$ and $C_{1,3} = 0$.

Lemma 5

Let F be an arbitrary field. If (E) occurs, then there exists no completely controllable pair of the form (7), with $C_{1,2}$, $C_{1,3}$ and $C_{2,1}$ prescribed.

Proof. This result is a particular situation of Proposition 3. □

The following theorem was established separately by E.M. Sá [23] and R.C. Thompson [29] and it is a very important result in Matrix Completion Problems. Usually in Matrix Theory this result is known as *The Interlacing Inequalities for the Invariant factors*.

Theorem 6 [23, 29]

Let F be an arbitrary field. Let $l_1(x), \ldots, l_s(x) \in F[x]$ be monic polynomials such that $l_1(x) \mid \cdots \mid l_s(x)$. Let $A(x) \in F[x]^{p\times q}$ and let $i_1(x), \ldots, i_r(x)$ be the invariant factors of $A(x)$. Then, there exist $B(x) \in F[x]^{p\times(n-q)}$, $C(x) \in F[x]^{(m-p)\times q}$, $D(x) \in F[x]^{(m-p)\times(n-q)}$ such that

$$\begin{bmatrix} A(x) & B(x) \\ C(x) & D(x) \end{bmatrix} \in F[x]^{m\times n}$$

has invariant factors $l_1(x), \ldots, l_s(x)$ if and only if the following conditions are satisfied:

(i_6) $r \leq s \leq r + (m-p) + (n-q)$;

(ii_6) $s \leq \min\{m, n\}$;

(iii_6) $l_k \mid i_k$, *for every* $k \in \{1, \ldots, r\}$;

(iv_6) $i_k \mid l_{k+(m-p)+(n-q)}$, *for every* $k \in \{1, \ldots, r\}$ *such that* $k + (m-p) + (n-q) \leq s$.

The following lemma is a consequence of a result established by F.C. Silva in [28].

Lemma 7 [28]

Let F be an infinite field. Let $A_{1,1} \in F^{q_1\times q_1}$, $A_{1,2} \in F^{q_1\times q_2}$, $A_{1,3} \in F^{q_1\times q_3}$, $A_{2,1} \in F^{q_2\times q_1}$ and let $v \in \{0, \ldots, q_1+q_2\}$. There exist $A_{2,2} \in F^{q_2\times q_2}$ and $A_{2,3} \in F^{q_2\times q_3}$ such that

$$i\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \end{bmatrix} \leq v$$

if and only if

$$\max\left\{ i\begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \end{bmatrix}, i\begin{bmatrix} A_{1,1} \\ A_{2,1} \end{bmatrix} - q_3 \right\} \leq v.$$

The following result was obtained by H.K. Wimmer in [30], where the author provided the complete answer to problem (P_3) presented in Section 1.

Theorem 8 [30]

Let F be an arbitrary field. Let p, q be positive integers such that $n = p + q$ and let $f \in F[x]$ be a monic polynomial of degree n. Let $A_{1,1} \in F^{p\times p}$, $A_{1,2} \in F^{p\times q}$. Let $f_1|\cdots|f_p$ be the invariant factors of the matrix

$$[xI_p - A_{1,1}| - A_{1,2}].$$

Then, there exist $A_{2,1} \in F^{q\times p}$, $A_{2,2} \in F^{q\times q}$ such that the matrix of the form (2) has characteristic polynomial f if and only if $f_1 \cdots f_p | f$.

The following result was established by F.C. Silva in [24], where the author obtained a partial solution to problem (P_5) presented in Section 1.

Theorem 9 [24]

Let F be an arbitrary field. Let p, q be positive integers such that $n = p + q$. Let $c_1, \ldots, c_n \in F$. Let $A_{1,2} \in F^{p\times q}$, $A_{2,1} \in F^{q\times p}$. Then there exist $A_{1,1} \in F^{p\times p}$, $A_{2,2} \in F^{q\times q}$ such that the matrix of the form (2) has eigenvalues $c_1, \ldots, c_n$ if and only if one of the following conditions is satisfied:

(i_9) *$p \neq 1$ or $q \neq 1$;*

(ii_9) *$p = q = 1$ and the equation*

$$x^2 - (c_1 + c_2)x + ab + c_1c_2 = 0$$

has one root in F, where $A_{1,2} = [a]$ and $A_{2,1} = [b]$.

