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Dimension free estimates for the bilinear Riesz transform

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Abstract

It is shown that for $1 < p_1, p_2 < \infty$, $1/p_3 = 1/p_1 + 1/p_2$, $p_3 \geq 1$ there exists C_1 (independent of n) such that

$$\|R_k(f,g)\|_{L^{p_3}(\mathbb{R}^n)} \leq C_1 \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)}$$

where

$$R_k(f,g)(x) = b_n \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} f(x-y)g(x+y) \frac{y_k}{|y|^{n+1}} dy,$$

and b_n is chosen so that R_k has norm 1 as a bilinear map from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$. In the case $p_3 > 1$ it is even shown that

$$\Big\| \Big(\sum_{k=1}^{n} |R_k(f,g)|^2 \Big)^{1/2} \Big\|_{L^{p_3}(\mathbb{R}^n)} \leq C_2 \|f\|_{L^{p_1}(\mathbb{R}^n)} \|g\|_{L^{p_2}(\mathbb{R}^n)}$$

for some constant C_2 independent of the dimension.

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1. Introduction

It is well known that the method of transference is a useful procedure for obtaining norm estimates independent of the dimension for classical operators acting on $L^p(\mathbb{R}^n, dx)$ (see for instance [1, 13, 14]) and even in the weighted situation (see for instance [7, 8, 9]). The aim of this note is to combine the techniques and methods at our disposal from the linear case (see [1, 6, 7, 18, 13]) and the "bilinear transference" method, introduced in [4] (and extended in [2, 3]), to show the boundedness of certain bilinear multipliers defined in $\mathbb{R}^n$ with the norm independent of the dimension n.

One particular case of interest in this note is the bilinear version of the classical *Riesz transforms* on $\mathbb{R}^n$, defined for $1 \le k \le n$ by

$$(R_k f)(x) = c_n \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<1/\varepsilon} f(x-y)\frac{y_k}{|y|^{n+1}}dy, \quad k = 1, 2, ..., n \tag{1}$$

where $c_n = \Gamma(\frac{n+1}{2})\pi^{-(n+1)/2}$, or equivalently, and more useful, by

$$(R_k f)\hat{}(\xi) = \frac{-i\xi_k}{(\sum_{j=1}^n \xi_j^2)^{1/2}}\hat{f}(\xi), \quad k = 1, 2, ..., n \tag{2}$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\langle x,\xi\rangle}dx$. These operators are known to satisfy, for $1 < p < \infty$, the estimate

$$\Big\|\Big(\sum_{k=1}^n |R_k(f)|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{3}$$

with a constant C independent of n.

The Riesz transforms are the basic examples of Calderón-Zygmund operators with kernels which are odd and homogeneous of degree 0.

Throughout the paper $K(x) = \frac{\Omega(x)}{|x|^n}$, where Ω is an odd function, homogeneous of degree 0 and integrable over Σ_{n-1}, i.e. $\Omega(-x) = -\Omega(x)$ and $\Omega(\lambda x) = \Omega(x)$ for $x \in \mathbb{R}^n$ and $\lambda > 0$, with $\Omega(u) \in L^1(\Sigma_{n-1})$. We define

$$T_\Omega(f) = c_n(\Omega) \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<1/\varepsilon} f(x-y)\frac{\Omega(y)}{|y|^n}dy$$

where $c_n(\Omega)$ is chosen such that $\|T_\Omega\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} = 1$, i.e.

$$c_n(\Omega)^{-1} = \|\hat{K}\|_{L^\infty(\mathbb{R}^n)}.$$

We use the notations $v_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ for the volume of the unit ball and write $d\sigma$ the normalized area measure of the sphere Σ_{n-1}. We shall see from our considerations that actually the following result holds true: The condition

$$n v_n c_n(\Omega)\|\Omega\|_{\Sigma_{n-1}} \le C \tag{4}$$

implies

$$\|T_\Omega(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$ with a constant C independent of n.

