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Spaces H^1 and BMO on $ax+b$–groups

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Abstract

Let S be the group $\mathbb{R}^d \ltimes \mathbb{R}^+$ endowed with the Riemannian symmetric space metric d and the right Haar measure ρ. The space (S,d,ρ) is a Lie group of exponential growth. In this paper we define an Hardy space H^1 and a BMO space in this context. We prove that the functions in BMO satisfy the John–Nirenberg inequality and that BMO may be identified with the dual space of H^1. We then prove that singular integral operators whose kernels satisfy a suitable integral Hörmander condition are bounded from H^1 to L^1 and from L^∞ to BMO. We also study the real interpolation between H^1, BMO and the L^p spaces.

1. Introduction

Let S be the group $\mathbb{R}^d \ltimes \mathbb{R}^+$ endowed with the product

$$(x,a)\cdot(x',a') = (x + a\,x', a\,a') \qquad \forall (x,a),\,(x',a') \in S\,.$$

We call S an *$ax+b$-group*. We endow S with the left-invariant Riemannian metric $ds^2 = a^{-2}(\,\mathrm{d}x^2 + \mathrm{d}a^2)$. We denote by d the corresponding metric, which is that of the $(d+1)$-dimensional hyperbolic space.

The group S is nonunimodular; the right and left Haar measures are given respectively by

$$\mathrm{d}\rho(x,a) = a^{-1}\,\mathrm{d}x\,\mathrm{d}a \qquad \text{and} \qquad \mathrm{d}\lambda(x,a) = a^{-(d+1)}\,\mathrm{d}x\,\mathrm{d}a\,.$$

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It is well known that the measure of the ball B_r centred at the identity and of radius r, behaves like

$$\rho(B_r) = \lambda(B_r) \sim \begin{cases} r^{d+1} & \text{if } r < 1 \\ \mathrm{e}^{dr} & \text{if } r \geq 1 \,. \end{cases}$$

This shows that the space (S, d, ρ) is of *exponential growth*. Throughout this paper, unless explicitly stated, we consider the right measure ρ on S and we denote by L^p the space $L^p(\rho)$ and by $\|\cdot\|_p$ the norm in this space, for all p in $[1, \infty]$.

Harmonic analysis on the space (S, d, ρ) has been the object of many investigations, mainly because it is an example of exponential growth group, where the classical theory of singular integral operators does not hold (see [5, 8, 9, 11, 13, 19]). In this context maximal operators, singular integrals and multiplier operators associated with a distinguished Laplacian have been studied. In particular, in the case when $d = 1$, S is the *affine group of the real line*, where the theory of singular integrals have been considered by many authors.

Recently W. Hebisch and T. Steger [13] adapted the classical *Calderón–Zygmund theory* to the space (S, d, ρ) and applied this theory to study singular integral operators in this context. The purpose of this paper is to develop a H^1–BMO theory in the space (S, d, ρ), which is a natural development of the Calderón–Zygmund theory introduced in [13] and which may be considered as an analogue of the classical theory.

The classical H^1–BMO theory holds in $(\mathbb{R}^n, d, m)$, where d is the Euclidean metric and m denotes the Lebesgue measure. In this context the spaces H^1 and BMO are defined as in [6, 16, 23] and satisfy the following properties:

(i) the space BMO may be identified with the dual space of H^1;
(ii) the functions in BMO satisfy the so-called John–Nirenberg inequality;
(iii) the Calderón–Zygmund operators are bounded from H^1 to L^1 and from L^∞ to BMO;
(iv) the real interpolation spaces between H^1 and BMO are the L^p spaces (see [6, 12, 15, 21, 22]).

We recall that there are several characterizations of the Hardy space H^1 in the classical setting. In particular, an atomic definition and a maximal characterization of H^1 are available. The properties (i)-(iv) involving H^1 were proved by using both its maximal characterization and its atomic definition.

Extensions of the H^1–BMO theory have been considered in the literature. In particular, a theory that parallels the Euclidean theory has been developed in *spaces of homogeneous type*. A space of homogeneous type is a measured metric space (X, d, μ) where the doubling condition is satisfied, i.e., there exists a constant C such that

$$\mu(B(x, 2r)) \leq C\, \mu(B(x, r)) \qquad \forall x \in X \qquad \forall r \in \mathbb{R}^+. \tag{1.1}$$

In the space (X, d, μ) a Calderón–Zygmund theory [3, 23] and a H^1–BMO theory [4, 7] have been studied. This theory is a generalization of the Euclidean one; in particular properties (i)-(iv) are satisfied.

It is natural to ask whether it is possible to develop a H^1–BMO theory in spaces which do not satisfy the doubling condition (1.1). This was done in the space $(\mathbb{R}^n, d, \mu)$, where d is the Euclidean metric and μ is a (possibly nondoubling) measure, which grows polinomially at infinity [17, 20, 25]. A space BMO was also introduced by A. Ionescu in symmetric spaces of the noncompact type and rank one: note that the BMO theory developed in [14] applies to the space (S,d) endowed with the Riemannian measure, i.e., the left Haar measure λ, but does not apply to the space (S,d,ρ), which we are considering in this paper.

G. Mauceri and S. Meda [18] introduced a H^1–BMO theory in the space $(\mathbb{R}^n,d,\gamma)$, where d is the Euclidean metric and γ is the Gauss measure, and applied this theory to study appropriate operators related to the Ornstein-Uhlenbeck operator.

In this paper we develop a H^1–BMO theory in the space (S,d,ρ) defined above. The starting point is the Calderón–Zygmund theory introduced in [13]. There exists a family of appropriate sets in S, which are called *Calderón–Zygmund sets*, which replaces the family of balls in the classical Calderón–Zygmund theory.

For each p in $(1,\infty]$, we define an *atomic Hardy space* $H^{1,p}$. Atoms are functions supported in Calderón–Zygmund sets, with vanishing integral and satisfying a certain size condition. An important feature of the classical theory is that all the spaces $H^{1,p}$, for p in $(1,\infty]$, are equivalent. We shall prove that this holds also in our setting. We define a space of *functions of bounded mean oscillation* BMO, whose definition is analogue to the classical one, where balls are replaced by Calderón–Zygmund sets. We shall prove that the John–Nirenberg inequality is satisfied and that BMO may be identified with the dual space of H^1.

Further, we show that a singular integral operator, whose kernel satisfies an integral Hörmander condition, extends to a bounded operator from H^1 to L^1 and from L^∞ to BMO. As a consequence of this result, we show that spectral multipliers of a distinguished Laplacian Δ extend to bounded operators from H^1 to L^1 and from L^∞ to BMO.

Finally, we find the real interpolation spaces between H^1 and L^p, L^p and BMO, H^1 and BMO, for p in $(1,\infty)$. The interpolation results which we prove are the analogues of the classical ones [12, 15, 21, 22], but the proofs are different. Indeed, in the classical setting the maximal characterization of the Hardy space is used to obtain the interpolation results, while the Hardy space H^1 introduced in this paper has only an atomic definition.

Positive constants are denoted by C; these may differ from one line to another, and may depend on any quantifiers written, implicitly or explicitly, before the relevant formula. Given two quantities f and g, by $f\sim g$ we mean that there exists a constant C such that $1/C\leq f/g\leq C$.

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2. The Hardy space

In this section, we give the definition of the Hardy space on S, where the Calderón–Zygmund sets are involved. Let us recall the definition of Calderón–Zygmund sets which appears in [13] and implicitly in [11].

Definition 2.1 A *Calderón–Zygmund set* is a set $R=Q\times[a\mathrm{e}^{-r},a\mathrm{e}^{r}]$, where Q is a dyadic cube in $\mathbb{R}^d$ of side L, $a \in \mathbb{R}^+$, $r > 0$ and

$$\mathrm{e}^2 a\, r \leq L < \mathrm{e}^8 a\, r \qquad \text{if } r < 1\,,$$

$$a\, \mathrm{e}^{2r} \leq L < a\, \mathrm{e}^{8r} \qquad \text{if } r \geq 1\,.$$

Let $\mathcal{R}$ denote the family of all Calderón–Zygmund sets.