Proof of Theorem 4. Exception (E) has already been justified. Now, suppose that condition (E) is not satisfied. Let $r = \text{rank}C_{1,3}$. Let $P \in F^{p_1\times p_1}, Q \in F^{p_3\times p_3}$ be nonsingular matrices such that

$$PC_{1,3}Q = \begin{bmatrix} 0 & 0 \\ I_r & 0 \end{bmatrix}.$$

Partition $PC_{1,2}$ as follows:

$$PC_{1,2} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where $B_1 \in F^{(p_1-r)\times p_2}$, $B_2 \in F^{r\times p_2}$. Let $s = \text{rank}B_1$. Bearing in mind that (E) is not satisfied, at least one of the numbers r, s is different from zero. Let $R \in F^{(p_1-r)\times(p_1-r)}$, $S \in F^{p_2\times p_2}$ be nonsingular matrices such that

$$RB_1S = \begin{bmatrix} 0 & 0 \\ 0 & I_s \end{bmatrix}.$$

Let

$$C'_{1,2} = \begin{bmatrix} RB_1S \\ 0 \end{bmatrix} \in F^{p_1\times p_2},$$

and $C'_{1,3} = PC_{1,3}Q$. Consider $C_{2,1}P^{-1}$ partitioned as follows:

$$C_{2,1}P^{-1} = \begin{bmatrix} E_1 & E_2 \end{bmatrix},$$

where $E_1 \in F^{p_2\times(p_1-r)}$, $E_2 \in F^{p_2\times r}$. Let $C'_{2,1} = \begin{bmatrix} S^{-1}E_1R^{-1} & S^{-1}E_2 \end{bmatrix}$. Let $C'_{1,1} = C(x^{p_1}) \in F^{p_1\times p_1}$. Since at least one of the numbers r, s is different from zero, it follows that the pair

$$\left(C'_{1,1}, \begin{bmatrix} C'_{1,2} & C'_{1,3} \end{bmatrix}\right)$$

is completely controllable, i.e.,

$$i\begin{bmatrix} C'_{1,1} & C'_{1,2} & C'_{1,3} \end{bmatrix} = 0.$$

Now suppose that $\alpha_1|\cdots|\alpha_{p_1}$ are the invariant polynomials of $C'_{1,1}$ and $\beta_1|\cdots|\beta_{p_1}$ are the invariant factors of the matrix

$$\begin{bmatrix} xI_{p_1} - C'_{1,1} \\ -C'_{2,1} \end{bmatrix}.$$

According to Theorem 6, it follows that $\beta_i|\alpha_i, i \in \{1,\ldots,p_1\}$. Since $\alpha_1 = \cdots = \alpha_{p_1-1} = 1$, then $\beta_1 = \cdots = \beta_{p_1-1} = 1$. Consequently

$$i\begin{bmatrix} C'_{1,1} \\ C'_{2,1} \end{bmatrix} \leq 1.$$

Then

$$\max\left\{ i\begin{bmatrix} C'_{1,1} & C'_{1,2} & C'_{1,3} \end{bmatrix}, i\begin{bmatrix} C'_{1,1} \\ C'_{2,1} \end{bmatrix} - p_3 \right\} = 0.$$

By Lemma 7 there exist $C'_{2,2} \in F^{p_2 \times p_2}$, $C'_{2,3} \in F^{p_2 \times p_3}$ such that

$$(A'_1, A'_2) = \left(\begin{bmatrix} C'_{1,1} & C'_{1,2} \\ C'_{2,1} & C'_{2,2} \end{bmatrix}, \begin{bmatrix} C'_{1,3} \\ C'_{2,3} \end{bmatrix} \right)$$

is completely controllable. And $\begin{bmatrix} A'_1 & A'_2 \end{bmatrix}$ is block-similar to