Proposition 1.1

Let $a \neq 0$ *and* $\Omega_a(x) = \langle a, \frac{x}{|x|}\rangle$. *Then*

$$c_n(\Omega_a) = |a|^{-1}\pi^{-(n+1)/2}\Gamma\Big(\frac{n+1}{2}\Big) = \frac{c_n}{|a|}.$$

Proof. It is elementary to show that if Ω is odd then

$$\hat{K}(\xi) = \frac{i\pi n v_n}{2}\int_{\Sigma_{n-1}} \Omega(u) sign\langle u, \xi\rangle d\sigma(u).$$

Hence $|\hat{K}(\xi)| \leq \frac{\pi n v_n}{2}\|\Omega\|_{L^1(\Sigma_{n-1})}$. In particular for $\Omega = \Omega_a$ one gets $\hat{K}(a) = \frac{i\pi n v_n}{2}\int_{\Sigma_{n-1}} |\Omega_a(u)| d\sigma(u)$.

Hence

$$\pi n v_n c_n(\Omega_a)\|\Omega_a\|_{L^1(\Sigma_{n-1})} = 2. \tag{5}$$

On the one hand, using polar coordinates, one has

$$\int_{|x|\leq 1} |\langle a, x\rangle| dx = \frac{n}{n+1} v_n \|\Omega_a\|_{L^1(\Sigma_{n-1})},$$

and, on the other hand, using Fubini's theorem, one also has

$$\int_{|x|\leq 1} |\langle a, x\rangle| dx = |a|\int_{|x|\leq 1} |x_1| dx = |a|\frac{2v_{n-1}}{n+1}.$$

Hence $n v_n \|\Omega_a\|_{L^1(\Sigma_{n-1})} = 2|a| v_{n-1}$ which gives

$$c_n(\Omega_a) = \frac{1}{|a|\pi v_{n-1}} = |a|^{-1}\pi^{-(n+1)/2}\Gamma\Big(\frac{n+1}{2}\Big).$$

□

In the last decade the *bilinear Hilbert transform*, given by

$$H(f,g)(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{|y|>\varepsilon}\frac{f(x-y)g(x+y)}{y} dy$$

for f, g belonging to the Schwarzt class $\mathcal{S}(\mathbb{R})$, was shown by M. Lacey and C. Thiele to be bounded from $L^2(\mathbb{R})\times L^2(\mathbb{R})$ into $L^1(\mathbb{R})$ solving an old question by A. Calderón. In their fundamental work they discover that the parameter p_3 in the range space could go even below 1.

Theorem 1.2 (see [11, 12])

Let $1 < p_1, p_2 < \infty$, $1/p_3 = 1/p_1 + 1/p_2$ *and* $2/3 < p_3 < \infty$. *Then there exists a constant* $C > 0$ *such that*

$$\|H(f,g)\|_{L^{p_3}(\mathbb{R})} \leq C\|f\|_{L^{p_1}(\mathbb{R})}\|g\|_{L^{p_2}(\mathbb{R})}. \tag{6}$$

Actually, the *bisublinear maximal Hilbert transform*, defined by

$$H^*(f,g)(x) = \sup_{\varepsilon>0} \frac{1}{\pi}\Big| \int_{|y|>\varepsilon} \frac{f(x-y)g(x+y)}{y} dy\Big|$$

for f, g belonging to the Schwarzt class $\mathcal{S}(\mathbb{R})$, was also shown to be bounded from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ into $L^1(\mathbb{R})$ by M. Lacey (see [10]), i.e. there exists a constant $C > 0$ such that

$$\|H^*(f,g)\|_{L^1(\mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}. \tag{7}$$

In a similar way we shall define the bilinear version of the operator T_Ω and shall try to get its boundedness from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ into $L^{p_3}(\mathbb{R}^n)$ under the same conditions on p_i. To analyze the independence of the dimension for the norm of the corresponding bilinear operator one needs to select the right normalization constant $b_n(\Omega)$. Let us introduce the natural choice in the following definition.

DEFINITION 1.3 Given Ω as above we define

$$B_\Omega(f,g)(x) = b_n(\Omega) \lim_{\varepsilon\to 0} \int_{\varepsilon<|y|<1/\varepsilon} f(x-y)g(x+y)\frac{\Omega(y)}{|y|^n} dy,$$

where $b_n(\Omega)$ is chosen in such a way that

$$\|B_\Omega\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n)} = 1.$$

Let us also mention the formulation in terms of Fourier transforms which is left to the reader.