In [13] the authors proved that the space (S, d, ρ) is a Calderón–Zygmund space with Calderón–Zygmund constant κ_0. More precisely, they proved that the following hold:

(i) for every set R in $\mathcal{R}$ there exist a point x_R and a positive number r_R such that $R \subseteq B(x_R, \kappa_0\, r_R)$;

(ii) for every set R in $\mathcal{R}$ its dilated set is defined as $R^* = \{x \in S :\ d(x, R) < r_R\}$; its right measure satisfies the following inequality:

$$\rho(R^*) \leq \kappa_0\, \rho(R)\,;$$

(iii) for every set R in $\mathcal{R}$ there exist mutually disjoint sets $R_1, \ldots, R_k$ in $\mathcal{R}$, with $2 \leq k \leq 2^d$, such that $R = \bigcup_{i=1}^k R_i$ and $\rho(R_i) = \rho(R)/k$, for $i = 1, \ldots, k$.

For any integrable function f and for any $\alpha > 0$, f admits a Calderón–Zygmund decomposition at level α, i.e., a decomposition $f = g + \sum_i b_i$, where g is bounded almost everywhere by $\kappa_0\, \alpha$ and the functions b_i have vanishing integral and are supported in Calderón–Zygmund sets R_i. The average of $|f|$ on each set R_i is comparable with α (see [13, Definition 1.1] for the details).

Suppose that p is in $(1, \infty]$. By replacing balls with Calderón–Zygmund sets in the classical definition of atoms, we say that a function a is a $(1, p)$-*atom* if it satisfies the following properties:

(i) a is supported in a Calderón–Zygmund set R;

(ii) $\|a\|_p \leq \rho(R)^{1/p-1}$;

(iii) $\int a \,\mathrm{d}\rho = 0$.

Observe that a $(1, p)$-atom is in L^1 and it is normalized in such a way that its L^1-norm does not exceed 1.

DEFINITION 2.2 The Hardy space $H^{1,p}$ is the space of all functions h in L^1 such that $h = \sum_j \lambda_j\, a_j$, where a_j are $(1, p)$-atoms and λ_j are complex numbers such that $\sum_j |\lambda_j| < \infty$. We denote by $\|h\|_{H^{1,p}}$ the infimum of $\sum_j |\lambda_j|$ over such decompositions.

The space $H^{1,p}$ endowed with the norm $\|\cdot\|_{H^{1,p}}$ is a Banach space.

For any p in $(1, \infty]$ we denote by $H^{1,p}_{\mathrm{fin}}$ the vector space of all finite linear combinations of $(1, p)$-atoms. Clearly, $H^{1,p}_{\mathrm{fin}}$ is dense in $H^{1,p}$.

It easily follows from the above definitions that $H^{1,\infty} \subseteq H^{1,p}$, whenever p is in $(1, \infty)$. Actually the following theorem holds.

Theorem 2.3

For any p in $(1, \infty)$, the spaces $H^{1,p}$ and $H^{1,\infty}$ coincide and their norms are equivalent.

To prove the Theorem 2.3 we follow the proof of [4, Theorem A]. We shall need the following preliminary result.

Proposition 2.4

Suppose that p is in $(1,\infty)$ and a is a $(1,p)$-atom. Then a is in $H^{1,\infty}$ and there exists a constant C_p, which depends only on p, such that

$$\|a\|_{H^{1,\infty}} \leq C_p\,.$$

Proof. Let a be a $(1,p)$-atom supported in the Calderón–Zygmund set R. We define $b := \rho(R)\,a$. Note that b is in L^p and $\|b\|_p \leq \rho(R)^{1/p}$.

Let α be a positive number such that $\alpha > \max\{1, 2^{-d/p}\,2^{1/(p-1)}\}$. We shall prove that for all $n \in \mathbb{N}$ there exist functions a_{j_ℓ}, h_{j_n} and Calderón–Zygmund sets R_{j_ℓ}, with $j_\ell \in \mathbb{N}^\ell$, $\ell = 0, ..., n$, such that

$$b = \sum_{\ell=0}^{n-1} 2^{(d(\ell+1))/p}\,2^\ell\,\alpha^{\ell+1} \sum_{j_\ell} \rho(R_{j_\ell})\; a_{j_\ell} + \sum_{j_n} h_{j_n}\,, \tag{2.1}$$

where the following properties are satisfied:

(i) a_{j_ℓ} is a $(1,\infty)$-atom supported in the Calderón–Zygmund set R_{j_ℓ};

(ii) h_{j_n} is supported in R_{j_n} and $\int h_{j_n}\,\mathrm{d}\rho = 0$;

(iii) $\left(\frac{1}{\rho(R_{j_n})}\int_{R_{j_n}} |h_{j_n}|^p\,\mathrm{d}\rho\right)^{1/p} \leq 2^{dn/p}\,2^n\,\alpha^n$;

(iv) $\sum_{j_n} \|h_{j_n}\|_p^p \leq 2^{pn}\,\|b\|_p^p$;

(v) $|h_{j_n}(x)| \leq |b(x)| + 2^{dn/p}\,2^n\,\alpha^n \qquad \forall x \in R_{j_n}$;

(vi) $\sum_{j_n} \rho(R_{j_n}) \leq 2^{d(-n+1)}\,\alpha^{-np}\,\|b\|_p^p$.

We first suppose that the decomposition (2.1) exists and we show that a lies in $H^{1,\infty}$. Set $H_n = \sum_{j_n} h_{j_n}$. By Hölder's inequality

$$\|H_n\|_1 \leq \sum_{j_n} \|h_{j_n}\|_1 \leq \sum_{j_n} \rho(R_{j_n})^{1/p'}\,\|h_{j_n}\|_p\,,$$

where p' is the conjugate exponent of p. Now by (iii) and (vi) we have

$$\begin{aligned}
\|H_n\|_1 &\leq \sum_{j_n} \rho(R_{j_n})^{1/p'}\rho(R_{j_n})^{1/p}\,2^{dn/p}\,2^n\,\alpha^n \\
&\leq 2^{d(-n+1)}\,\alpha^{-np}\,\|b\|_p^p\,2^{dn/p}\,2^n\,\alpha^n \\
&\leq 2^d\big(2\,2^{(d(1-p))/p}\,\alpha^{1-p}\big)^n \rho(R)\,.
\end{aligned}$$

Then, since $\alpha > 2^{-d/p}\,2^{1/(p-1)}$, the functions H_n converge to 0 in L^1 when n goes to ∞.

This shows that the series $\sum_{\ell=0}^\infty 2^{(d(\ell+1))/p}\,2^\ell\,\alpha^{\ell+1}\sum_{j_\ell}\rho(R_{j_\ell})\,a_{j_\ell}$ converges to b in L^1. Moreover, by (vi) we deduce that

$$\sum_{\ell=0}^{\infty} 2^{(d(\ell+1))/p}\, 2^{\ell}\, \alpha^{\ell+1} \sum_{j_\ell} \rho(R_{j_\ell}) \leq \sum_{\ell=0}^{\infty} 2^{(d(\ell+1))/p}\, 2^{\ell}\, \alpha^{\ell+1}\, 2^{d(-\ell+1)}\, \alpha^{-\ell p}\, \|b\|_p^p$$
$$\leq 2^{(d(1+1))/p}\, \alpha \sum_{\ell=0}^{\infty} \left(2\, 2^{(d(1-p))/p}\, \alpha^{1-p}\right)^{\ell} \rho(R)$$
$$= C_p\, \rho(R)\,,$$

because $\alpha > 2^{-d/p}\, 2^{1/(p-1)}$, where C_p depends only on d, p, α.

It follows that b is in $H^{1,\infty}$ and $\|b\|_{H^{1,\infty}} \leq C_p\, \rho(R)$. Thus $a = \rho(R)^{-1}\, b$ is in $H^{1,\infty}$ and $\|a\|_{H^{1,\infty}} \leq C_p$, as required.

It remains to prove that the decomposition (2.1) exists. This can be done by induction on n, following closely the proof of [4, Theorem A]. For the reader's convenience we give the proof in the case $n = 1$, and we shall omit the details of the inductive step.

We construct a partition $\mathcal{P}$ of S in Calderón–Zygmund sets which contains the set R (see [13, Proof of 5.1]).

