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A remark on the rational cohomology of $\overline{S}_{1,n}$

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Abstract

We focus on the rational cohomology of Cornalba's moduli space of spin curves of genus 1 with n marked points. In particular, we show that both its first and its third cohomology group vanish and the second cohomology group is generated by boundary classes.

1. Introduction

The moduli space of spin curves $\overline{S}_g$ was constructed by Cornalba in [6] in order to compactify the moduli space of pairs $\{$smooth genus g complex curve C, theta-characteristic on $C\}$. Cornalba's compactification turns out to be a normal projective variety equipped with a finite morphism:

$$\chi : \overline{S}_g \longrightarrow \overline{\mathcal{M}}_g$$

onto the Deligne-Mumford moduli space of stable curves of genus g (see [6, Proposition (5.2)]). The geometry of $\overline{S}_g$ (in particular, its Picard group) was investigated by Cornalba himself in [6, 7]; here instead we begin the study of the rational cohomology of $\overline{S}_g$.

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As shown by Arbarello and Cornalba in [3], the rational cohomology of $\overline{\mathcal{M}}_g$ vanishes in low odd degree, so it seems reasonable to expect that the same holds also for $\overline{S}_g$; however, a priori it is not clear at all that the morphism χ does not increase cohomology. The inductive method of [3] provides indeed an effective tool to check our guess, but the set up of the induction requires to work with moduli of pointed spin curves. Namely, for all integers g, n such that $2g-2+n>0$, we consider the moduli spaces

$\overline{S}_{g,n} := \{[(C, p_1, \ldots, p_n; \zeta; \alpha)] : (C, p_1, \ldots, p_n)$ is a genus g quasi-stable projective curve with n marked points; ζ is a line bundle of degree $g-1$ on C having degree 1 on every exceptional component of C, and $\alpha : \zeta^{\otimes 2} \to \omega_C$ is a homomorphism which is not zero at a general point of every non-exceptional component of $C\}$.

In order to put an analytic structure on $\overline{S}_{g,n}$, we can easily adapt Cornalba's construction in [6]: from the universal deformation of the stable model of $(C, p_1, \ldots, p_n)$ we obtain exactly as in [6, § 4], a universal deformation $\mathcal{U}_X \to B_X$ of $X = (C, p_1, \ldots, p_n; \zeta; \alpha)$; next, we transplant on $\overline{S}_{g,n}$ the structure of $B_X/\mathrm{Aut}(X)$ following [6, § 5]. Alternatively, we can regard $\overline{S}_{g,n}$ as the coarse moduli space associated in the easiest case $r=2$ to the stack of r-spin curves constructed by Jarvis in [10] and revisited by Abramovich and Jarvis in [1].

We recall that $\overline{S}_{g,n}$ is the union of two connected components, $\overline{S}^+_{g,n}$ and $\overline{S}^-_{g,n}$, which correspond to even and odd theta-characteristics, respectively. The main result of the present paper, which completes the research project started in [4] and continued in [5], is the following:

Theorem 1

For every n,

$$H^1(\overline{S}^+_{1,n}, \mathbb{Q}) = H^3(\overline{S}^+_{1,n}, \mathbb{Q}) = 0\,,$$

and $H^2(\overline{S}^+_{1,n}, \mathbb{Q})$ is generated by boundary classes.

We note that a similar statement holds true for the moduli space of odd theta-characteristics (see [8]) since $\overline{S}^-_{1,n} \cong \overline{\mathcal{M}}_{1,n}$.

In what follows, we work over the field $\mathbb{C}$ of complex numbers; all cohomology groups are implicitly assumed to have rational coefficients.