Remark 1.1 Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then

$$B_\Omega(f,g)(x) = b_n(\Omega) \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\hat{K}(\xi-\eta)e^{2\pi i\langle(\xi+\eta),x\rangle} d\xi d\eta. \tag{8}$$

Proposition 1.4

Let $a \in \mathbb{R}^n \setminus \{0\}$ and $\Omega_a(x) = \frac{\langle a,x\rangle}{|x|}$. Then, for $e_1 = (1,0,...,0)$,

$$b_n(\Omega_a) = |a|^{-1} b_n(\Omega_{e_1}).$$

Proof. Let A be an orthogonal transformation of $\mathbb{R}^n$ such as $Ae_1 = \frac{a}{|a|}$ and write $f_A(x) = f(Ax)$. Then, for $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} \tilde{B}_{\Omega_a}(f,g)(Ax) &= \lim_{\varepsilon\to 0}\int_{\varepsilon<|y|<1/\varepsilon} f(Ax-y)g(Ax+y)\frac{\langle a,y\rangle}{|y|^{n+1}} dy \\ &= |a| \lim_{\varepsilon\to 0}\int_{\varepsilon<|u|<1/\varepsilon} f_A(x-u)g_A(x+u)\frac{u_1}{|u|^{n+1}} du \\ &= |a|\tilde{B}_{\Omega_{e_1}}(f_A, g_A)(x). \end{aligned}$$

This allows to conclude the result. □

Definition 1.5 For $a = e_k$, $\Omega(x) = \frac{x_k}{|x|}$, $k = 1, 2, ..., n$, the bilinear Riesz transform is given by

$$(R_k(f,g))(x) = b_n \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} f(x-y)g(x+y)\frac{y_k}{|y|^{n+1}}dy \tag{9}$$

$$= -i\frac{b_n}{c_n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\eta)\frac{\xi_k-\eta_k}{|\xi-\eta|}e^{2\pi i\langle(\xi+\eta),x\rangle}d\xi d\eta, \tag{10}$$

where

$$b_n^{-1} = \|\tilde{B}_{\Omega_{e_1}}\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n)}.$$

Hence

$$B_{\Omega_a} = |a|^{-1}\sum_{k=1}^n a_k R_k,\ a \in \mathbb{R}^n \setminus \{0\}.$$

Our aim is to show that the transforms R_k (and more generally B_Ω for certain Ω) define bounded bilinear maps from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ into $L^{p_3}(\mathbb{R}^n)$ for $\frac{1}{p_3} = \frac{1}{p_1}+\frac{1}{p_2}$ for $1 < p_1, p_2 < \infty$ and certain values of p_3 with norm independent of the dimension. As in the linear case we shall make use of the method of rotations and some transference results.

Let us mention the transference results we shall need later on. Let G be a l.c.a group with Haar measure m, let $R : G \to \mathcal{L}(L^p(\mu), L^p(\mu))$ be a representation of G into the space of bounded linear operators on $L^p(\mu)$ for some measure space (Ω, Σ, μ), i.e. $t \to R_t$ verifies $R_tR_s = R_{t+s}$ for $t, s \in G$, $\lim_{t\to 0} R_tf = f$ for $f \in L^p(\mu)$ and $\sup_{t\in G}\|R_t\| < \infty$. For a given $K \in L^1(G)$ with compact support we denote

$$C_K(\phi,\psi)(s) = \int_G \phi(s-t)\psi(s+t)K(t)dm(t)$$

(defined for nice functions ϕ, ψ defined on G). We consider the transferred operator by the formula

$$T_K(f,g)(w) = \int_G R_{-t}f(w)R_tg(w)K(t)dm(t)$$

where f and g are functions defined on Ω.