Step $n = 1$. We choose $R_0 = R$. Since $\|b\|_p \leq \rho(R)^{1/p}$,

$$\frac{1}{\rho(R)} \int_R |b|^p \,\mathrm{d}\rho \leq \frac{1}{\rho(R)}\, \|b\|_p^p \,\mathrm{d}\rho \leq 1 \leq \alpha^p\,.$$

We split up the set R in at most 2^d Calderón–Zygmund subsets. If the average of $|b|^p$ on a subset is greater than α^p, then we stop; otherwise we divide again the subset until we find sets on which the average of $|b|^p$ is greater than α^p. We denote by $\mathcal{C}$ the collection of the stopping sets. We distinguish two cases.

Case $\mathcal{C} = \emptyset$. In this case it suffices to define $R_0 = R$, $a_0 = 2^{-d/p}\, \alpha^{-1}\, \rho(R_0)^{-1}\, b$ and $h_i = 0$ for all $i \in \mathbb{N}$.

Case $\mathcal{C} \neq \emptyset$. Let $\mathcal{C} = \{R_i :\ i \in \mathbb{N}\}$. The average of $|b|^p$ on each set R_i is comparable with α^p. Indeed, by construction we have

$$\frac{1}{\rho(R_i)} \int_{R_i} |b|^p \,\mathrm{d}\rho > \alpha^p\,.$$

On the other hand, there exists a set R_i', which contains R_i, such that $\rho(R_i) \geq \frac{\rho(R_i')}{2^d}$ and $\frac{1}{\rho(R_i')} \int_{R_i'} |b|^p \,\mathrm{d}\rho \leq \alpha^p$. It follows that

$$\frac{1}{\rho(R_i)} \int_{R_i} |b|^p \,\mathrm{d}\rho \leq \frac{2^d}{\rho(R_i')} \int_{R_i'} |b|^p \,\mathrm{d}\rho \leq 2^d\, \alpha^p\,.$$

We define

$$g(x) = \begin{cases} b(x) & \text{if } x \notin \bigcup_i R_i \\ \frac{1}{\rho(R_i)} \int_{R_i} b \,\mathrm{d}\rho & \text{if } x \in R_i \end{cases}$$

$$h_i(x) = \begin{cases} 0 & \text{if } x \notin R_i \\ b(x) - \frac{1}{\rho(R_i)} \int_{R_i} b \,\mathrm{d}\rho & \text{if } x \in R_i \qquad \forall i \in \mathbb{N}\,. \end{cases}$$

Obviously

$$b = g + \sum_i h_i = 2^{d/p}\,\alpha\,\rho(R_0)\,a_0 + \sum_i h_i\,,$$

where $a_0 = 2^{-d/p}\,\alpha^{-1}\,\rho(R_0)^{-1}\,g$.

The function a_0 is supported in R and has vanishing integral. By Hölder's inequality for any x in R_i

$$|g(x)| \leq \frac{1}{\rho(R_i)}\int_{R_i}|b|\,\mathrm{d}\rho \leq \frac{1}{\rho(R_i)}\rho(R_i)^{1/p'}\Big(\int_{R_i}|b|^p\,\mathrm{d}\rho\Big)^{1/p} \leq 2^{d/p}\,\alpha\,.$$

If x is in the complement of $\bigcup_i R_i$, then all the averages of $|b|^p$ on the sets of the partition $\mathcal{P}$ which contain x are $\leq \alpha^p$. Thus $|g(x)| \leq \alpha$ for almost every x in the complement of $\bigcup_i R_i$. Then $\|a_0\|_\infty \leq \rho(R_0)^{-1}$, so that a_0 is a $(1,\infty)$-atom.

We now verify that the functions h_i satisfy properties (ii)-(vi). Each function h_i is supported in R_i and has vanishing integral. Moreover, by Hölder's inequality

$$\|h_i\|_p \leq \|b\|_{L^p(R_i)} + \rho(R_i)^{1/p}\,\frac{1}{\rho(R_i)}\int_{R_i}|b|\,\mathrm{d}\rho \leq 2\,\|b\|_{L^p(R_i)}\,. \tag{2.2}$$

Since the sets R_i are mutually disjoint, by summing estimates (2.2) over $i \in \mathbb{N}$, we obtain

$$\sum_i \|h_i\|_p^p \leq 2^p \sum_i \|b\|^p_{L^p(R_i)} \leq 2^p\,\|b\|_p^p\,,$$

which proves (iv). From (2.2) we also have

$$\frac{1}{\rho(R_i)}\int_{R_i}|h_i|^p\,\mathrm{d}\rho \leq 2^p\,\frac{1}{\rho(R_i)}\int_{R_i}|b|^p\,\mathrm{d}\rho \leq M\,2^p\,\alpha^p\,,$$

which proves (iii). The pointwise estimate (v) of h_i is an easy consequence of Hölder's inequality, since for all x in R_i

$$\begin{aligned}|h_i(x)| &\leq |b(x)| + \frac{1}{\rho(R_i)}\int_{R_i}|b|\,\mathrm{d}\rho\\ &\leq |b(x)| + \rho(R_i)^{-1}\,\rho(R_i)^{1/p'}\Big(\int_{R_i}|b|^p\,\mathrm{d}\rho\Big)^{1/p}\\ &\leq |b(x)| + M^{1/p}\,2\,\alpha\,.\end{aligned}$$

It remains to prove property (vi):

$$\sum_i \rho(R_i) \leq \alpha^{-p}\sum_i\int_{R_i}|b|^p\,\mathrm{d}\rho \leq \alpha^{-p}\,\|b\|_p^p\,.$$

This concludes the proof of the first step in the case when $\mathcal{C} \neq \emptyset$.

Inductive step. Suppose that

$$b = \sum_{\ell=0}^{n-1} 2^{(d(\ell+1))/p}\,2^\ell\,\alpha^{\ell+1}\sum_{j_\ell}\rho(R_{j_\ell})a_{j_\ell} + \sum_{j_n} h_{j_n}\,,$$

where the functions a_{j_ℓ}, h_{j_ℓ} and the sets R_{j_ℓ} satisfy properties (i)-(vi). We shall prove that a similar decomposition of b holds with $(n+1)$ in place of n. To do so, we decompose each function h_{j_n} by following the same construction used in the case when $n=1$ and the proof of [4, Theorem A]. We omit the details.

This concludes the proof of the proposition. □

Theorem 2.3 is an easy consequence of Proposition 2.4.

In the sequel, we denote by H^1 the space $H^{1,\infty}$ and by $\|\cdot\|_{H^1}$ the norm $\|\cdot\|_{H^{1,\infty}}$.

3. The space BMO

In this section, we introduce the space of functions of bounded mean oscillation and we investigate its properties. For every locally integrable function f and every set R we denote by f_R the average of f on R, i.e., $f_R = \frac{1}{\rho(R)}\int_R f\,\mathrm{d}\rho$.

DEFINITION 3.1 The space $\mathcal{BMO}$ is the space of all functions in L^1_{loc} such that

$$\sup_R \frac{1}{\rho(R)}\int_R |f-f_R|\,\mathrm{d}\rho < \infty\,,$$

where the supremum is taken over all Calderón–Zygmund sets in the family $\mathcal{R}$. The space BMO is the quotient of $\mathcal{BMO}$ module constant functions. It is a Banach space endowed with the norm

$$\|f\|_* = \sup\Big\{\frac{1}{\rho(R)}\int_R |f-f_R|\,\mathrm{d}\rho :\ R\in\mathcal{R}\Big\}\,.$$

We now prove that the functions in BMO satisfy the John–Nirenberg inequality.

Theorem 3.2 (John–Nirenberg inequality)

There exist two positive constants η and A such that for any f in BMO

$$\sup_{R\in\mathcal{R}} \frac{1}{\rho(R)}\int_R \exp\Big(\frac{\eta}{\|f\|_*}|f-f_R|\Big)\,\mathrm{d}\rho \le A\,.$$

Proof. We first take f in L^∞. Let R_0 be a fixed Calderón–Zygmund set.