2. The inductive approach

As pointed out in the Introduction, we are going to apply the inductive strategy developed by Arbarello and Cornalba in [3] for the moduli space of curves. Namely, we consider the long exact sequence of cohomology with compact supports:

$$\ldots \to H^k_c(S_{1,n}) \to H^k(\overline{S}_{1,n}) \to H^k(\partial S_{1,n}) \to \ldots \tag{1}$$

Hence, whenever $H^k_c(S_{1,n}) = 0$, there is an injection $H^k(\overline{S}_{1,n}) \hookrightarrow H^k(\partial S_{1,n})$. Moreover, from [6, § 3], it follows that each irreducible component of the boundary of $\overline{S}_{1,n}$ is the image of a morphism:

$$\mu_i : X_i \to \overline{S}_{1,n}$$

where either

$$X_i = \overline{\mathcal{M}}_{0,s+1} \times \overline{S}_{1,t+1}$$

where $s + t = n$; or

$$X_i = \overline{\mathcal{M}}_{0,n+2} \,.$$

Finally, exactly as in [3, Lemma 2.6], a bit of Hodge theory implies that the map $H^k(\overline{S}_{1,n}) \to \oplus_i H^k(X_i)$ is injective whenever $H^k(\overline{S}_{1,n}) \to H^k(\partial S_{1,n})$ is. Thus, we obtain the first claim of Theorem 1 by induction, provided we show that $H^1_c(S_{1,n}) = H^3_c(S_{1,n}) = 0$ for almost all values of n, and we check that $H^1(\overline{S}_{1,n}) = H^3(\overline{S}_{1,n}) = 0$ for all remaining values of n. The first task is accomplished by the following.

Lemma 1

We have $H_k(S_{1,n}) = 0$ *for* $k > n$.

Indeed, $\mathcal{M}_{1,1} \cong \mathbb{A}^1$ is affine. Moreover, it is well-known that the forgetful morphism $\mathcal{M}_{1,n} \to \mathcal{M}_{1,1}$ is affine. Finally, the morphism $S_{1,n} \to \mathcal{M}_{1,n}$ is finite since it is the restriction of the finite morphism $\overline{S}_{1,n} \to \overline{\mathcal{M}}_{1,n}$, hence the claim holds.

Now, we give a closer inspection to $\overline{S}_{1,n}$. Of course, it is the disjoint union of $\overline{S}^+_{1,n}$ and $\overline{S}^-_{1,n}$, corresponding to even and odd spin structures respectively. However, since the unique odd theta characteristic on a smooth elliptic curve E is $\mathcal{O}_E$, there is a natural isomorphism $\overline{S}^-_{1,n} \cong \overline{\mathcal{M}}_{1,n}$, so we may restrict our attention to $\overline{S}^+_{1,n}$. First of all, the following holds:

Proposition 2

$H^1(\overline{S}^+_{1,n}) = 0$.

Proof. By the above argument, it is enough to check that $H^1(\overline{S}^+_{1,n})$ vanishes for $n = 1$. In order to do so, we claim that there is a surjective morphism

$$f : \overline{\mathcal{M}}_{0,4} \longrightarrow \overline{S}^+_{1,1} \,.$$

Indeed, let $(C; p_1, p_2, p_3, p_4)$ be a 4-pointed stable genus zero curve. The morphism f associates to it the admissible covering E of C branched at the p_i's, pointed at q_1 and equipped with the line bundle $\mathcal{O}_E(q_1 - q_2)$, where q_i denotes the point of E lying above p_i. It follows that

$$H^1(\overline{S}^+_{1,1}) \hookrightarrow H^1(\overline{\mathcal{M}}_{0,4}) = H^1(\mathbb{P}^1) = 0$$

and Proposition 2 is completely proved. □

Recall that the boundary components of $\overline{\mathcal{M}}_{1,n}$ are Δ_{irr}, whose general member is an irreducible n-pointed curve C of geometric genus zero with exactly one node, and $\Delta_{1,I}$, whose general member is the union of two smooth curves meeting at one node, C_1 of genus 1 with marked points labelled by $I \subseteq \{1, \ldots, n\}$, and C_2 of genus 0 with marked points labelled by $\{1, \ldots, n\} \setminus I$ (of course $|I| \leq n - 2$). The corresponding boundary components of $\overline{S}^+_{1,n}$ are:

- A^+_{irr}, with an even spin structure on C;
- B^+_{irr}, with an even spin structure on C blown up at the node;
- $A^+_{1,I}$, with even theta-characteristics on C_1 and C_2.

Notice that in this case $B^+_{1,I}$, whose general member should carry odd theta-characteristics on both C_1 and C_2, is empty since a smooth rational curve has no odd theta-characteristic.