Theorem 1.6 (see [4])

Let $1 \le p_1, p_2 < \infty$ and $1/p_3 = 1/p_1 + 1/p_2$ and let R be a representation of $\mathbb{R}$ on acting $L^{p_i}(\mu)$ for $i = 1, 2$. Assume that there exists a map $S : \mathbb{R} \to \mathcal{L}(L^{p_3}(\mu), L^{p_3}(\mu))$ given by $t \to S_t$ such that S_t are invertible with $\sup_{t\in\mathbb{R}}\|S_t\| = 1$ and

$$S_s\big((R_{-t}f)(R_tg)\big) = (R_{s-t}f)(R_{s+t}g)$$

for $s, t \in \mathbb{R}$, $f \in L^{p_1}(\mu)$ and $g \in L^{p_2}(\mu)$.

If $K \in L^1(G)$ has compact support and the bilinear operator C_K is bounded from $L^{p_1}(G) \times L^{p_2}(G)$ into $L^{p_3}(G)$ with "norm" $N_{p_1,p_2}(C_K)$ then T_K is also bounded from $L^{p_1}(\mu) \times L^{p_2}(\mu)$ to $L^{p_3}(\mu)$ and with norm bounded by $CN_{p_1,p_2}(C_K)$.

Here is its maximal counterpart.

Theorem 1.7 (see [2, 3])

Let us assume the hypotheses in Theorem 1.6 and that S_u^{-1} are positive operators. Let $\{K_j\}_j$ be a family of kernels in $L^1(G)$ with compact supports $\{C_j\}_j$, such that the corresponding bisublinear maximal operator

$$C_K^*(\phi,\psi)(s) = \sup_{j\in\mathbb{N}} \Big| \int_G \phi(s-t)\psi(s+t)K_j(t)dm(t)\Big|, \tag{11}$$

is bounded from $L^{p_1}(G)\times L^{p_2}(G)$ to $L^{p_3}(G)$ with norm less than or equal to $N(\{K_j\}_j)$.

Then we have that the maximal operator

$$T_K^*(f,g) = \sup_j |T_{K_j}(f,g)(w)| = \sup_j \Big| \int_G R_{-t}f(w)R_t g(w)K_j(t)dm(t)\Big|$$

is bounded from $L^{p_1}(\mu)\times L^{p_2}(\mu)$ to $L^{p_3}(\mu)$ and it has norm bounded by $CN(\{K_j\}_j)$.

We now define the directional bilinear Hilbert transform $\mathbb{R}^n$ and bisublinear maximal Hilbert transform as follows: Given $u\in\Sigma_{n-1}$ we denote

$$H^u(f,g)(x) = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\varepsilon<|t|<1/\varepsilon} f(x-tu)g(x+tu)\frac{dt}{t}$$

and

$$H^{*u}(f,g)(x) = \sup_{\varepsilon>0}\frac{1}{\pi}\Big|\int_{\varepsilon<|t|<1/\varepsilon} f(x-tu)g(x+tu)\frac{dt}{t}\Big|.$$

We also use the notation

$$H(f,g)(x,y) = H^{y/|y|}(f,g)(x), x\in\mathbb{R}^n, y\in\mathbb{R}^n, y\neq 0.$$

For each $u\in\Sigma_{n-1}$ denote $R^u:\mathbb{R}\to\mathcal{L}(L^p(\mathbb{R}^n),L^p(\mathbb{R}^n))$ the representation given by $R_t^u f(x) = f(x-tu)$. Hence Theorem 1.6 and Theorem 1.7 can be applied, using $S_t = R_t^u$ together with the estimates (6) and (7) to obtain the following result.

Corollary 1.8

Let $1<p_1,p_2<\infty, p_3>2/3$ and $1/p_3 = 1/p_1+1/p_2$. Then H^u and H^{*u} are bounded from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ into $L^{p_3}(\mathbb{R}^n)$ with norm independent of u if $|u|=1$.

In particular $H^{*u}(f,g)\in L^1(\mathbb{R}^n)$ for each $f,g\in\mathcal{S}(\mathbb{R}^n)$ and

$$\sup_{|u|=1}\|H^{*u}(f,g)\|_{L^1}\leq C\|f\|_{L^2}\|g\|_{L^2}.$$

Here is now our version of the method of rotations in the bilinear case.