Note that $\frac{1}{\rho(R_0)}\int_{R_0}|f-f_{R_0}|\,\mathrm{d}\rho \le 2\,\|f\|_*$. We split up R_0 in at most 2^d Calderón–Zygmund sets. If the average of $|f-f_{R_0}|$ on a subset is $> 2\,\|f\|_*$, then we stop. Otherwise we go on by splitting the sets that we obtain, until we find Calderón–Zygmund sets contained in R_0 where the average of $|f-f_{R_0}|$ is $> 2\,\|f\|_*$. Let $\{R_i\}_i$ be the collection of the stopping sets. We have that:

(i) $|(f-f_{R_0})\chi_{R_0}| \le 2\|f\|_*$ on $(\cup_i R_i)^c$;

(ii) $\rho(\cup_i R_i) \le \frac{\|(f-f_{R_0})\chi_{R_0}\|_1}{2\|f\|_*} \le \frac{\rho(R_0)\,\|f\|_*}{2\|f\|_*} = \frac{\rho(R_0)}{2}$;

(iii) $\frac{1}{\rho(R_i)}\int_{R_i}|f-f_{R_0}|\chi_{R_0}\,\mathrm{d}\rho > 2\,\|f\|_*$;

(iv) for each set R_i there exists a Calderón–Zygmund set R_i' which contains R_i, whose measure is $\leq 2^d\,\rho(R_i)$ and such that $\frac{1}{\rho(R_i')}\int_{R_i'}|f-f_{R_0}|\chi_{R_0}\,\mathrm{d}\rho \leq 2\,\|f\|_*$. Thus

$$\begin{aligned}|f_{R_i}-f_{R_0}| &\leq |f_{R_i}-f_{R_i'}|+|f_{R_i'}-f_{R_0}|\\ &\leq \frac{1}{\rho(R_i)}\int_{R_i}|f-f_{R_i'}|\,\mathrm{d}\rho+\frac{1}{\rho(R_i')}\int_{R_i'}|f-f_{R_0}|\,\mathrm{d}\rho\\ &\leq \frac{2^d}{\rho(R_i')}\int_{R_i'}|f-f_{R_i'}|\,\mathrm{d}\rho+2\,\|f\|_*\\ &\leq (2^d+2)\,\|f\|_*\,.\end{aligned}$$

For any positive t we define

$$F(t)=\sup_R\frac{1}{\rho(R)}\int_R\exp\Big(\frac{t}{\|f\|_*}|f-f_R|\Big)\,\mathrm{d}\rho,$$

which is finite, since we are assuming that f is bounded. From (i)-(iv) above we obtain that

$$\begin{aligned}&\frac{1}{\rho(R_0)}\int_{R_0}\exp\Big(\frac{t}{\|f\|_*}|f-f_{R_0}|\Big)\,\mathrm{d}\rho\\ &\quad\leq \frac{1}{\rho(R_0)}\int_{R_0-\cup_iR_i}\mathrm{e}^{2t}\,\mathrm{d}\rho+\frac{1}{\rho(R_0)}\sum_i\int_{R_i}\\ &\qquad\times\exp\Big(\frac{t}{\|f\|_*}(|f-f_{R_i}|+|f_{R_i}-f_{R_0}|)\Big)\,\mathrm{d}\rho\\ &\quad\leq \mathrm{e}^{2t}+\frac{1}{\rho(R_0)}\sum_i\int_{R_i}\mathrm{e}^{(2^d+2)t}\,\exp\Big(\frac{t}{\|f\|_*}|f-f_{R_i}|\Big)\,\mathrm{d}\rho\\ &\quad\leq \mathrm{e}^{2t}+\mathrm{e}^{(2^d+2)t}\,\frac{1}{\rho(R_0)}\,\frac{\rho(R_0)}{2}\,F(t)\,.\end{aligned}$$

By taking the supremum over all Calderón–Zygmund sets R_0 we deduce that

$$F(t)(1-\mathrm{e}^{(2^d+2)t}/2)\leq \mathrm{e}^{2t}\,.$$

This implies that there exists a sufficiently small positive η such that $F(\eta)\leq C$.

This proves the theorem for all bounded functions. Now let f be in BMO and for $k\in\mathbb{N}$ define $f_k:S\to\mathbb{C}$ by

$$f_k(x)=\begin{cases}f(x) & \text{if } |f(x)|\leq k\\ k\,\frac{f(x)}{|f(x)|} & \text{if } |f(x)|>k\,.\end{cases}$$

Then $\|f_k\|_\infty\leq k$ and $\|f_k\|_*\leq C\,\|f\|_*$. Moreover $|f_k-f|$ tends monotonically to zero when k tends to ∞. We have that

$$\begin{aligned}\frac{1}{\rho(R)}\int_R \exp\Big(\frac{\eta}{\|f\|_*}|f-f_R|\Big)\,\mathrm{d}\rho &\le \frac{1}{\rho(R)}\int_R \exp\Big(\frac{\eta}{\|f\|_*}|f-f_k|\Big)\,\mathrm{d}\rho\\ &\quad+\frac{1}{\rho(R)}\int_R \exp\Big(\frac{\eta}{\|f_k\|_*}|f_k-(f_k)_R|\Big)\,\mathrm{d}\rho\\ &\quad+\frac{1}{\rho(R)}\int_R \exp\Big(\frac{\eta}{\|f\|_*}|(f_k)_R-f_R|\Big)\,\mathrm{d}\rho\\ &\le C+\frac{1}{\rho(R)}\int_R \exp\Big(\frac{\eta}{\|f_k\|_*}|f_k-(f_k)_R|\Big)\,\mathrm{d}\rho\\ &\le A\,,\end{aligned}$$

if k is sufficiently large. Thus the theorem is proved for all functions in BMO. □

A standard consequence of the John–Nirenberg inequality is the following.

Corollary 3.3

The following hold:

(i) *there exist two positive constants η and A such that for any $t>0$*

$$\rho(\{x\in R:\ |f(x)-f_R|>t\,\|f\|_*\})\le A\,\mathrm{e}^{-\eta\,t}\,\rho(R)\qquad \forall R\in\mathcal{R},\ \forall f\in BMO\,;$$

(ii) *for any q in $(1,\infty)$ there exists a constant C_q, which depends only on q, such that*

$$\Big(\frac{1}{\rho(R)}\int_R |f-f_R|^q\,\mathrm{d}\rho\Big)^{1/q}\le C_q\,\|f\|_*\qquad \forall R\in\mathcal{R},\ \forall f\in BMO\,.$$

Proof. Let f be in BMO, R be a Calderón–Zygmund set and take $t>0$.

To prove (i) we observe that by Theorem 3.2

$$\begin{aligned}\rho(\{x\in R:\ |f(x)-f_R|>t\,\|f\|_*\}) &= \rho\Big(\{x\in R:\ \exp\Big(\frac{\eta}{\|f\|_*}|f(x)-f_R|\Big)>\mathrm{e}^{\eta\,t}\}\Big)\\ &\le \frac{\int_R \exp\Big(\frac{\eta}{\|f\|_*}|f-f_R|\Big)\,\mathrm{d}\rho}{\mathrm{e}^{\eta\,t}}\\ &\le A\,\mathrm{e}^{-\eta\,t}\,\rho(R)\,,\end{aligned}$$

where η and A are the constants which appear in Theorem 3.2.

We now prove (ii). If q is in $(1,\infty)$, then there exists C such that $x^q\le C\,\mathrm{e}^{\eta\,x}$ for $x>0$. It clearly follows that

$$\int_R \frac{|f-f_R|^q}{\|f\|_*^q}\,\mathrm{d}\rho\le \int_R \exp\Big(\frac{\eta}{\|f\|_*}|f-f_R|\Big)\,\mathrm{d}\rho\le C\,\rho(R)\,.$$

Thus

$$\Big(\frac{1}{\rho(R)}\int_R |f-f_R|^q\,\mathrm{d}\rho\Big)^{1/q}\le C_q\,\|f\|_*\,,$$

where C_q only depends on q. □

For any q in $[1,\infty)$ and for every function f in L^q_{loc} define

$$\|f\|_{q,*}=\sup_{R\in\mathcal{R}}\Big(\frac{1}{\rho(R)}\int_R |f-f_R|^q\,\mathrm{d}\rho\Big)^{1/q}\,,$$

and $BMO_q=\{f\in L^q_{\mathrm{loc}}:\ \|f\|_{q,*}<\infty\}$. Note that $BMO_1=BMO$ and $\|\cdot\|_{1,*}=\|\cdot\|_*$.