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} = Y_1^{-1} Y_2^{-1} \begin{bmatrix} A'_1 & A'_2 \end{bmatrix} Z_1 Z_2 Z_3,$$

where

$$Z_1 = \begin{bmatrix} I_{p_1+p_2} & 0 \\ A & I_{p_3} \end{bmatrix}, \text{ with}$$

$$A = \begin{bmatrix} 0 & X \end{bmatrix} \in F^{p_3 \times (p_1+p_2)}, \; X = \begin{bmatrix} B_2 S \\ 0 \end{bmatrix} \in F^{p_3 \times p_2},$$

$$Z_2 = \begin{bmatrix} Y_2 & 0 \\ 0 & I_{p_3} \end{bmatrix}, Z_3 = \begin{bmatrix} Y_1 & 0 \\ 0 & Q^{-1} \end{bmatrix}, \text{ with}$$

$$Y_1 = \begin{bmatrix} P & 0 \\ 0 & S^{-1} \end{bmatrix}, Y_2 = \begin{bmatrix} R & 0 \\ 0 & I_{r+p_2} \end{bmatrix}.$$

Since $[A_1\ A_2]$ and $[A'_1\ A'_2]$ are block-similar, then the matrix pencils

$$[xI_{p_1+p_2} - A_1| - A_2]$$

and

$$[xI_{p_1+p_2} - A'_1| - A'_2]$$

have the same invariant factors. As (A'_1, A'_2) is completely controllable, i.e., $i\,[A'_1\ A'_2] = 0$, then $i\,[A_1\ A_2] = 0$, i.e., (A_1, A_2) is completely controllable. Clearly the pair (A_1, A_2) has the prescribed form. □

Proposition 10

Let F be an arbitrary field. Let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $f \in F[x]$ be a monic polynomial of degree n. Let $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$ and $C_{2,1} \in F^{p_2 \times p_1}$. If the exceptional condition (E) is satisfied, then there exist $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (4) has characteristic polynomial f if and only if f has a divisor of degree p_1.

Proof. Suppose that condition (E) occurs and assume that there exist $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (4) has characteristic polynomial f. Let $\alpha_1 | \cdots | \alpha_{p_1}$ be the invariant polynomials of $C_{1,1}$ and let $\beta_1 | \cdots | \beta_n$ be the invariant polynomials of (4). Note that $\alpha_1 | \cdots | \alpha_{p_1}$ are the invariant factors of the matrix

$$\begin{bmatrix} xI_{p_1} - C_{1,1} & -C_{1,2} & -C_{1,3} \end{bmatrix}.$$

According to Theorem 6, $\alpha_i | \beta_{i+p_2+p_3}$, $i \leq p_1$. Therefore,

$$\alpha_1 \cdots \alpha_{p_1} | \beta_{1+p_2+p_3} \cdots \beta_n | \beta_1 \cdots \beta_n = f.$$

As $\alpha_1 \cdots \alpha_{p_1}$ is the characteristic polynomial of $C_{1,1}$, then $\deg(\alpha_1 \cdots \alpha_{p_1}) = p_1$ and so the result is satisfied.

Conversely, let $g(x)$, $h(x) \in F[x]$ such that $f = gh$, with $\deg(g) = p_1$. Let $G = C(g) \in F^{p_1 \times p_1}$ and $H = C(h) \in F^{(p_2+p_3) \times (p_2+p_3)}$. Then for every $C_{3,1} \in F^{p_3 \times p_1}$ the matrix

$$\begin{bmatrix} G & B \\ D & H \end{bmatrix}, \tag{8}$$

where

$$B = \begin{bmatrix} C_{1,2} & C_{1,3} \end{bmatrix} \tag{9}$$

and

$$D = \begin{bmatrix} C_{2,1} \\ C_{3,1} \end{bmatrix} \tag{10}$$

has characteristic polynomial f. □

Corollary 11

Let F be an infinite field. Let n, p_1, p_2, p_3 be positive integers such that $n = p_1 + p_2 + p_3$. Let $f \in F[x]$ be a monic polynomial of degree n. Let $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$ and $C_{2,1} \in F^{p_2 \times p_1}$. If (E) is not satisfied, then there exist $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (4) has characteristic polynomial f.