Hence on $\overline{S}^+_{1,n}$ we have the *boundary classes* α^+_{irr}, β^+_{irr}, and $\alpha^+_{1,I}$; there are also the classes

$$\begin{aligned} \delta_{\mathrm{irr}} &= p^*(\delta_{\mathrm{irr}}) \\ \delta_{i,I} &= p^*(\delta_{i,I}) \end{aligned}$$

where

$$p : \overline{S}^+_{1,n} \to \overline{\mathcal{M}}_{1,n}$$

is the natural projection. Exactly as in [6, § 7], there are relations

$$\delta_{\mathrm{irr}} = \alpha^+_{\mathrm{irr}} + 2\beta^+_{\mathrm{irr}} \tag{2}$$

$$\delta_{1,I} = 2\alpha^+_{1,I}\,. \tag{3}$$

Lemma 3

The vector space $H^2(\overline{S}^+_{1,2})$ is generated by boundary classes.

Proof. We are going to deduce this from an Euler characteristic computation. Indeed, we are going to show that

$$\chi(\overline{S}^+_{1,2}) = 4\,. \tag{4}$$

Since

$$\begin{aligned} \chi(\overline{S}^+_{1,2}) &= 2h^0(\overline{S}^+_{1,2}) - 2h^1(\overline{S}^+_{1,2}) + h^2(\overline{S}^+_{1,2}) \\ &= 2 + h^2(\overline{S}^+_{1,2}) \end{aligned}$$

from (4) we may deduce that $h^2(\overline{S}^+_{1,2}) = 2$. On the other hand, since the natural projection $\overline{S}^+_{1,2} \to \overline{\mathcal{M}}_{1,2}$ is surjective, $H^2(\overline{\mathcal{M}}_{1,2})$ injects into $H^2(\overline{S}^+_{1,2})$. It follows that $H^2(\overline{S}^+_{1,2})$ is generated by δ_{irr} and $\delta_{1,\emptyset}$, which are linear combinations of α^+_{irr}, β^+_{irr}, and $\alpha^+_{1,\emptyset}$ by (2) and (3).

First of all, we compute $\chi(S^+_{1,1})$. It is clear that

$$\chi(\overline{S}^+_{1,1}) = 2h^0(\overline{S}^+_{1,1}) - h^1(\overline{S}^+_{1,1}) = 2\,.$$

On the other hand, $\partial S^+_{1,1}$ consists of exactly two points, corresponding to a 3-pointed rational curve with two marked points either identified or joined by an exceptional component. Hence

$$\chi(S^+_{1,1}) = \chi(\overline{S}^+_{1,1}) - \chi(\partial S^+_{1,1}) = 0\,. \tag{5}$$

Next, we compute $\chi(S^+_{1,2})$. The natural projection $S^+_{1,2} \to \mathcal{M}_{1,2}$ is generically three-to-one, but there are a few special fibers with less than three points. Indeed, let $(E; p_1, p_2)$ be a smooth 2-pointed elliptic curve.

The linear series $|2p_1|$ provides a realization of E as a two-sheeted covering of $\mathbb{P}^1$ ramified over ∞, 0, 1 and λ. Denote by q_0, q_1, and q_λ the points of E lying above 0, 1, and λ, so that the three even theta-characteristics of E are given by $\mathcal{O}_E(p_1 - q_0)$, $\mathcal{O}_E(p_1 - q_1)$, and $\mathcal{O}_E(p_1 - q_\lambda)$. If $\lambda = \frac{1}{2}$ and $p_2 = q_\lambda$, then the projectivity of $\mathbb{P}^1$ defined by $z \mapsto 1 - z$ induces an automorphism of $(E; p_1, p_2)$ exchanging $\mathcal{O}_E(p_1 - q_0)$ and $\mathcal{O}_E(p_1 - q_1)$. If $\lambda = -\omega$ (with $\omega^3 = 1$) and p_2 is one point lying above $\frac{\omega}{\omega - 1}$, then the projectivity of $\mathbb{P}^1$ defined by $z \mapsto \frac{z+\omega}{\omega}$ induces an automorphism of $(E; p_1, p_2)$ that exchanges ciclically its three even theta-characteristics. Since it is clear (for instance, from [9, IV, proof of Corollary 4.7]) that the above ones are the only exceptional cases, we have:

$$\chi(S^+_{1,2}) = 3\chi(\mathcal{M}_{1,2} \setminus \{2 \text{ points}\}) + 2\chi(\text{point}) + \chi(\text{point}) = 0\,. \tag{6}$$