By Corollary 3.3(ii), if f is in BMO, then $f \in BMO_q$ and $\|f\|_{q,*} \leq C_q \|f\|_*$, for any q in $(1,\infty)$.

Conversely, for any q in $(1,\infty)$, if f is in BMO_q, then trivially f is in BMO and $\|f\|_* \leq \|f\|_{q,*}$.

This means that all the spaces BMO_q, with q in $(1,\infty)$, are equivalent to BMO.

We now prove that the dual space of $H^{1,2}$ may be identified with BMO_2.

Theorem 3.4 (duality theorem)

The following hold:

(i) *for any f in BMO_2 the functional ℓ defined on $H^{1,2}_{\mathrm{fin}}$ by*

$$\ell(g) = \int f\, g \,\mathrm{d}\rho \qquad \forall g \in H^{1,2}_{\mathrm{fin}},$$

extends to a bounded functional on $H^{1,2}$. Furthermore, there exists a constant C such that

$$\|\ell\|_{(H^{1,2})^*} \leq C\,\|f\|_{2,*};$$

(ii) *there exists a constant C such that for any bounded linear functional ℓ on $H^{1,2}$ there exists a function f^ℓ in BMO_2 such that $\|f^\ell\|_{2,*} \leq C\,\|\ell\|_{(H^{1,2})^*}$ and $\ell(g) = \int f^\ell\, g\,\mathrm{d}\rho$ for any g in $H^{1,2}_{\mathrm{fin}}$.*

Proof. The proof of (i) follows the proof of the analogue result in the classical setting [4, 23]. We omit the details.

We now prove (ii). For any $n \in \mathbb{N}$ let R_n be the Calderón–Zygmund set $Q_n \times [\mathrm{e}^{-n}, \mathrm{e}^{n}]$, where Q_n is a dyadic cube in $\mathbb{R}^d$ centred at 0 of side L_n, such that $\mathrm{e}^{2n} \leq L_n < \mathrm{e}^{8n}$. Obviously, $\bigcup_n R_n = S$.

For any $n \in \mathbb{N}$ let X_n be the space $L^2_0(R_n)$ of all functions in L^2 which are supported in R_n and have vanishing integral. The space $(X_n, \|\cdot\|_2)$ is a Banach space. We denote by X the space $L^2_{c,0}(S)$ of all functions in L^2 with compact support and vanishing integral, interpreted as the strict inductive limit of the spaces X_n (see [2, II, p. 33] for the definition of the strict inductive limit topology). Observe that $H^{1,2}_{\mathrm{fin}}$ and X agree as vector spaces.

For any g in X_n the function $\rho(R_n)^{-1/2}\,\|g\|_2^{-1}\,g$ is a $(1,2)$-atom, so that g is in $H^{1,2}$ and $\|g\|_{H^{1,2}} \leq \rho(R_n)^{1/2}\,\|g\|_2$. Hence $X \subset H^{1,2}$ and the inclusion is continuous.

Now take a bounded linear functional ℓ on $H^{1,2}$. Since $X \subset H^{1,2}$, ℓ lies in the dual of X, i.e, the quotient space $L^2_{\mathrm{loc}}/\mathbb{C}$. Then there exists a function f^ℓ in L^2_{loc} such that

$$\ell(g) = \int f^\ell\, g\,\mathrm{d}\rho \qquad \forall g \in X.$$

It remains to show that f^ℓ is in BMO_2. Let R be a Calderón–Zygmund set. For any function g in X which is supported in R the function $\|g\|_2^{-1}\,\rho(R)^{-1/2}\,g$ is a $(1,2)$-atom. Thus

$$\Big|\int_R f^\ell\, g\,\mathrm{d}\rho\Big| = |\ell(g)| \leq \|\ell\|_{(H^{1,2})^*}\,\|g\|_2\,\rho(R)^{1/2}.$$

It easily follows that $\left(\int_R |f^\ell - f^\ell_R|^2 \,\mathrm{d}\rho\right)^{1/2} \le \|\ell\|_{(H^{1,2})^*}\,\rho(R)^{1/2}$, i.e., f^ℓ is in BMO_2 and $\|f^\ell\|_{2,*} \le \|\ell\|_{(H^{1,2})^*}$. $\square$

Since we already proved that the space H^1 is equivalent to $H^{1,2}$, and the space BMO is equivalent to BMO_2, the Theorem 3.4 means that BMO may be identified with the dual space of H^1.

4. H^1–L^1-boundedness of integral operators

We now prove that integral operators whose kernels satisfy a suitable integral Hörmander condition are bounded from H^1 to L^1 and from L^∞ to BMO. Note that the integral Hörmander condition which we require below is weaker than the integral conditions in the hypothesis of [13, Theorem 1.2].

Theorem 4.1

Let T be a linear operator which is bounded on L^2 and admits a locally integrable kernel K off the diagonal which satisfies the condition

$$\sup_{R\in\mathcal{R}}\ \sup_{y,\,z\in R} \int_{(R^*)^c} |K(x,y)-K(x,z)|\,\mathrm{d}\rho(x) < \infty\,. \tag{4.1}$$

Then T extends to a bounded operator from H^1 to L^1.

If the kernel K satisfies the condition

$$\sup_{R\in\mathcal{R}}\ \sup_{y,\,z\in R} \int_{(R^*)^c} |K(y,x)-K(z,x)|\,\mathrm{d}\rho(x) < \infty\,, \tag{4.2}$$

then T extends to a bounded operator from L^∞ to BMO.

Proof. Suppose that (4.1) is satisfied. We first show that there exists a constant C such that for any $(1,2)$-atom a

$$\|Ta\|_1 \le C\,. \tag{4.3}$$

Let a be a $(1,2)$-atom supported in the Calderón–Zygmund set R. Recall that $R \subseteq B(x_R, \kappa_0\, r_R)$, for some x_R in S and $r_R > 0$, and that R^* denotes the dilated set $\{x \in S:\ d(x,R) < r_R\}$.

We estimate the integral of Ta on R^* by the Cauchy–Schwarz inequality:

$$\begin{aligned}\int_{R^*} |Ta|\,\mathrm{d}\rho &\le \|Ta\|_2\,\rho(R^*)^{1/2}\\ &\le \kappa_0^{1/2}\,|\!|\!|T|\!|\!|_2\,\|a\|_2\,\rho(R)^{1/2}\\ &\le \kappa_0^{1/2}\,|\!|\!|T|\!|\!|_2\,.\end{aligned} \tag{4.4}$$

We consider the integral of $|Ta|$ on the complement of R^*:

$$\begin{aligned}
\int_{R^{*c}} |Ta| \,\mathrm{d}\rho &\leq \int_{(R^*)^c} \Big| \int_R K(x,y)\, a(y) \,\mathrm{d}\rho(y) \Big| \,\mathrm{d}\rho(x) \\
&= \int_{(R^*)^c} \Big| \int_R [K(x,y) - K(x,x_R)]\, a(y) \,\mathrm{d}\rho(y) \Big| \,\mathrm{d}\rho(x) \\
&\leq \int_{(R^*)^c} \int_R |K(x,y) - K(x,x_R)|\, |a(y)| \,\mathrm{d}\rho(y) \,\mathrm{d}\rho(x) \\
&= \int_R |a(y)| \Big(\int_{(R^*)^c} |K(x,y) - K(x,x_R)| \,\mathrm{d}\rho(x) \Big) \,\mathrm{d}\rho(y) \\
&\leq \|a\|_1 \sup_{y\in R} \int_{(R^*)^c} |K(x,y) - K(x,x_R)| \,\mathrm{d}\rho(x) \\
&\leq C\,.
\end{aligned} \tag{4.5}$$

By (4.4) and (4.5), the inequality (4.3) follows.