Proof. Assume that condition (E) is not satisfied. According to Theorem 4 there exist $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, such that the pair of the form (7) is completely controllable. Since $1 | f$, applying Theorem 8, there exist $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (4) has characteristic polynomial f. □

The following result is an immediate consequence of Proposition 10 and Corollary 11.

Corollary 12

Let F be an infinite field. Let $c_1, \ldots, c_n \in F$. Let $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$ and $C_{2,1} \in F^{p_2 \times p_1}$. Then there exist $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (4) has eigenvalues $c_1, \ldots, c_n$.

Note that Corollary 12 is still valid for arbitrary fields, as we show in the following result, with a different approach.

Proposition 13

Let F be an arbitrary field. Let $c_1, \ldots, c_n \in F$. Let $C_{1,2} \in F^{p_1 \times p_2}$, $C_{1,3} \in F^{p_1 \times p_3}$ and $C_{2,1} \in F^{p_2 \times p_1}$. Then there exist $C_{1,1} \in F^{p_1 \times p_1}$, $C_{2,2} \in F^{p_2 \times p_2}$, $C_{2,3} \in F^{p_2 \times p_3}$, $C_{3,1} \in F^{p_3 \times p_1}$, $C_{3,2} \in F^{p_3 \times p_2}$, $C_{3,3} \in F^{p_3 \times p_3}$ such that the matrix of the form (4) has eigenvalues $c_1, \ldots, c_n$.

Proof. *Case* 1. Suppose that condition (E) is satisfied. Let $G = \operatorname{diag}(c_1, \ldots, c_{p_1})$, $H = \operatorname{diag}(c_{p_1+1}, \ldots, c_n)$ and let $C_{3,1} \in F^{p_3 \times p_1}$ be an arbitrary matrix. Let $B \in F^{p_1 \times (p_2+p_3)}$ and $D \in F^{(p_2+p_3) \times p_1}$ with the forms (9) and (10), respectively. Then, the matrix of the form (8) has eigenvalues $c_1, \ldots, c_n$.

Case 2. Suppose that condition (E) is not satisfied and $C_{2,1} = 0$. Let G and H defined as in the previous case. Let $C_{3,1} = 0 \in F^{p_3 \times p_1}$. Let $B \in F^{p_1 \times (p_2+p_3)}$ and $D \in F^{(p_2+p_3) \times p_1}$ with the forms (9) and (10), respectively. Then the matrix of the form (8) has eigenvalues $c_1, \ldots, c_n$.

Case 3. Suppose that condition (E) is not satisfied and $C_{2,1} \neq 0$. Let $C_{3,1} \in F^{p_3 \times p_1}$ be an arbitrary matrix. Let $B \in F^{p_1 \times (p_2+p_3)}$ and $D \in F^{(p_2+p_3) \times p_1}$ with the forms (9) and (10), respectively. Since $p_2 + p_3 \neq 1$, according to Theorem 9 there exist $G \in F^{p_1 \times p_1}$ and $H \in F^{(p_2+p_3) \times (p_2+p_3)}$ such that the matrix of the form (8) has eigenvalues $c_1, \ldots, c_n$. □

4. Concluding Remarks

In this paper we establish conditions under which the system of the form (1) is completely controllable, when some entries of $[A\ B]$ are prescribed and the others are unknown. This is an advance in this type of problems. However the general problem of finding a completely controllable pair of the form (5), when k of its blocks are fixed and the remaining are unknown, is still open.

Our approach allows to solve a special question on Matrix Completion Problems. Considerable work has been done in this type of problems, however many questions still have only partial solutions and others remain open. Further research is required to solve this type of problems.

The general problem of describing the possible characteristic polynomials of a matrix of the form (3) when $k > 3$, and some of its blocks are prescribed and the remaining are unknown is still open. When the prescribed positions correspond to "large" submatrices, there are necessary interlacing inequalities involving invariant factors [23, 29]. The technique used to prove these inequalities can be very hard.

In this paper we establish new results for the case $k = 3$, which is an advance concernig this question. On the other hand, our approach unifies important problems in this area. In particular, the work developed in this paper is an extension of Oliveira's problem [18].

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