Theorem 1.9

Let $\Omega\in L^1(\Sigma_{n-1})$ be odd and homogeneous of degree 0 and let $\psi_n\in L^1(\mathbb{R}^+,\frac{dr}{r})$. Define $d\mu_n(x)=\psi_n(|x|)dx$ and $\langle f,g\rangle_{\mu_n} = \int_{\mathbb{R}^n} f(x)g(x)\psi_n(|x|)dx$. Then

$$B_\Omega(f,g)(x) = \frac{\pi}{2}\, n v_n b_n(\Omega)\int_{\Sigma_{n-1}} H(f,g)(x,u)\Omega(u)d\sigma(u), x\in\mathbb{R}^n \tag{12}$$

$$B_\Omega(f,g)(x) = \frac{\pi b_n(\Omega)}{2\|\psi_n\|_{L^1(\mathbb{R}^+,\frac{dr}{r})}}\langle H(f,g)(x,.),K\rangle_{\mu_n}, x\in\mathbb{R}^n \tag{13}$$

for $f,g\in\mathcal{S}(\mathbb{R}^n)$.

Proof. Note that if f and g belong to $\mathcal{S}(\mathbb{R}^n)$ then for each $x \in \mathbb{R}^n$ the function

$$\begin{aligned}\phi_x(y) &= (f(x-y)g(x+y) - f(x)g(x))\frac{\Omega(y)}{|y|^n}\chi_{\{|y|\leq 1\}}\\ &\quad + f(x-y)g(x+y)\frac{\Omega(y)}{|y|^n}\chi_{\{|y|>1\}}\end{aligned}$$

belongs to $L^1(\mathbb{R}^n)$ and one actually has

$$B_\Omega(f,g)(x) = b_n(\Omega)\int_{\mathbb{R}^n}\phi_x(y)dy.$$

Similarly for each $x \in \mathbb{R}^n$ and $u \in \Sigma_{n-1}$

$$\begin{aligned}\psi_{x,u}(t) &= (f(x-tu)g(x+tu) - f(x)g(x))\frac{\chi_{\{0<|t|\leq 1\}}}{t}\\ &\quad + f(x-tu)g(x+tu)\frac{\chi_{\{|t|>1\}}}{t}\end{aligned}$$

belongs to $L^1(\mathbb{R})$ (in fact, since f and g belong to $\mathcal{S}(\mathbb{R}^n)$, $|\psi_{x,u}|$ is majorized by an integrable function for all $|u|=1$, one has that $\sup_{|u|=1}\int_{\mathbb{R}}|\psi_{x,u}(t)|dt < \infty$) and

$$H(f,g)(x,u) = \frac{1}{\pi}\int_{\mathbb{R}}\psi_{x,u}(t)dt.$$

We use the spherical coordinates to obtain (12).

$$\begin{aligned}B_\Omega(f,g)(x) &= nv_nb_n(\Omega)\int_{\Sigma_{n-1}}\int_{0<t<\infty}t^{n-1}\phi_x(tu)dtd\sigma(u)\\ &= \frac{nv_n}{2}b_n(\Omega)\int_{\Sigma_{n-1}}\int_{0<|t|<1}(f(x-tu)g(x+tu) - f(x)g(x))\frac{dt}{t}\Omega(u)d\sigma(u)\\ &\quad + \frac{nv_n}{2}b_n(\Omega)\int_{\Sigma_{n-1}}\int_{1<|t|<\infty}f(x-tu)g(x+tu)\frac{dt}{t}\Omega(u)d\sigma(u)\\ &= \frac{1}{2}nv_nb_n(\Omega)\int_{\Sigma_{n-1}}\Big(\int_{\mathbb{R}}\psi_{x,u}(t)dt\Big)\Omega(u)d\sigma(u)\\ &= \frac{\pi}{2}nv_nb_n(\Omega)\int_{\Sigma_{n-1}}\Omega(u)H(f,g)(x,u)d\sigma(u).\end{aligned}$$

The previous arguments allow us to apply Fubini argument and to conclude that

$$\begin{aligned}\langle H(f,g)(x,\cdot),K\rangle_{\mu_n} &= \int_{\mathbb{R}^n}H(f,g)(x,y)\frac{\Omega(y)}{|y|^n}\psi_n(|y|)dy\\ &= nv_n\Big(\int_0^\infty\frac{\psi_n(r)}{r}dr\Big)\Big(\int_{\Sigma_{n-1}}H(f,g)(x,u)\Omega(u)d\sigma(u)\Big)\\ &= \frac{2\|\psi_n\|_{L^1(\mathbb{R}^+,\frac{dr}{r})}}{\pi b_n(\Omega)}B_\Omega(f,g)(x).\end{aligned}$$

□

Let us now get the basic estimate on $b_n(\Omega)$.