We shall deduce from (4.3) that T is bounded from H^1 to L^1. Indeed, by [13, Remark 1.4] T is bounded from L^1 to the Lorentz space $L^{1,\infty}$. Now take a function f in H^1 and suppose that $f = \sum_{j=1}^{\infty} \lambda_j a_j$ is an atomic decomposition of f with $\sum_j |\lambda_j| \sim \|f\|_{H^1}$. Define $f_N = \sum_{j=1}^{N} \lambda_j a_j$. Since f_N converges to f in L^1, $Tf_N = \sum_{j=1}^{N} \lambda_j Ta_j$ converges to Tf in $L^{1,\infty}$. On the other hand, by (4.3)

$$\Big\| Tf_N - \sum_{j=1}^{\infty} \lambda_j Ta_j \Big\|_1 \leq \sum_{j=N+1}^{\infty} |\lambda_j|\, \|Ta_j\|_1 \leq C \sum_{j=N+1}^{\infty} |\lambda_j|\,,$$

so that Tf_N converges to $\sum_{j=1}^{\infty} \lambda_j Ta_j$ in L^1. This implies that $Tf = \sum_{j=1}^{\infty} \lambda_j Ta_j \in L^1$ and $\|Tf\|_1 \leq C\, \|f\|_{H^1}$, i.e., T is bounded from H^1 to L^1.

Suppose now that (4.2) is satisfied. By arguing as before, we may prove that the adjoint operator T' of T is bounded from H^1 to L^1. By duality it follows that T is bounded from L^∞ to BMO. □

We can apply the previous results to the multipliers of a distinguished Laplacian Δ on S. Let

$$X_0 = a\partial_a \qquad X_i = a\partial_{x_i} \qquad i = 1, \ldots, d$$

be a basis of left-invariant vector fields of the Lie algebra of S and $\Delta = -\sum_{i=0}^{d} X_i^2$ be the corresponding left-invariant Laplacian, which is essentially self-adjoint on L^2. In [13] the authors studied a class of multipliers of Δ. More precisely, let ψ be a function in $C_c^\infty(\mathbb{R}^+)$, supported in $[1/4, 4]$, such that

$$\sum_{j\in\mathbb{Z}} \psi(2^{-j}\lambda) = 1 \qquad \forall \lambda \in \mathbb{R}^+\,.$$

Let m be a bounded measurable function on $\mathbb{R}^+$. We say that m satisfies a *mixed Mihlin-Hörmander condition of order* (s_0, s_∞) if

$$\sup_{t<1} \|m(t\cdot)\, \psi(\cdot)\|_{H^{s_0}(\mathbb{R})} < \infty \qquad \text{and} \qquad \sup_{t\geq 1} \|m(t\cdot)\, \psi(\cdot)\|_{H^{s_\infty}(\mathbb{R})} < \infty\,,$$

where $H^s(\mathbb{R})$ denotes the L^2-Sobolev space of order s on $\mathbb{R}$. By [13, Theorem 2.4] if m satisfies a mixed Mihlin-Hörmander condition of order (s_0, s_∞), with $s_0 > 3/2$ and $s_\infty > \max\{3/2, (d+1)/2\}$, then the operator $m(\Delta)$ is bounded from L^1 to $L^{1,\infty}$ and bounded on L^p, for p in $(1, \infty)$. We now prove a boundedness result for the same multipliers.

Proposition 4.2

Suppose that $s_0 > 3/2$ and $s_\infty > \max\{3/2, (d+1)/2\}$. If m satisfies a mixed Mihlin–Hörmander condition of order (s_0, s_∞), then the operator $m(\Delta)$ is bounded from H^1 to L^1 and from L^∞ to BMO.

Proof. The kernel of the operator $m(\Delta)$ satisfies the conditions (4.1) and (4.2) [13, Theorem 2.4]. By Theorem 4.1 the result follows. □

5. Real interpolation

In this section, we study the real interpolation of H^1, BMO and the L^p spaces. We first recall some notation of the real interpolation of normed spaces, focusing on the K-method. For the details see [1].

Given two compatible normed spaces X_0 and X_1, for any $t > 0$ and for any $x \in X_0 + X_1$ we define

$$K(t, x; X_0, X_1) = \inf\left\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : \ x = x_0 + x_1,\, x_i \in X_i\right\}.$$

Take q in $[1, \infty]$ and θ in $(0, 1)$. The *real interpolation space* $[X_0, X_1]_{\theta,q}$ is defined as the set of the elements $x \in X_0 + X_1$ such that

$$\|x\|_{\theta,q} = \begin{cases} \left(\int_0^\infty \left[t^{-\theta} K(t, x; X_0, X_1)\right]^q \frac{dt}{t}\right)^{1/q} & \text{if } q \in [1, \infty) \\ \|t^{-\theta} K(t, x; X_0, X_1)\|_\infty & \text{if } q = \infty\,, \end{cases}$$

is finite. The space $[X_0, X_1]_{\theta,q}$ endowed with the norm $\|\cdot\|_{\theta,q}$ is an exact interpolation space of exponent θ.

We refer the reader to [15] for an overview of the real interpolation results which hold in the classical setting. Our aim is to prove the same results in our context. Note that in our case a maximal characterization of the Hardy space is not available, so that we cannot follow the classical proofs but we shall only use the atomic definition of H^1 to prove the results.

We shall first estimate the K functional of L^p-functions with respect to the couple of spaces (H^1, L^{p_1}), with p_1 in $(1, \infty]$.

Lemma 5.1

Suppose that $1 < p < p_1 \le \infty$ and $\frac{1}{p} = 1 - \theta + \frac{\theta}{p_1}$, with θ in $(0, 1)$. Let f be in L^p. The following hold:

(i) *for every $\lambda > 0$ there exists a decomposition $f = g^\lambda + b^\lambda$ in $L^{p_1} + H^1$ such that*

 (a) *$\|g^\lambda\|_\infty \le C\,\lambda$;*

(b) *if* $p_1<\infty$, *then* $\|g^\lambda\|_{p_1}^{p_1}\leq C\,\lambda^{p_1-p}\,\|f\|_p^p$;

(c) $\|b^\lambda\|_{H^1}\leq C\,\lambda^{1-p}\,\|f\|_p^p$;

(ii) *for any* $t>0$, $K(t,f;H^1,L^{p_1})\leq C\,t^\theta\,\|f\|_p$;

(iii) $f\in[H^1,L^{p_1}]_{\theta,\infty}$ *and* $\|f\|_{\theta,\infty}\leq C\,\|f\|_p$.

Proof. Let f be in L^p. We first prove (i). Given a positive λ, let $\{R_j\}$ be the collection of sets associated with the Calderón–Zygmund decomposition of $|f|^p$ corresponding to the value λ^p. We write

$$f=g^\lambda+b^\lambda=g^\lambda+\sum_j b_j^\lambda=g^\lambda+\sum_j\,(f-f_{R_j})\,\chi_{R_j}\,.$$

We then have

$$\|g^\lambda\|_\infty\leq C\,\lambda\,,\qquad \frac{1}{\rho(R_j)}\int_{R_j}|f|^p\,\mathrm{d}\rho\sim\lambda^p\qquad\text{and}\qquad |f_{R_j}|\leq C\,\lambda.$$

If $p_1<\infty$, then

$$\begin{aligned}\|g^\lambda\|_{p_1}^{p_1}&\leq\sum_j\int_{R_j}|f_{R_j}|^{p_1}\,\mathrm{d}\rho+\int_{(\bigcup R_j)^c}|f|^{p_1}\,\mathrm{d}\rho\\&\leq C\,\lambda^{p_1}\sum_j\rho(R_j)+\int_{(\bigcup R_j)^c}|f|^{p_1-p}\,|f|^p\,\mathrm{d}\rho\\&\leq C\,\lambda^{p_1}\,\frac{\|f\|_p^p}{\lambda^p}+\lambda^{p_1-p}\,\|f\|_p^p\\&\leq C\,\lambda^{p_1-p}\,\|f\|_p^p\,.\end{aligned}$$

We now prove that b is in $H^{1,p}$. For any j, b_j^λ is supported in R_j, has vanishing integral and

$$\Big(\int_{R_j}|b_j^\lambda|^p\,\mathrm{d}\rho\Big)^{1/p}\leq C\,\rho(R_j)^{1/p}\,\lambda=C\,\lambda\,\rho(R_j)\,\rho(R_j)^{-1+1/p}\,.$$

This shows that $b_j^\lambda\in H^{1,p}=H^1$ and $\|b_j^\lambda\|_{H^1}\leq C\,\lambda\,\rho(R_j)$. Since $b^\lambda=\sum_j b_j^\lambda$, b^λ is in H^1 and

$$\|b^\lambda\|_{H^1}\leq C\,\lambda\sum_j\rho(R_j)\leq C\,\lambda\,\frac{\|f\|_p^p}{\lambda^p}\,,$$

as required.