In fact, $\chi(\mathcal{M}_{1,2}) = 1$, as observed in [3, (5.4)]. Finally we turn to the Euler characteristic of $\overline{S}^+_{1,2}$. From [6, Examples (3.2)], and (3.3), and [3], Figure 1, we may deduce that

$$\chi(\overline{S}^+_{1,2}) = \chi(S^+_{1,2}) + 2\chi(\mathcal{M}'_{0,4}) + \chi(S^+_{1,1}) + 4\,,$$

where $\mathcal{M}'_{0,n}$ denotes the quotient of $\mathcal{M}_{0,n}$ modulo the operation of interchanging the labelling of two of the marked points.

Since $\chi(\mathcal{M}'_{0,4}) = 0$ (see [3, (5.4)]), relation (4) follows from (5) and (6). □

Let P a finite set with $|P| = n$ and let x and y be distinct and not belonging to P; define

$$\xi : \overline{\mathcal{M}}_{0,P\cup\{x,y\}} \longrightarrow B^+_{\text{irr}} \hookrightarrow \overline{S}^+_{1,n}$$

by joining the points labelled x and y with an exceptional component and taking the unique even theta characteristic on the resulting curve. Then the analogue of [3, Lemma 4.5] holds:

Lemma 4

The kernel of

$$\xi^* : H^2(\overline{S}^+_{1,n}) \longrightarrow H^2(\overline{\mathcal{M}}_{0,P\cup\{x,y\}})$$

is one-dimensional and generated by δ_{irr}.

Proof. By [3, Lemma 3.16], it is clear that $\xi^*(\delta_{\text{irr}}) = 0$. Moreover, from Lemma 3 it follows that $H^2(\overline{S}^+_{1,2})$ is generated by δ_{irr} and $\delta_{1,\emptyset}$; since $\delta_{1,\emptyset}$ pulls back to $\delta_{0,\{x,y\}}$, which is not zero, the claim holds for $n = 2$. Hence we can apply the inductive argument of [3, pp. 113–114]. It follows that if $\xi^*(\alpha) = 0$ for $\alpha \in H^2(\overline{S}^+_{1,n})$ then there exists a constant a such that $\alpha - a\delta_{\text{irr}}$ restricts to zero on all boundary components of $\overline{S}^+_{1,n}$ different from A^+_{irr}. However, we claim that A^+_{irr} is linearly equivalent to $2B^+_{\text{irr}}$. Indeed, this is clear in $\overline{S}^+_{1,1} \cong \mathbb{P}^1$. If $\pi : \overline{S}^+_{1,n} \to \overline{S}^+_{1,1}$ is the natural forgetful map, then $A^+_{\text{irr}} = \pi^*(A^+_{\text{irr}})$ and $B^+_{\text{irr}} = \pi^*(B^+_{\text{irr}})$. Hence $\alpha - a\delta_{\text{irr}}$ restricts to zero on all boundary components of

$\overline{S}^+_{1,n}$ and the claim follows exactly as in [3, Lemma 4.5], from Proposition 1 and the analogue of [3, Proposition 2.8]. □

Proposition 5

The vector space $H^2(\overline{S}^+_{1,n})$ is generated by boundary classes.

Proof. Let V be the subspace of $H^2(\overline{S}^+_{1,n})$ generated by the elements $\alpha^+_{1,I}$. In view of Lemma 4 and (2), it will be sufficient to show that the morphism ξ^* vanishes modulo V. The proof is by induction on n: for the inductive step we refer to [3, pp. 114–118], while the basis of the induction is provided by Lemma 3. □

Finally, we are also able to prove the last part of Theorem (1).

Lemma 6

We have $H^3(\overline{S}^+_{1,n}) = 0$.