Proposition 1.10

Let Ω be defined as in the introduction. Then

$$b_n(\Omega) \leq c_n(\Omega).$$

Proof. Denote

$$\tilde{B}^{\varepsilon}_{\Omega}(f,g)(x) = \int_{\varepsilon<|y|<1/\varepsilon} f(x-y)g(x+y)\frac{\Omega(y)}{|y|^n}dy$$

and

$$\tilde{T}^{\varepsilon}_{\Omega}(f)(x) = \int_{\varepsilon<|y|<1/\varepsilon} f(x-y)\frac{\Omega(y)}{|y|^n}dy.$$

If $f, g \in \mathcal{S}(\mathbb{R}^n)$ and $\varepsilon > 0$ then the function

$$f(x-y)g(x+y)\frac{\Omega(y)}{|y|^n}\chi_{\{\varepsilon<|y|<1/\varepsilon\}} \in L^1(\mathbb{R}^n \times \mathbb{R}^n).$$

Hence we have

$$\begin{aligned}
\|\tilde{B}^{\varepsilon}_{\Omega}(f,g)\|_{L^1(\mathbb{R}^n)} &\geq \Big|\int_{\mathbb{R}^n}\int_{\varepsilon<|y|<1/\varepsilon} f(x-y)g(x+y)\frac{\Omega(y)}{|y|^n}dydx\Big| \\
&= \Big|\int_{\mathbb{R}^n}\int_{\varepsilon<|y|<1/\varepsilon} f(x')g(x'+2y)\frac{\Omega(y)}{|y|^n}dydx'\Big| \\
&= \Big|\int_{\mathbb{R}^n}\int_{2\varepsilon<|y'|<1/2\varepsilon} f(x)g(x-y')\frac{\Omega(y')}{|y'|^n}dy'dx\Big| \\
&= \Big|\int_{\mathbb{R}^n} f(x)\tilde{T}^{2\varepsilon}_{\Omega}(g)(x)dx\Big|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\tilde{B}^{*}_{\Omega}(f,g)(x) &= \sup_{\varepsilon>0}|\tilde{B}^{\varepsilon}_{\Omega}(f,g)(x)| \\
&\leq \sup_{\varepsilon>0}\Big|\int_{\Sigma_{n-1}}\int_{\varepsilon<t<1/\varepsilon} f(x-tu)g(x+tu)\frac{\Omega(u)}{t}dtd\sigma(u)\Big| \\
&\leq \sup_{\varepsilon>0}\Big|\int_{\Sigma_{n-1}}\int_{\varepsilon<t<1/\varepsilon} f(x-tu)g(x+tu)\frac{\Omega(u)}{t}dtd\sigma(u)\Big| \\
&\leq \int_{\Sigma_{n-1}}\Big|\sup_{\varepsilon>0}\int_{\varepsilon<t<1/\varepsilon} f(x-tu)g(x+tu)\frac{dt}{t}\Big||\Omega(u)|d\sigma(u) \\
&= C\int_{\Sigma_{n-1}}|H^{*u}(f,g)(x)||\Omega(u)|d\sigma(u).
\end{aligned}$$

Using the previous estimate and (7) one concludes that $B^{*}_{\Omega}(f,g) \in L^1(\mathbb{R}^n)$. Given $(\varepsilon_n)_n$ with $\varepsilon_n > 0$ and $\lim_n \varepsilon_n = 0$ one has that

$$\tilde{B}_{\Omega}(f,g)(x) = \lim_{n\to\infty}|\tilde{B}^{\varepsilon_n}_{\Omega}(f,g)(x)| \in L^1(\mathbb{R}^n)$$

and $\tilde{T}_{\Omega}(g) = \lim_n \tilde{T}^{\varepsilon_n}_{\Omega}(g)$.