We now prove (ii). Fix $t>0$. For any positive λ, let $f=g^\lambda+b^\lambda$ be the decomposition of f in $L^{p_1}+H^1$ given by (i). Thus

$$\begin{aligned}K(t,f;H^1,L^{p_1})&=\inf\{\|f_0\|_{H^1}+t\,\|f_1\|_{p_1}:\ f=f_0+f_1,\ f_0\in H^1,\ f_1\in L^{p_1}\}\\&\leq\inf_{\lambda>0}\big(\|b^\lambda\|_{H^1}+t\,\|g^\lambda\|_{p_1}\big)\\&\leq C\,\inf_{\lambda>0}\big(\lambda^{1-p}\,\|f\|_p^p+t\,\lambda^{1-p/p_1}\|f\|_p^{p/p_1}\big)\\&\leq C\,\|f\|_p^{p/p_1}\,\inf_{\lambda>0}\big(\lambda^{1-p}\,\|f\|_p^{p(1-1/p_1)}+t\,\lambda^{1-p/p_1}\big)\\&=C\,\|f\|_p^{p/p_1}\,\inf_{\lambda>0}G(t,\lambda)\,,\end{aligned}$$

where $G(t,\lambda) = \lambda^{1-p}\,\|f\|_p^{p(1-1/p_1)} + t\,\lambda^{1-p/p_1}$. We now compute the infimum of the function G with respect to the variable λ. Note that

$$\begin{aligned}\partial_\lambda G(t,\lambda) &= (1-p)\lambda^{-p}\,\|f\|_p^{p(1-1/p_1)} + (1-p/p_1)t\,\lambda^{-p/p_1}\\ &= \lambda^{-p}\big[(1-p)\,\|f\|_p^{p(1-1/p_1)} + (1-p/p_1)t\,\lambda^{-p/p_1+p}\big]\,.\end{aligned}$$

If $p_1 < \infty$, then

$$\inf_{\lambda>0} G(t,\lambda) = G\big(t, C_p\|f\|_p\, t^{p_1/p-pp_1}\big) = C_p\,\|f\|_p^{1-p/p_1}\, t^{\frac{p_1(p-1)}{p(p_1-1)}}\,.$$

If $p_1 = \infty$, then

$$\inf_{\lambda>0} G(t,\lambda) = G\big(t, C_p\|f\|_p\, t^{-1/p}\big) = C_p\,\|f\|_p\, t^{1-1/p}\,.$$

Hence,

$$K(t,f;H^1,L^{p_1}) \le C_p\,\|f\|_p\, t^{\theta}\,,$$

which proves (ii). This implies that $\|t^{-\theta}\,K(t,f;H^1,L^{p_1})\|_\infty \le C_p\,\|f\|_p$, so that $f \in [H^1,L^{p_1}]_{\theta,\infty}$ and $\|f\|_{\theta,\infty} \le C_p\|f\|_p$, as required in (iii). □

Theorem 5.2

Suppose that $1 < p < p_1 \le \infty$ *and* $\frac{1}{p} = 1-\theta+\frac{\theta}{p_1}$, *with* θ *in* $(0,1)$. *Then*

$$\big[H^1,L^{p_1}\big]_{\theta,p} = L^p\,.$$

Proof. Since $H^1 \subset L^1$, we have that $\big[H^1,L^{p_1}\big]_{\theta,p} \subset \big[L^1,L^{p_1}\big]_{\theta,p} = L^p$ [1, Theorem 5.2.1]. It remains to prove the converse inclusion.

To do so, we choose r, s, θ_0, θ_1 such that $1<r<p<s<p_1$, $\frac{1}{r} = 1-\theta_0+\frac{\theta_0}{p_1}$ and $\frac{1}{s} = 1-\theta_1+\frac{\theta_1}{p_1}$. By Lemma 5.1

$$L^r \subset [H^1,L^{p_1}]_{\theta_0,\infty} \qquad \text{and} \qquad L^s \subset [H^1,L^{p_1}]_{\theta_1,\infty}\,.$$

Choose η in $(0,1)$ such that $\frac{1}{p} = \frac{1-\eta}{r}+\frac{\eta}{s}$. Then by [1, Theorem 5.2.1]

$$L^p = [L^r,L^s]_{\eta,p} \subset \big[[H^1,L^{p_1}]_{\theta_0,\infty},[H^1,L^{p_1}]_{\theta_1,\infty}\big]_{\eta,p}\,.$$

It is easy to show that $\theta = (1-\eta)\,\theta_0 + \eta\,\theta_1$, so that by the reiteration theorem [1, Theorem 3.5.3]

$$\big[[H^1,L^{p_1}]_{\theta_0,\infty},[H^1,L^{p_1}]_{\theta_1,\infty}\big]_{\eta,p} = [H^1,L^{p_1}]_{\theta,p}\,.$$

Thus $L^p \subset [H^1,L^{p_1}]_{\theta,p}$, as required. □

We shall apply the duality theorem [1, Theorem 3.7.1] to deduce a corresponding interpolation result involving BMO and the L^p spaces. To do so, we shall need the following technical lemma.

Lemma 5.3

For any p_1 *in* $(1,\infty)$, $H^1\cap L^{p_1}$ *is dense in* H^1 *and in* L^{p_1}.

Proof. Since H^1_{fin} is contained in $H^1 \cap L^{p_1}$ and H^1_{fin} is dense in H^1, it is obvious that $H^1 \cap L^{p_1}$ is dense in H^1.

It remains to prove that $H^1 \cap L^{p_1}$ is dense in L^{p_1}.

Let $L^\infty_{c,0}$ denote the space of all functions in L^∞ with compact support and integral 0. If f is in $L^\infty_{c,0}$, then f is in L^{p_1} and f is a multiple of a $(1,\infty)$-atom, so that $f \in H^1$. Thus $L^\infty_{c,0} \subset H^1 \cap L^{p_1}$. It is easy to see that

(i) $L^\infty_{c,0}$ is dense in L^∞_c with respect to the L^{p_1}-norm;
(ii) L^∞_c is dense in L^{p_1}, since L^∞_c contains C_c which is dense in L^{p_1}.

Thus $L^\infty_{c,0}$ is dense in L^{p_1}. This implies that $H^1 \cap L^{p_1}$ is dense in L^{p_1}, as required. □

Corollary 5.4

Suppose that $1 < q_1 < q < \infty$ *and* $\frac{1}{q} = \frac{1-\theta}{q_1}$, *with* θ *in* $(0,1)$. *Then*

$$\left[L^{q_1}, BMO\right]_{\theta,q} = L^q\,.$$

Proof. Let p and p_1 be the conjugate exponents of q and q_1, respectively. Then $1 < p < p_1 < \infty$ and $\frac{1}{p} = \theta + \frac{1-\theta}{p_1}$. By Theorem 5.2

$$\left[H^1, L^{p_1}\right]_{1-\theta,p} = L^p\,.$$

Since by Lemma 5.3 $H^1 \cap L^{p_1}$ is dense in H^1 and in L^{p_1}, we can apply the duality theorem [1, Theorem 3.7.1] and conclude that

$$L^q = L^{p'} = \left[H^1, L^{p_1}\right]'_{1-\theta,p} = \left[(H^1)', (L^{p_1})'\right]_{1-\theta,p'} = \left[BMO, L^{q_1}\right]_{1-\theta,q}\,.$$

By [1, Theorem 3.4.1] it follows that

$$\left[L^{q_1}, BMO\right]_{\theta,q} = \left[BMO, L^{q_1}\right]_{1-\theta,q} = L^q\,,$$

as required. □

Note that Theorem 5.2 also concerns the limit case $p_1 = \infty$, showing that $[H^1, L^\infty]_{\theta,p} = L^p$, where $1/p = 1-\theta$. The Corollary 5.4 does not give a result for the limit case $q_1 = 1$, since it is not possible to deduce it by applying [1, Theorem 3.7.1]. To find the interpolation space $[L^1, BMO]_{\theta,q}$, where $1/q = 1-\theta$, we shall apply the reiteration theorem by T. Wolff. To do so we shall need the following technical lemma.