Proof. Once again, by the exact sequence (1) and Lemma 1, it is enough to check that $H^3(\overline{S}^+_{1,n}) = 0$ for $n \leq 3$. First, we deal with the case $n = 3$. By Proposition 5, $H^2(\overline{S}^+_{1,3})$ is generated by the six boundary classes α^+_{irr}, β^+_{irr}, $\alpha^+_{1,\emptyset}$, $\alpha^+_{1,\{1\}}$, $\alpha^+_{1,\{2\}}$, and $\alpha^+_{1,\{3\}}$. Notice further that α^+_{irr} and β^+_{irr} are linearly dependent. Indeed, if $\pi : \overline{S}^+_{1,3} \to \overline{S}^+_{1,1}$ is the natural forgetful map, from [3, Lemma 3.1 (iii)], it follows that $\alpha^+_{\mathrm{irr}} = \pi^*(\alpha^+_{\mathrm{irr}})$ and $\beta^+_{\mathrm{irr}} = \pi^*(\beta^+_{\mathrm{irr}})$, while Poincaré duality yields $H^2(\overline{S}^+_{1,1}) \cong H^0(\overline{S}^+_{1,1}) \cong \mathbb{Q}$. Hence we deduce $h^2(\overline{S}^+_{1,3}) \leq 5$; next, we claim that

$$\chi(\overline{S}^+_{1,3}) = 12\,. \tag{7}$$

The statement is a direct consequence of the claim, since

$$\begin{aligned}\chi(\overline{S}^+_{1,3}) &= 2h^0(\overline{S}^+_{1,3}) + 2h^2(\overline{S}^+_{1,3}) - 2h^1(\overline{S}^+_{1,3}) \\ &\quad - h^3(\overline{S}^+_{1,3}) \leq 12 - h^3(\overline{S}^+_{1,3})\,.\end{aligned}$$

First of all, we compute $\chi(S^+_{1,3})$. The natural projection $S^+_{1,3} \to \mathcal{M}_{1,3}$ is generically three-to-one, but there is a special fiber with only one point. Indeed, if $(E; p_1, p_2, p_3)$ is a smooth 3-pointed elliptic curve realized by the linear series $|2p_1|$ as a two-sheeted covering of $\mathbb{P}^1$ ramified over ∞, 0, 1 and $-\omega$ (with $\omega^3 = 1$) and p_2, p_3 are the two points lying above $\frac{\omega}{\omega-1}$, then the projectivity of $\mathbb{P}^1$ defined by $z \mapsto \frac{z+\omega}{\omega}$ induces automorphisms of $(E; p_1, p_2, p_3)$ exchanging ciclically its three even theta-characteristics. Therefore we have:

$$\chi(S^+_{1,3}) = 3\chi(\mathcal{M}_{1,3} \setminus \{\text{ point}\}) + \chi(\text{point}) = -2\,. \tag{8}$$

Recall that $\chi(\mathcal{M}_{1,3}) = 0$, as observed in [3, (5.4)]. Finally we turn to the Euler characteristic of $\overline{S}^+_{1,3}$. From [6, Examples (3.2)], and (3.3), and [3, Figure 2], it is clear that

$$\begin{aligned}\chi(\overline{S}^+_{1,3}) &= \chi(S^+_{1,3}) + 2\chi(\mathcal{M}'_{0,5}) + \chi(S^+_{1,1})\chi(\mathcal{M}_{0,4}) \\ &\quad + 3\chi(S^+_{1,2}) + 2\chi(\mathcal{M}_{0,4}) + 12\chi(\mathcal{M}'_{0,4}) + 3\chi(S^{(0),+}_{1,1}) + 14\,.\end{aligned}$$

Since $\chi(\mathcal{M}_{0,4}) = -1$, $\chi(\mathcal{M}'_{0,4}) = 0$ and $\chi(\mathcal{M}'_{0,5}) = 1$ (see [3, (5.4)]), now (7) follows from (5), (6) and (8). Finally, by Hodge theory of complex projective orbifolds, the surjective morphism $\overline{S}_{1,3} \to \overline{S}_{1,n}$ for $n \leq 3$ induces an injective morphism $H^3(\overline{S}_{1,n}) \to H^3(\overline{S}_{1,3})$ for $n \leq 3$. Hence the claim follows. $\square$

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