Therefore, from the Lebesgue dominated convergence theorem one has

$$\begin{aligned}\|\tilde{B}_\Omega(f,g)\|_{L^1(\mathbb{R}^n)} &= \lim_{n\to\infty} \|\tilde{B}_\Omega^{\varepsilon_n}(f,g)\|_{L^1(\mathbb{R}^n)} \\ &\geq \liminf_{n\to\infty} \Big| \int_{\mathbb{R}^n} f(x)\tilde{T}_\Omega^{2\varepsilon_n}(g)(x)dx \Big| \\ &\geq \Big| \int_{\mathbb{R}^n} f(x)\tilde{T}_\Omega(g)(x)dx \Big|.\end{aligned}$$

Now taking the supremum over $f, g \in \mathcal{S}(\mathbb{R}^n)$ with $\|f\|_{L^2(\mathbb{R}^n)} = \|g\|_{L^2(\mathbb{R}^n)} = 1$ we obtain

$$\|\tilde{B}_\Omega\|_{L^2(\mathbb{R}^n)\times L^2(\mathbb{R}^n)\to L^1(\mathbb{R}^n)} \geq \|\tilde{T}_\Omega\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)}$$

and the result follows. □

An application of Minkowski's inequality in Theorem 1.9, combined with Theorem 1.6, allows us to conclude the following boundedness result.

Theorem 1.11

Let Ω be an odd kernel, homogeneous of degree 0, and let $1 < p_1, p_2 < \infty, p_3 \geq 1$ and $1/p_3 = 1/p_1 + 1/p_2$. Then $B_\Omega : L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n)$ is bounded with

$$\|B_\Omega\|_{L^{p_1}\times L^{p_2}\to L^{p_3}} \leq \frac{\pi}{2}\|H\|_{L^{p_1}\times L^{p_2}\to L^{p_3}} n v_n b_n(\Omega)\|\Omega\|_{L^1(\Sigma_{n-1})}.$$

Finally combining Theorem 1.11, Proposition 1.10 and (5) one obtains our main result.

Corollary 1.12

Let $|a| = 1$, $1 < p_1, p_2 < \infty, p_3 \geq 1$ and $1/p_3 = 1/p_1 + 1/p_2$. Then $\sum_{k=1}^n a_k R_k$ is bounded from $L^{p_1}(\mathbb{R}^n) \times L^{p_2}(\mathbb{R}^n)$ to $L^{p_3}(\mathbb{R}^n)$ with norm independent of the dimension.

Remark 1.2 Observe that Theorems 1.9 and 1.11 are valid for vector-valued kernels. We can consider $\bar{\Omega}(x) = (\Omega_1(x), ..., \Omega_n(x)) = \frac{x}{|x|}$ as a ℓ_2^n-valued kernel, where $\Omega_i = \Omega_{e_i}$.

Defining

$$B_{\bar{\Omega}}(f,g) = \big(R_1(f,g), ..., R_n(f,g)\big) = b_n \int_{\mathbb{R}^n} f(x-y)g(x+y)\frac{y}{|y|}dy,$$

the previous method does not give the analogue of (3). Note that $\|\bar{\Omega}(x)\|_{\ell_2^n} = 1$ for each $x \in \mathbb{R}^n$ gives

$$\|\bar{\Omega}\|_{L^1(\Sigma_{n-1},\ell_2^n)} = 1$$

and now, using $b_n \leq c_n$, one can only estimate $\frac{4\pi^{n/2}b_n(\bar{\Omega})}{\Gamma(\frac{n}{2})}\|\bar{\Omega}\|_{L^1(\Sigma_{n-1},\ell^2)} \leq C\sqrt{n}$.

Our aim is now to show that in spite of this observation, also the norm for the ℓ_2^n-valued formulation of the bilinear Riesz transform, at least for $p_3 > 1$, is independent of the dimension.