Lemma 5.5

For any p *in* $(1,\infty)$, $L^1 \cap BMO$ *is contained in* L^p.

Proof. Let p' denote the conjugate exponent of p. For any f in $L^{p'}$, by applying Lemma 5.1(i) with $\lambda = \|f\|_{p'}$, we may decompose f into a sum $f = g + b$ such that $\|g\|_\infty \le C_p\,\|f\|_{p'}$ and $\|b\|_{H^1} \le C_p\,\|f\|_{p'}$. Thus $f \in L^\infty + H^1$ and

$$\|f\|_{L^\infty + H^1} \le C_p\,\|f\|_{p'}\,.$$

This proves that $L^{p'} \subset L^\infty + H^1$. By duality we deduce that $L^p \supset \left(L^\infty + H^1\right)'$. It is easy to show that $\left(L^\infty + H^1\right)' \supset L^1 \cap BMO$, which concludes the proof of the lemma. □

We can now apply the reiteration theorem by T. Wolff [26, Theorem 1] to study the real interpolation between L^1 and BMO.

Proposition 5.6

Suppose that $1 < q < \infty$ *and* $\frac{1}{q} = 1 - \psi$, *with* ψ *in* $(0, 1)$. *Then*

$$[L^1, BMO]_{\psi,q} = L^q .$$

Proof. We choose r in $(1, q)$. By [1, Theorem 5.2.1] and Corollary 5.4

$$[L^1, L^q]_{\phi,r} = L^r \qquad \text{and} \qquad [L^r, BMO]_{\theta,q} = L^q ,$$

where $\frac{1}{r} = 1 - \phi + \frac{\phi}{q}$ and $\frac{1}{q} = \frac{1-\theta}{r}$. By Lemma 5.5, $L^1 \cap BMO \subset L^r \cap L^q$; then we can apply the reiteration theorem [26, Theorem 1] to conclude that

$$[L^1, BMO]_{\eta,q} = L^q ,$$

where $\psi = \frac{\theta}{1-\phi+\phi\theta}$. It is easy to verify that $\frac{1}{q} = 1 - \psi$, as required. □

We easily deduce a real interpolation result for H^1 and BMO.

Corollary 5.7

Suppose that $1 < q < \infty$ *and* $\frac{1}{q} = 1 - \psi$, *with* ψ *in* $(0, 1)$. *Then*

$$[H^1, BMO]_{\psi,q} = L^q .$$

Proof. Since $H^1 \subset L^1$, $[H^1, BMO]_{\psi,q} \subset [L^1, BMO]_{\psi,q} = L^q$. On the other hand, since $L^\infty \subset BMO$,

$$L^q = [H^1, L^\infty]_{\psi,q} \subset [H^1, BMO]_{\psi,q} ,$$

as required. □

By applying the reiteration theorem we may also deduce some real interpolation results involving Lorentz spaces. For the definition of the Lorentz spaces $L^{p,q}$ we refer the reader to [24, Chapter V].

Corollary 5.8

The following hold:

(i) *if* $1 < p < p_1 \leq \infty$, $1 \leq q$, $q_1 \leq \infty$, $\theta \in (0, 1)$ *and* $\frac{1}{p} = 1 - \theta + \frac{\theta}{p_1}$, *then*

$$[H^1, L^{p_1,q_1}]_{\theta,q} = L^{p,q} ;$$

(ii) *if* $1 \leq s, s_1 \leq \infty$, $1 \leq q_1 < q < \infty$, $\theta \in (0, 1)$ *and* $\frac{1}{q} = \frac{1-\theta}{q_1}$, *then*

$$[L^{q_1,s_1}, BMO]_{\theta,s} = L^{q,s} ;$$

(iii) *if* $1 < q < \infty$, $\theta \in (0, 1)$ *and* $\frac{1}{p} = 1 - \theta$, *then*

$$[H^1, BMO]_{\theta,q} = L^{p,q} .$$

References

1. J. Bergh and J. Löfström, *Interpolation Spaces, An Introduction*, Grundlehren der Mathematischen Wissenschaften **223**, Springer-Verlag, Berlin-New York, 1976.
2. N. Bourbaki, *Topological Vector Spaces, Chapters 1–5*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1987.
3. R.R. Coifman and G. Weiss, *Analyse Harmonique Non Commutative Sur Certains Espaces Homogenes*, Lecture Notes in Mathematics **242**, Springer-Verlag, Berlin-New York, 1971.
4. R.R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in Analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.
5. M. Cowling, S. Giulini, A. Hulanicki, and G. Mauceri, Spectral multipliers for a distinguished Laplacian on certain groups of exponential growth, *Studia Math.* **111** (1994), 103–121.
6. C. Fefferman and E.M. Stein, H^p spaces of several variables, *Acta Math.* **129** (1972), 137–193.
7. G.B. Folland and E.M. Stein, *Hardy Spaces on Homogeneous Groups*, Mathematical Notes **28**, Princeton University Press, Princeton, N.J., 1982.
8. G.I. Gaudry, T. Qian, and P. Sjögren, Singular integrals associated to the Laplacian on the affine group $ax + b$, *Ark. Mat.* **30** (1992), 259–281.
9. G.I. Gaudry and P. Sjögren, Singular integrals on Iwasawa NA groups of rank 1, *J. Reine Angew. Math.* **479** (1996), 39–66.
10. S. Giulini and G. Mauceri, Analysis of a distinguished Laplacian on solvable Lie groups, *Math. Nachr.* **163** (1993), 151–162.
11. S. Giulini and P. Sjögren, A note on maximal functions on a solvable Lie group, *Arch. Math. (Basel)* **55** (1990), 156–160.
12. R. Hanks, Interpolation by the real method between BMO, $L^\alpha(0 < \alpha < \infty)$ and $H^\alpha(0 < \alpha < \infty)$, *Indiana Univ. Math. J.* **26** (1977), 679–689.
13. W. Hebisch and T. Steger, Multipliers and singular integrals on exponential growth groups, *Math. Z.* **245** (2003), 37–61.
14. A.D. Ionescu, Singular integrals on symmetric spaces of real rank one, *Duke Math. J.* **114** (2002), 101–122.
15. P.W. Jones, Interpolation between Hardy spaces, *Conference on harmonic analysis in honor of Antoni Zygmund I, II, (Chicago, 1981)*, 437–451, Wadsworth, Belmont, CA, 1983.
16. J.L. Journé, *Calderón–Zygmund Operators, Pseudo-Differential Operators and the Cauchy Integral of Calderón*, Lecture Notes in Mathematics **994**, Springer-Verlag, Berlin, 1983.
17. J. Mateu, P. Mattila, A. Nicolau, and J. Orobitg, BMO for nondoubling measures, *Duke Math. J.* **102** (2000), 533–565.
18. G. Mauceri and S. Meda, A BMO space for the Gauss measure and applications, *J. Funct. Anal.* **252** (2007), 278–313.
19. D. Müller and Ch. Thiele, Wave equation and multiplier estimates on $ax+b$ groups, *Studia Math.* **179** (2007), 117–148.
20. F. Nazarov, S. Treil, and A. Volberg, The Tb-theorem on non-homogeneous spaces, *Acta Math.* **190** (2003), 151–239.
21. J. Peetre, Two observations on a theorem by Coifman, *Studia Math.* **64** (1979), 191–194.
22. N.M. Rivière and Y. Sagher, Interpolation between L^∞ and H^1, the real method *J. Functional Analysis* **14** (1973), 401–409.
23. E.M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series **43**, Princeton University Press, Princeton, N.J., 1993.
24. E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Mathematical Series **32**, Princeton University Press, Princeton, N.J., 1971.
25. X. Tolsa, BMO, H^1, and Calderón-Zygmund operators for non doubling measures, *Math. Ann.* **319** (2001), 89–149.
26. T.H. Wolff, A note on interpolation spaces, *Harmonic Analysis (Minneapolis, Minneapolis, Minn., 1981),* Lecture Notes in Math. **908**, Springer, Berlin-New York, 1982.