Let us select $\psi_n(r) = (2\pi)^{-n/2}r^{n+1}e^{-r^2/2}$ and $\Omega(x) = \Omega_a(x)$, $|a| = 1$, in Theorem 1.9. Observe that

$$\|\psi_n\|_{L^1(\frac{dr}{r})} = (2\pi)^{-n/2}\int_0^\infty r^n e^{-r^2/2}dr = (2\pi)^{-n/2}2^{(n-1)/2}\Gamma\Big(\frac{n+1}{2}\Big) = \sqrt{\frac{\pi}{2}}c_n$$

which gives

$$\frac{2\|\psi_n\|_{L^1(\frac{dr}{r})}}{\pi b_n(\Omega)} = \sqrt{\frac{2}{\pi}}\frac{c_n}{b_n}.$$

In particular, denoting by $d\gamma_n(y) = (2\pi)^{-n/2}e^{-|y|^2/2}dy$ the Gaussian measure our formula (13) becomes

$$\langle H(f,g)(x,\cdot), \langle a,\cdot\rangle\rangle_{\gamma_n} = \sqrt{\frac{2}{\pi}}\frac{c_n}{b_n}B_{\Omega_a}(f,g)(x). \tag{14}$$

Observing that the coordinate functions y_k are an orthonormal system in $L^2(\gamma_n)$ and following G. Pisier ([13]) we define $\mathcal{A}_n$ to be the subspace generated by $\{y_1, ..., y_n\}$ in $L^2(\gamma_n)$ and by $Q : L^2(\gamma_n) \to \mathcal{A}_n$ the orthogonal projection, that is

$$Q(f)(y) = \sum_{k=1}^{n}\Big(\int_{\mathbb{R}^n} f(y)y_k d\gamma_n(y)\Big)y_k. \tag{15}$$

Hence applying (14) to this particular case one gets the following analogue to the result given in [13]

$$Q(H(f,g))(x,y) = \sqrt{\frac{2}{\pi}}\frac{c_n}{b_n}\sum_{k=1}^{n} y_k R_k(f,g)(x). \tag{16}$$

This allows us to repeat Pisier's argument ([13]) and get the following analogue of (3).

Theorem 1.13

Let $1 < p_1, p_2 < \infty$, $1/p_3 = 1/p_1 + 1/p_2$, $p_3 > 1$. There exists C independent of n such that

$$\Big\|\Big(\sum_{k=1}^{n}|R_k(f,g)|^2\Big)^{1/2}\Big\|_{L^{p_3}(\mathbb{R}^n)} \leq C\|f\|_{L^{p_1}(\mathbb{R}^n)}\|g\|_{L^{p_2}(\mathbb{R}^n)}. \tag{17}$$

Proof. Following Pisier's proof one first uses the fact that

$$\Big\|\sum_{k=1}^{n}\lambda_k y_k\Big\|_{L^p(\gamma_n)} = \Big(\sum_{k=1}^{n}|\lambda_k|^2\Big)^{1/2}\gamma(p) \tag{18}$$

where

$$\gamma(p) = \Big(\int_{\mathbb{R}}|t|^p e^{-t^2/2}\frac{dt}{\sqrt{2\pi}}\Big)^{1/p}.$$

$$\begin{aligned}
&\Big\|\Big(\sum_{k=1}^{n}|R_k(f,g)|^2\Big)^{1/2}\Big\|^{p_3}_{L^{p_3}(\mathbb{R}^n)}\\
=\;& \gamma(p_3)^{-p_3}\Big\|\sum_{k=1}^{n}y_k R_k(f,g)\Big\|^{p_3}_{L^{p_3}(\mathbb{R}^n\times\gamma_n)}\\
\leq\;& C\Big(\frac{b_n}{c_n}\Big)^{p_3}\|Q(H(f,g))\|^{p_3}_{L^{p_3}(\mathbb{R}^n\times\gamma_n)}\\
\leq\;& C\|Q\|^{p_3}_{L^{p_3}(\gamma_n)\to L^{p_3}(\gamma_n)}\|H(f,g)\|^{p_3}_{L^{p_3}(\mathbb{R}^n\times\gamma_n)}\\
\leq\;& C\|Q\|^{p_3}_{L^{p_3}(\gamma_n)\to L^{p_3}(\gamma_n)}\|f\|^{p_3}_{L^{p_1}(\mathbb{R}^n)}\|g\|^{p_3}_{L^{p_2}(\mathbb{R}^n)}.
\end{aligned}$$

□

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